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EINSTEIN METRICS AND MAGNETIC MONOPOLES

by

Nigel Hitchin

For Jean Pierre Bourguignon on his 60th birthday

Abstract. — We investigate the geometry of the moduli space of centred magnetic monopoles on hyperbolic three-space, and derive using twistor methods some (incomplete) quaternionic Kähler metrics of positive scalar curvature. For the group SU(2) these have an orbifold compactification but we show that this is not the case for SU(3).

***Résumé* (Métriques d'Einstein et monopoles magnétiques).** — Nous étudions la géométrie des espaces de modules des monopoles maghétiques sur le 3-espace hyperbolique et nous en dérivons quelques métriques kähleriennes quaternioniques (incomplètes) de courbure scalaire positive, en utilisant des méthodes *twistor*. Celles-ci ont une compactification orbifolde pour le groupe SU(2) et nous montrons qu'il n'en est rien dans le cas du groupe SU(3).

1. Introduction

Over 20 years ago Jean Pierre Bourguignon and I were part of the team helping Arthur Besse to produce a state-of-the-art book on Einstein manifolds [**3**]. As might have been expected, the subject proved to be a moving target, and the contributors had to quickly assemble a number of appendices to cover material that came to light after all the initial planning. The last sentence of the final appendix refers to: "hyperkählerian metrics on finite dimensional moduli spaces", and so it seems appropriate to write here about some of the results which have followed on from that, and some questions that remain to be answered.

There is by now a range of gauge-theoretical moduli spaces which have natural hyperkähler metrics: the moduli space of instantons on $\mathbf{R}^4$ or the 4-torus or a K3

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surface [**16**], magnetic monopoles on $\mathbf{R}^3$ [**2**] and Higgs bundles on a Riemann surface [**12**]. The latter structure features prominently in the recent work of Kapustin and Witten on the Geometric Langlands correspondence [**15**]. Some of these metrics, in low dimensions, can be explicitly calculated, but even when this is not possible, the fact that these spaces are moduli spaces enables us to observe some geometrical properties which reflect their physical origin. In this paper we shall concentrate on the case of magnetic monopoles.

For monopoles in Euclidean space $\mathbf{R}^3$, there exist in certain cases explicit formulae (for example [**5**]), but in general we cannot write the metric down. Instead we can seek a geometrical means to describe the metrics; such a technique is provided by the use of twistor spaces, spectral curves and the symplectic geometry of the space of rational maps. This is documented in [**2**]. We review this in Section 2, drawing on new approaches to the symplectic structure.

We then shift attention to the hyperbolic version. The serious study of monopoles in hyperbolic space $\mathbf{H}^3$ was initiated long ago by Atiyah [**1**], who showed that there were many similarities with the Euclidean case. Yet the differential-geometric structure of the moduli space is still elusive, despite recent efforts [**18**], [**19**]. One would expect some type of quaternionic geometry which in the limit where the curvature of the hyperbolic space becomes zero approaches hyperkähler geometry. In Section 3 we give one approach to this, and show, following [**17**], how to resolve one of the problems that arises in attempting this – assigning a centre to a hyperbolic monopole.

The other problem, concerning a real structure on the putative twistor space, can currently be avoided only in the case of charge 2 and in Section 4 we produce, for the groups SU(2) and SU(3), quaternionic Kähler metrics on the moduli spaces of centred hyperbolic monopoles, generalizing the Euclidean cases computed in [**2**] and [**8**]. These metrics are expressed initially in twistor formalism, using the holomorphic contact geometry of certain spaces of rational maps, but we obtain some very explicit formulae as well.

For SU(2), these concrete self-dual Einstein metrics, originally introduced in [**14**], have nowadays found a new life in the area of 3-Sasakian geometry, Kähler-Einstein orbifolds and manifolds of positive sectional curvature. We consider briefly these aspects in the final section, and suggest where new examples might be found.

2. Euclidean monopoles

All of the hyperkähler moduli spaces mentioned above arise through the hyperkähler quotient construction. Recall that a hyperkähler metric on a manifold M^{4n} is defined by three closed 2-forms $\omega_1, \omega_2, \omega_3$ whose joint stabilizer at each point is conjugate to $Sp(n) \subset GL(4n, \mathbf{R})$. If a Lie group G acts on M, preserving the forms, then there usually exists a hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$. The quotient construction is the statement that the induced metric on $\mu^{-1}(0)/G$ is also hyperkähler.

For the moduli space of monopoles we use an infinite-dimensional version of this. The objects consist of connections A on a principal G-bundle over $\mathbf{R}^3$ together with a Higgs field ϕ, a section of the adjoint bundle. There are boundary conditions at infinity [**2**], in particular that $\|\phi\| \sim 1 - k/2r$, which imply that the connection on the sphere of radius R approaches a standard homogeneous connection as $R \to \infty$. The manifold M to which we apply the quotient construction then consists of pairs (A, ϕ) which differ from this standard connection by terms which decay appropriately, and in particular are in $\mathcal{L}^2$. This is formally an affine flat hyperkähler manifold where the closed forms ω_i are given by

$$\omega_i((\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)) = \int_{\mathbf{R}^3} dx_i \wedge \operatorname{tr}(\dot{A}_1 \dot{A}_2) + \int_{\mathbf{R}^3} *dx_i \wedge [\operatorname{tr}(\dot{\phi}_1 \dot{A}_2) - \operatorname{tr}(\dot{\phi}_2 \dot{A}_1)].$$

For the symplectic action of a group we take the group of gauge transformations which approach the identity at infinity suitably fast.

The zero set of the moment map in this case consists of solutions to the Bogomolny equations $F_A = *d_A\phi$, and the hyperkähler quotient is a bundle over the true moduli space of solutions – it is a principal bundle with group the automorphisms of the homogeneous connection at infinity. This formal framework has to be supported by analytical results of Taubes to make it work rigorously.

When $G = \mathrm{SU}(2)$, the connection on a large sphere has structure group $U(1)$ and Chern class k, which is called the monopole charge. The hyperkähler quotient is a manifold of dimension $4k$ which is a circle bundle over the true moduli space. It has a complete metric which is invariant under the Euclidean group and the circle action (completeness comes from the Uhlenbeck weak compactness theorem, one use of gauge theoretical results to shed light on metric properties). The gauge circle action in fact preserves the hyperkähler forms $\omega_1, \omega_2, \omega_3$, and its moment map defines a centre in $\mathbf{R}^3$. The $(4k-4)$-dimensional hyperkähler quotient can then be thought of as the moduli space of centred monopoles.

For charge 2, by using a variety of techniques [**2**], one can determine the metric explicitly. It has an action of SO(3) and may be written as

$$g = (abc)^2 d\eta^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{1}$$

where

$$ab = -2k(k')^2 K \frac{dK}{dk} \quad bc = ab - 2(k'K)^2 \quad ca = ab - 2(k'K)^2$$

$$\eta = -K'/\pi K \qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$$

and $\sigma_1, \sigma_2, \sigma_3$ are the standard left-invariant forms on SO(3).

Differentiably, this manifold can be understood in terms of the unit sphere in the irreducible 5-dimensional representation space of SO(3). For each axis there is, up to a scalar multiple, a unique axially symmetric vector in this representation and these trace out two copies of $\mathbf{R}\mathrm{P}^2 \subset S^4$. The centred moduli space is the complement of one of these, the removed point being the axis joining two widely separated monopoles. The other $\mathbf{R}\mathrm{P}^2$ parametrizes axially symmetric monopoles, which are (for any value of charge) uniquely determined by their axis.

For the group $G = \mathrm{SU}(3)$ we consider a Higgs field which asymptotically has two equal eigenvalues. On a large sphere the eigenspace is a rank two bundle with first Chern class k, again called the charge. When $k = 2$, Dancer computed this metric [**7**]. For centred monopoles it is eight-dimensional with an $\mathrm{SO}(3)\times\mathrm{PSU}(2)$ action, the first factor from the geometric action of rotations and the second from the automorphisms of the connection at infinity. Explicitly it can be written as follows:

$$g = \frac{1}{4}\sum_i (x(1+px)m_i m_i + y(1+py)n_i n_i + 2pxy m_i n_i)$$

where

$$\begin{aligned}
m_1 &= -f_1 df_1 + f_2 df_2 \quad m_2 = (f_1^2 - f_2^2)\sigma_3 \\
m_3 &= \frac{1}{px}[(pyf_3^2 - (1+py)f_1^2)\sigma_2 + f_3 f_1 \Sigma_2] \\
m_4 &= -\frac{1}{1+px+py}[(pyf_3^2 - (1+py)f_2^2)\sigma_1 + f_2 f_3 \Sigma_1] \\
n_1 &= \frac{1}{py}(-pxf_2 df_2 + (1+px)f_1 df_1) \\
n_2 &= \frac{1}{py}[((1+px)f_1^2 - pxf_2^2)\sigma_3 - f_1 f_2 \Sigma_3] \\
n_3 &= (f_1^2 - f_3^2)\sigma_2 \\
n_4 &= \frac{1}{1+px+py}[(pxf_2^2 - (1+px)f_3^2)\sigma_1 + f_2 f_3 \Sigma_1]
\end{aligned}$$

with σ_i, Σ_i invariant one-forms on $\mathrm{SO}(3)\times\mathrm{SU}(2)$, and

$$f_1 = -\frac{D\,\mathrm{cn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_2 = -\frac{D\,\mathrm{dn}(3D,k)}{\mathrm{sn}(3D,k)} \quad f_3 = -\frac{D}{\mathrm{sn}(3D,k)},$$

$$x = \frac{1}{D^3}\int_0^{3D} \frac{\mathrm{sn}^2(u)}{\mathrm{dn}^2(u)}du \quad y = \frac{1}{D^3}\int_0^{3D} \mathrm{sn}^2(u)du.$$

and $p = f_1 f_2 f_3$ for $D < 2K/3$.

Clearly there are limits to extracting information from formulae like these. Nevertheless, the restriction to certain submanifolds can be useful.

2.1. Twistor spaces. — Penrose's twistor theory provides a method for transforming the equations of a hyperkähler metric into holomorphic geometry. The idea is that the three closed two-forms of a hyperkähler manifold M can be arranged as $\omega_1, \omega_2 + i\omega_3$ which define a complex structure I for which $\omega_2 + i\omega_3$ is a holomorphic symplectic two-form and ω_1 a Kähler form. The other choices give complex structures J, K; more generally for a point $\mathbf{x} \in S^2$, $(x_1 I + x_2 J + x_3 K)^2 = -1$ and defines a complex structure.

The twistor space is the product $Z = M \times S^2$. It has a complex structure $((x_1 I + x_2 J + x_3 K), \mathbf{I})$ where $\mathbf{I}$ is the complex structure on $S^2 = \mathbf{C}\mathrm{P}^1$. The projection $p : Z \to \mathbf{C}\mathrm{P}^1$ to the second factor is holomorphic, and the fibre is M with the structure of a holomorphic symplectic manifold. There is a real structure $(m, (x_1, x_2, x_3)) \to (m, -(x_1, x_2, x_3))$. To recover the space M one sees that for $m \in M$, (m, S^2) is a holomorphic section of the projection p and M is then a component of the space of real sections.

We shall describe here how to construct the twistor space for the moduli space of SU(2) monopoles on $\mathbf{R}^3$ (see [**2**]). This involves the link with rational maps. Consider a straight line $\mathbf{x} = \mathbf{a} + t\mathbf{u}$ and the ordinary differential equation along the line $(\nabla_{\mathbf{u}} - i\phi)s = 0$. Since asymptotically $\phi \sim \mathrm{diag}(i, -i)$, there is a solution s_0 which decays exponentially at $+\infty$. Choose another solution s_1 with $\langle s_0, s_1 \rangle = 1$ using the SU(2)-invariant skew form. This is well-defined modulo the addition of a multiple of s_0. Now take s_0', a solution which decays at $-\infty$, then $s_0' = as_0 + bs_1$. There are normalizations at infinity which make the coefficient b well-defined.

Now take all lines in a fixed direction $(1,0,0)$. We split $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ with coordinates $(z, t) = (x_1 + ix_2, x_3)$, and then write $s_0'(z,t) = a(z)s_0(z,t) + b(z)s_1(z,t)$. The Bogomolny equations imply that the coefficients are holomorphic in z, and furthermore the boundary conditions tell us that for a charge k monopole $b(z)$ is a polynomial of degree k. Take $p(z)$ to be the unique polynomial of degree $k-1$ such that $p(z) = a(z)$ modulo $b(z)$ and define

$$S(z) = \frac{p(z)}{b(z)} = \frac{a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}}{b_0 + b_1 z + \ldots \cdots + b_{k-1} z^{k-1} + z^k}.$$

It is a theorem that this gives a diffeomorphism between the moduli space of monopoles and the space R_k of rational maps $S : \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^1$ of degree k which take ∞ to 0. Note that the denominator vanishes when $s_0' = as_0$ – when a solution exists which decays at both ends of the line. Such lines are called spectral lines.

We can carry out the above isomorphism for lines in any direction in $\mathbf{R}^3$ which yields a 2-sphere of complex symplectic structures. The set of spectral lines then forms a k-fold cover of S^2 which is called the spectral curve. It is more than just an abstract Riemann surface, though: it sits naturally as an algebraic curve in the total space of $\mathcal{O}(2)$, which we can identify with the space of straight lines in $\mathbf{R}^3$.

In order to construct the twistor space we need to do two things: first identify the symplectic structure, and secondly see how to glue the different complex structures together to give a bundle over $\mathbf{CP}^1$.

The original approach of the authors of [**2**] was to think more in terms of the physics of the monopoles rather than the geometry of rational maps. For charge $k = 1$ we know that the moduli space is flat $S^1 \times \mathbf{R}^3$, simply a gauge phase S^1 and a centre, a point of $\mathbf{R}^3$. As a complex symplectic manifold this is $\mathbf{C} \times \mathbf{C}^*$ with holomorphic symplectic form $dz \wedge dw/w$. Now there are solutions to the Bogomolny equations (the original existence theorem of Jaffe and Taubes) which approximate k widely separated charge one monopoles, so it is reasonable to think that asymptotically the moduli space approximates the symmetric product $S^k(\mathbf{C} \times \mathbf{C}^*)$ with symplectic form

$$\sum_1^k dz_i \wedge \frac{dw_i}{w_i}. \tag{2}$$

This symmetric product is singular but R_k gives in fact a smooth resolution of it: if $S(z) = p(z)/q(z)$ and the zeros of q are $z_1, \ldots, z_k$, then

$$S \mapsto ((z_1, p(z_1)), \ldots, (z_k, p(z_k)))$$

is the map (note that p, q being coprime means that $p(z_i) \neq 0$).

It is shown in [**2**] that the symplectic form extends, but there is now a more attractive way of defining this form (see [**10**],[**22**],[**23**]). Note that fixing all z_i gives a Lagrangian submanifold, as does fixing all w_i. In other words fixing the numerator or denominator gives two transverse Lagrangian foliations. Given $x \in \mathbf{C}$ define $f_x(S) = p(x)$, $g_x(S) = q(x)$. Then from the previous remark the Poisson brackets $\{f_x, f_y\}, \{g_x, g_y\}$ vanish. We can determine the symplectic structure by Poisson brackets of the form $\{f_x, g_y\}$ and this is defined in [**10**] by

$$\{f_x, g_y\} = \frac{p(x)q(y) - q(x)p(y)}{x - y}$$

which is the classical invariant known as the Bezoutian. Taking k points in general give local coordinates to write down the form. Clearly as $y \to x$ we get the Wronskian $p'(x)q(x) - p(x)q'(x)$, so x and y don't have to be distinct.

The expression in (2) consists of taking the points to be very special – the zeros z_i of the denominator – for then $\{f_{z_i}, g_{z_j}\} = (p(z_i)q(z_j) - q(z_j)p(z_i))/(z_i - z_j) = 0$ if $i \neq j$ and $\{f_{z_i}, g_{z_i}\} = -p(z_i)q'(z_i)$ and hence the symplectic form is

$$\sum_i \frac{1}{p(z_i)q'(z_i)} df_{z_i} \wedge dg_{z_i} = \sum_i dz_i \wedge \frac{dp(z_i)}{p(z_i)}.$$

since $q(z_i) = 0$ implies $q'(z_i)dz_i + dg_{z_i} = 0$.

To define the monopole twistor space we stick together two copies of $R_k \times \mathbf{C}$ over $\mathbf{C}^*$ by the following patching:

$$\tilde{\zeta} = \frac{1}{\zeta} \qquad \tilde{q}\left(\frac{z}{\zeta^2}\right) = \frac{1}{\zeta^{2k}} q(z) \qquad p\left(\frac{z}{\zeta^2}\right) = e^{-2z/\zeta} p(z) \mod q(z).$$

To see that this preserves the symplectic form note that if $H : R_k \times \mathbf{C}^* \to \mathbf{C}$ is defined by

$$H(S, \zeta) = \frac{1}{\zeta} \sum z_i^2$$

we obtain the Hamiltonian vector field

$$\frac{dq(z)}{dt} = 0 \qquad \frac{dp(z)}{dt} = -\frac{2z}{\zeta} p(z) \mod q(z).$$

The transformation law for p is obtained by integrating this.

To find a holomorphic section of $p : Z \to \mathbf{CP}^1$, the transformation rule for $q(z)$ shows that we must have $q(z, \zeta) = z^k + a_1(\zeta) z^{k-1} + \cdots + a_k(\zeta)$ where $a_i(\zeta)$ is a polynomial of degree $2i$. The rule for $p(z)$ (and the fact that $p(z) \neq 0$) means that on the curve $q(z, \zeta) = 0$, the line bundle with transition function $e^{-2z/\zeta}$ must be trivial. Globally, noting the transformation $z \mapsto z/\zeta^2$, this makes sense in the total space of the line bundle $\mathcal{O}(2)$ over $\mathbf{CP}^1$. But the spectral curve of a monopole is defined by the equation $q(z, \zeta) = 0$ and satisfies precisely this constraint (see [**2**]). Since the spectral curve determines the monopole we can, then, in principle find the metric on the moduli space from just two pieces of information – the spectral curve and the symplectic geometry of the space of rational functions.

3. Hyperbolic monopoles

If we replace $\mathbf{R}^3$ by hyperbolic space $\mathbf{H}^3$, then some features of the Bogomolny equations remain the same, others are radically different. The main complication is that, with the analogous boundary conditions, the SU(2) connection A has a limiting $U(1)$ connection on the boundary two-sphere at infinity which is not homogeneous. In fact the solution is uniquely determined by its boundary value [**17**]. This means that there is no obvious $\mathcal{L}^2$ metric to define on the moduli space, and no analogue of the hyperkähler quotient to suggest what sort of geometric structure the moduli space might have. Another difference is the appearance of an extra parameter, the mass m, defined as the limit of $\|\phi\|$ as $R \to \infty$. In the Euclidean case one can rescale the metric to make $m = 1$ but in the hyperbolic case this will change the value of the curvature to $-1/m$. It is convenient to have the curvature of $\mathbf{H}^3$ fixed as -1 and vary the mass. Occasionally we shall consider a limit as $m \to \infty$, and interpret it as a limit through hyperbolic metrics with curvature tending to zero.

Some features are quite similar to the Euclidean case and discussed in the original paper [**1**]. In particular, the two end-points give a parametrization of the geodesics in hyperbolic space by $S^2 \times S^2 \setminus \{x=y\}$. We give this a complex structure by letting V be the standard 2-dimensional representation space of $\mathrm{SL}(2,\mathbf{C})$ (the isometries of $\mathbf{H}^3$) and take $P(V) \times P(\bar{V}) \setminus \{x=\bar{y}\}$. By considering the equation $(\nabla_{\mathbf{u}} - i\phi)s = 0$ along a geodesic we also obtain a spectral curve for an SU(2) monopole of charge k which is the divisor of a section of $\mathcal{O}(k,k)$ on $P(V) \times P(\bar{V})$, and is therefore given by $H \in S^kV^* \otimes S^k\bar{V}^*$ where S^kV is the kth symmetric power of V. By reality $\bar{H} = H^t$ but it is shown in [**17**] that H actually defines a positive definite Hermitian form on S^kV.

The spectral curve satisfies a constraint analogous to that of a Euclidean monopole – instead of the triviality of the line bundle with transition function $e^{-2z/\zeta}$ we have the triviality of $\mathcal{O}(k+2m, -k-2m)$. Note that by removing the graph of complex conjugation from $P(V) \times P(\bar{V})$, this line bundle makes sense for any real value of m. Nonetheless, there are special reasons for considering half-integral mass, in particular any formulas we derive will be algebraic in appropriate coordinates.

Given the lack of any direct introduction of a metric structure on the moduli space, we shall attempt to use the spectral curve to generate a metric by twistor means. But problems arise even here. In the Euclidean situation the one-monopole space was flat $S^1 \times \mathbf{R}^3$; in the hyperbolic case it is $S^1 \times \mathbf{H}^3$. This carries no $\mathrm{SL}(2,\mathbf{C})$-invariant Einstein metric. If one introduces singularities then, as pointed out by Kronheimer, the spectral curves for charge one Euclidean monopole moduli spaces generate non-trivial hyperkähler metrics of A_k ALF type, which is evidence for the type of geometry to be expected in general. In the hyperbolic case one obtains this way conformal structures related to LeBrun metrics [**19**] – non-trivial geometry but still not Einstein. These low-dimensional examples therefore provide no suggestions as to what geometry to expect. On the other hand, charge one is firmly rooted in the notion of a centre – each of these four-dimensional moduli spaces has a map to Euclidean or hyperbolic space which we can regard as assigning a centre to the monopole. The problem of centres for hyperbolic monopoles has a solution given in [**17**] which we describe (in slightly different terms) next.

3.1. Centres. — Let ϵ be a skew form on V preserved by $\mathrm{SL}(2,\mathbf{C})$. Hyperbolic space is the quotient $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ which we interpret as the space of Hermitian forms ω on V such that $\omega^2 = -2\epsilon\bar{\epsilon}$. Thus the standard SU(2) preserves the two forms $\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$ and $\epsilon = dz_1 \wedge dz_2$.

From ω we can form

$$\omega^{\otimes k} \in S^kV^* \otimes S^k\bar{V}^*$$

and, using the isomorphism $V \cong V^*$ given by ϵ, define the real-valued function

$$h(\omega) = H(\omega^{\otimes k}).$$

Theorem 3.1. — *The function $h : \mathbf{H}^3 \to \mathbf{R}$ has a unique critical point which we call the centre of the monopole.*

Proof. — First consider the meaning of a critical point. The derivative of h in the direction $\dot{\omega}$ is $kH(\dot{\omega} \otimes \omega^{\otimes k-1})$. But the volume form of ω is fixed so using a Lagrange multiplier λ a critical point corresponds to

$$H(\omega^{\otimes k-1}) = \lambda\omega. \tag{3}$$

Here $H(\omega^{\otimes k-1}) \in V^* \otimes \bar{V}^*$ is the contraction of $H \in S^kV^* \otimes S^k\bar{V}^*$ with $\omega^{\otimes k-1} \in S^{k-1}V^* \otimes S^{k-1}\bar{V}^*$ using the skew form on V^*.

More explicitly, use ω at a critical point to identify $V \cong \bar{V}^*$, then $H \in S^kV \otimes S^kV$. The contraction using ω is now a contraction using the skew form ϵ which gives $H(\omega^{\otimes k-1}) \in V \otimes V$. The condition (3) says that this is a multiple of ϵ^{-1}. In other words, if $H \in S^kV \otimes S^kV$, the S^2V component in the Clebsch-Gordan decomposition of this tensor product vanishes. Now we essentially follow [**17**], showing that this condition is the vanishing of a moment map.

Choose a Hermitian metric ω on V (hence an origin in $\mathbf{H}^3$), then H can be considered as a self-adjoint endomorphism of S^kV. Since it is positive definite, we can write it as $H = Q^*Q$ for an invertible endomorphism Q. Now consider the right action of $\mathrm{SL}(2,\mathbf{C})$ on Q. This gives the transformation $H \mapsto A^*HA$ on H which is the natural isometric action of hyperbolic isometries.

Consider now $\operatorname{End} S^kV$ as a complex vector space with the right action of $\mathrm{SU}(2)$. Then the moment map is the projection of Q^*Q onto the Lie algebra of $\mathrm{SU}(2)$ in $\operatorname{End} S^kV$, and the vanishing of this is just the condition for a critical point above. Now, as shown in [**17**], an invertible Q is stable for the $\mathrm{SL}(2,\mathbf{C})$ action so by the theorem of Kempf and Ness there is a point on the $\mathrm{SL}(2,\mathbf{C})$ orbit of Q for which the moment map vanishes, and this point is unique modulo $\mathrm{SU}(2)$. Thus for any positive-definite H, an isometry, well-defined modulo the stabilizer of the origin, takes it to another H whose centre is the origin: in other words a monopole has a unique centre. $\square$

Remark 3.2. — Widely separated monopoles have a spectral curve which approximates the union of twistor lines for k distinct points, so we may try and apply the definition of centre above in this case. We therefore have k Hermitian forms $H_i \in V \otimes \bar{V}^*$ and the function h is then given by the product

$$h(\omega) = (H_1, \omega)(H_2, \omega) \dots (H_k, \omega).$$

If ω is a critical point then we rewrite the form as Hermitian matrices. Replacing h by $\log h$, the condition (3) is

$$\sum_1^k \frac{H_i}{\operatorname{tr} H_i} = \lambda I. \tag{4}$$

To reinterpret this, we use the projective (Beltrami-Klein) model of $\mathbf{H}^3$. Let P be the three-dimensional real projective space of the four-dimensional vector space of Hermitian 2×2 matrices with the quadric defined by $\det H = 0$. The interior of this quadric ($\det H > 0$) is hyperbolic space. The polar plane of the identity matrix is defined by $\operatorname{tr} H = 0$ (which of course lies outside the quadric). Removing this plane gives an affine space where the hermitian matrices H_i are represented as vectors $H_i / \operatorname{tr} H_i$. The centroid in the affine sense is

$$\frac{1}{k}\sum_1^k \frac{H_i}{\operatorname{tr} H_i}$$

and from (4) we see that with our definition this is the origin.

Given two points, there is a hyperbolic isometry interchanging the two points and preserving the geodesic joining them and whose fixed point on the line is the hyperbolic midpoint. In the projective model this is a projective transformation which preserves the polar plane of that midpoint and is hence affine, so it fixes the affine midpoint. For two points, it follows that our centre coincides with the hyperbolic midpoint of the geodesic joining the points.

3.2. Rational normal curves. — For a hyperbolic monopole, we can associate to the spectral curve a certain rational curve in projective space as follows.

Given $v \in \bar{V}$, define $H(-, v^{\otimes k}) \in S^k V^*$. Because H is invertible this defines a map of degree k from $P(\bar{V})$ to $P(S^k V^*)$, or using the $\mathrm{SL}(2, \mathbf{C})$-invariant isomorphism $V \cong V^*$, to $P(S^k V)$. The spectral curve S thus naturally defines a curve $C(S) \subset P(S^k V)$. This is a rational normal curve: the image of $v \mapsto v^{\otimes k}$ is a canonical rational normal curve Δ in $P(S^k \bar{V})$ (the diagonal when we identify $S^k P(\bar{V}) = P(S^k \bar{V})$) and $C(S)$ is its image under the projective transformation H.

We lose information in passing from S to $C(S)$ – indeed for $k = 1$ $C(S)$ is the whole space. However:

Proposition 3.3. — *The spectral curve of a centred monopole is uniquely determined by the rational normal curve $C(S)$.*

Proof. — Suppose H and H_0 define the same rational curve $C(S) = C(S_0)$ and have the same centre. Rescale the forms H, H_0 so that they have determinant one. Then H and H_0 define invertible maps from Δ to $C(S)$ and so differ by a projective transformation of Δ (this is the action of $A \in \mathrm{SL}(2, \mathbf{C})$ on the representation space

$S^k\bar{V}$). Thus, considering $H, H_0 : S^k\bar{V} \to S^kV^*$ we have $H_0 = HA$. But H, H_0 are Hermitian so

$$HA = H_0 = \bar{H}_0^t = \bar{A}^t H.$$

Equivalently, A is self-adjoint with respect to the Hermitian form H and in particular its eigenspaces are orthogonal with respect to H. But if A is not a scalar then its eigenspaces in S^kV are one-dimensional and A acts as μ^{k-2i} for $0 \leq i \leq k$. Since H and H_0 are positive definite, $\mu > 0$ and hence has a positive square root, so that $A = B^2$ for $B \in \mathrm{SL}(2, \mathbf{C})$ and B is self-adjoint with respect to H. But then

$$H_0(u, v) = H(u, B^2v) = H(Bu, Bv)$$

so H_0 is obtained from H by the isometric action on $\mathbf{H}^3$ of $B \in \mathrm{SL}(2, \mathbf{C})$. However, from the centring argument this means $B \in \mathrm{SU}(2)$ and since it has positive eigenvalues $B = 1$ and $H = H_0$. □

The spectral curve $S \subset P(V) \times P(\bar{V})$ is constrained by the condition that the bundle $\mathcal{O}(k + 2m, -k - 2m)$ is trivial. This, as we shall see next, imposes a constraint on the curve $C(S)$. From Proposition 3.3 we may consider monopoles with fixed centre, which means that we have a chosen isomorphism $V \cong \bar{V}$ and so can consider H as a linear map from S^kV to S^kV, and the spectral curve S as lying in $P(V) \times P(V)$. Its equation is then $\langle H(v^{\otimes k}), w^{\otimes k}\rangle = 0$ where the brackets denote the $\mathrm{SL}(2, \mathbf{C})$-invariant bilinear form on S^kV built from the skew form ϵ on V.

Consider the map $p : P(V) \times P(S^{k-1}V) \to P(S^kV)$ defined by symmetrizing $v \otimes q$. This is a k-fold covering (in terms of polynomials in u this is the map $(z, q(u)) \mapsto (u - z)q(u)$ so the inverse image of $r(u)$ is defined by the k roots z_i). Now if $H(v^{\otimes k}) = \mathrm{Sym}(w \otimes q)$ then $\langle H(v^{\otimes k}), w^{\otimes k}\rangle = 0$ so, restricted to the rational normal curve $C(S)$, this map is the covering $\pi : S \to P(V)$ of $P(V)$ by the spectral curve $S \subset P(V) \times P(V)$ with respect to projection on the second factor.

For convenience set $n = k + 2m$. On the spectral curve S we have a non-vanishing section of $\mathcal{O}(n, -n)$. Let $\mathcal{O}(E) = \pi_*\mathcal{O}(n, -n)$ be the direct image sheaf on $P(V)$, so that E is a rank k vector bundle. Then tautologically there is a section s of E over $P(V)$, which defines a section of the projective bundle $P(E)$.

Now

$$\pi_*\mathcal{O}(n, 0) = \pi_*(\mathcal{O}(n, -n) \otimes \mathcal{O}(0, n)) = \pi_*(\mathcal{O}(n, -n) \otimes \pi^*\mathcal{O}(n)) = \pi_*\mathcal{O}(n, -n) \otimes \mathcal{O}(n)$$

so $P(E)$ can also be written as $P(\pi_*\mathcal{O}(n, 0))$. We can then extend this definition to define a bundle E_n over $P(S^kV)$ by taking the direct image of $\mathcal{O}(n, 0)$ on $P(V) \times P(S^{k-1}V)$ under the projection p to obtain a $2k - 1$-dimensional manifold $P(E_n)$. The constraint on the spectral curve S then defines a lift of the rational normal curve $C(S)$ to a rational curve $\tilde{C}$ in this projective bundle.

We calculate now the degree of the normal bundle of $\tilde{C}$. In the fibration $P(E_n) \to P(S^kV)$, the rational curve $\tilde{C}$ is a section over $C(S)$, so its normal bundle is an extension

$$0 \to T_F \to N \to N_P \to 0$$

where N_P is the normal bundle of $C(S)$ in $P(S^kV)$. But $C(S)$ has degree k and $c_1(T\mathbf{C}\mathrm{P}^k) = k+1$ so $\deg N_P = k(k+1) - 2$.

For T_F, by Grothendieck-Riemann-Roch for the map $S \to P(V)$, the degree of E is $-k^2 + k$. The tangent bundle along the fibres T_F fits into the Euler sequence

$$0 \to \mathcal{O} \to p^*E \otimes H \to T_F \to 0$$

where H is the fibrewise hyperplane bundle (H^{-1} is the tautological bundle). On $s(P(V)) \subset P(E)$, H^{-1} coincides with the trivial subbundle of E consisting of multiples of the non-vanishing section, and hence is trivial. From the Euler sequence it follows that, restricted to the section $s(P(V))$, $\deg(T_F) = \deg(E) = -k^2 + k$. Hence

$$\deg N = \deg N_P + \deg T_F = 2k - 2.$$

Generically, we expect the holomorphic structure of this rank $2k-2$ normal bundle to be $\mathbf{C}^{2k-2}(1)$ in which case the full space of deformations of the rational curve has complex dimension $4k-4$. Indeed, if there is a real structure on the complex manifold, then this is the situation where the twistor theory for a $4k-4$-dimensional quaternionic manifold becomes a theory of rational curves [**20**], a particular case being hyperkähler geometry. One might therefore expect that the complex manifold we have defined gives some type of quaternionic geometry for the moduli space of centred monopoles. There is a problem though, which involves the real structure.

By centring, we have a quaternionic structure on V and hence a real structure on $P(S^kV)$, and we have a rational normal curve $C(S) = H(\Delta)$ depending on a spectral curve S which has a real structure. However, $C(S)$ in general is not preserved by the real structure on $P(S^kV)$. In fact the reality condition on the spectral curve implies that $\bar{C}(S) = H^t(\Delta)$, so unless $H = H^t$ we do not have reality for $C(S)$.

However for charge $k = 2$, the Clebsch-Gordan decomposition is

$$S^2V \otimes S^2V = S^4V \oplus S^2V \oplus 1$$

where $S^4V \oplus 1$ are the symmetric forms and S^2V the skew-symmetric forms on S^2V. Centring sets the S^2V component to zero and so here we do in fact have $H = H^t$.

4. Charge 2 hyperbolic monopoles

4.1. SU(2) **monopoles.** — The programme in Section 3.2 for constructing a twistor space has been carried out in [**14**] to yield a quaternionic Kähler structure on the moduli space of centred charge 2 hyperbolic monopoles. Recall (see [**3**] Chapter

14) that a quaternionic Kähler manifold is of dimension $4n$ and has a rank three bundle $\mathcal{Q}$ of 2-forms $\omega_1, \omega_2, \omega_3$ whose stabilizer at each point is conjugate to $Sp(n) \cdot Sp(1) \subset GL(4n, \mathbf{R})$. The ideal generated by the ω_i should be closed under exterior differentiation. The bundle $\mathcal{Q}$ can also be thought of as the imaginary part of a bundle of quaternion algebras. The standard example is quaternionic projective space $\mathbf{HP}^n$.

The twistor space of a quaternionic Kähler manifold is a complex manifold Z^{2n-1} with a family of rational curves C with normal bundle $\mathbf{C}^{2n-2}(1)$ and a holomorphic contact form; there must also be a real structure compatible with these. It is the contact form which is the new feature here, replacing the symplectic geometry in the hyperkähler case.

Example 4.1. — A simple example is to take $Z = P(T^*\mathbf{CP}^{n+1})$, which has a canonical contact structure. Each rational curve is determined by a line $L \subset \mathbf{CP}^{n+1}$: using a Hermitian metric on $\mathbf{C}^{n+2}$, we take $L^\perp$ to be the orthogonal projective $(n-1)$-space and then for each $x \in L$, the join $x + L^\perp$ is a hyperplane in $\mathbf{CP}^{n+1}$ with a distinguished point x on it – hence a point in $P(T^*\mathbf{CP}^{n+1})$. As x moves along the line L this defines a rational curve in Z. The corresponding quaternionic Kähler manifold is the $4n$-dimensional Wolf space $U(n+2)/U(2) \times U(n)$ – the Grassmannian of lines L in $\mathbf{CP}^{n+1}$.

Here is how to derive the metric from the twistor data. A contact structure on Z is given invariantly by a holomorphic section φ of $T^* \otimes K^{-1/n}$ where $L = K^{-1/n}$ is a line bundle such that $L^n \cong K^{-1}$, the anticanonical bundle. In this formalism $\varphi \wedge (d\varphi)^{n-1}$ is a well-defined section of the trivial bundle and is therefore allowed to be everywhere non-vanishing, the contact condition. If the normal bundle N of C is isomorphic to $\mathbf{C}^{2n-2}(1)$ then the degree of K^{-1} on C is $2n$. Restricting φ to C gives a homomorphism from the tangent bundle T_C of the curve to $K^{-1/n}$. These are both of degree 2 and we consider rational curves for which this is non-zero and hence an isomorphism. It follows that $\varphi : TZ \to K^{-1/n} \cong T_C$ is a splitting of the sequence of bundles on C:

$$0 \to T_C \leftrightarrows TZ \to N \to 0.$$

A tangent vector to the space of rational curves at a curve C is a holomorphic section Y of N, which using the above splitting we can regard as a subbundle of $TZ|_C$. Again since $N \cong \mathbf{C}^{2n-2}(1)$, we have

$$H^0(C, N) \cong H^0(C, \mathcal{O}(1)) \otimes H^0(C, N(-1)) \tag{5}$$

where $\mathcal{O}(1) = K_C^{-1/2}$.

Now since φ is a contact form, $d\varphi$ restricted to the kernel of φ is a non-degenerate skew form with values in $K^{-1/n}$ which we have just identified with $T_C = \mathcal{O}(2)$. Hence it defines a skew form on $H^0(C, N(-1))$. There is a natural skew form (the

Wronskian) on $H^0(C, \mathcal{O}(1))$, and these two define a symmetric inner product on the tensor product, which from (5) is the tangent space to the space of rational curves.

Remark 4.2. — Given a point $x \in C$, there is, up to a constant multiple, a unique section v of $\mathcal{O}(1)$ which vanishes at x and so the sections of N which vanish at x are, in the decomposition (5), of the form $s \otimes v$. Since $\langle v, v\rangle = 0$ it follows from our description of the metric that such complex vectors are null.

The above is not the approach of [**14**], which is heavily focused on an alternative viewpoint: centred charge 2 monopoles form a 4-dimensional manifold with an isometric action of SO(3), which has generically codimension one orbits. Differentiably, and equivariantly, this is the same space as the Euclidean two-monopole space mentioned in Section 2: the 4-sphere with a copy of $\mathbf{R}\mathrm{P}^2$ removed.

The ODE which defines the metric is a particular form of the Painlevé VI equation. Together with some algebraic geometry [**13**],[**14**] it gives explicit formulae for the metric like the following (the charge 2, mass 2 case)

$$g = \frac{1+r+r^2}{r(r+2)^2(2r+1)^2}dr^2 + \frac{(1-r^2)^2}{(1+r+r^2)(r+2)(2r+1)}\sigma_1^2 + \\ + \frac{1+r+r^2}{(r+2)(2r+1)^2}\sigma_2^2 + \frac{r(1+r+r^2)}{(r+2)^2(2r+1)}\sigma_3^2$$

where $\sigma_1, \sigma_2, \sigma_3$ form a standard basis of left-invariant forms on SO(3). (The interested reader should beware of typos in some of the formulae in [**14**], and cross-check with [**24**], for example). The above defines a self-dual Einstein 4-manifold with positive scalar curvature, which is how one interprets the quaternionic Kähler condition in four dimensions.

The advantage of explicit formulas (in this case either algebraic or using elliptic functions) is that the behavior of the metric can be analyzed in more detail, and in [**14**] it is shown that, for all values of n, each of these (incomplete) metrics has an orbifold singularity of angle $2\pi/(n-2)$ around the removed $\mathbf{R}\mathrm{P}^2$. This is quite unlike the Euclidean moduli space, which is complete. Another feature is that, as $n \to \infty$ the metric approaches the Euclidean monopole metric, consistent with the idea that rescaling the mass to 1 is equivalent to changing the curvature to $-1/m$.

In fact, the orbifold behaviour can be detected without calculating the metric, as can be seen by looking at the twistor space afresh.

4.2. Orbifold twistor spaces.

— We consider as in Section 3.2, for $k = 2$, the projective bundle $P(E_n) \to P(S^2V)$ obtained from the direct image of $\mathcal{O}(n, 0)$ under the projection $p : P(V) \times P(V) \to P(S^2V)$, which in this case is just the quotient by interchange of the two factors. These bundles were first introduced by Schwarzenberger

[**21**]. By Grothendieck-Riemann-Roch $c_1(E_n) = (n-1)x, c_2(E_n) = (n(n-1)/2)x^2$ where x is the positive generator of $H^2(P(S^2V), \mathbf{Z})$. In this case, the rational normal curve $C(S)$ is a conic in $P(S^2V)$ and the spectral curve is an elliptic curve, the double covering of $C(S)$ over the four points of intersection with the conic $\Delta \subset P(S^2V)$.

The twistor space itself is not the whole of $P(E_n)$, but an open subset. There is a divisor $D \subset P(E_n)$ defined by the section of $P(E_n)$ over $P(V) \times P(V)$ given by the kernel of the natural evaluation map

$$\mathrm{ev} : p^*E_n \to \mathcal{O}(n, 0). \tag{6}$$

Because the lift $\tilde{C}$ of $C(S)$ is defined by a non-vanishing section of $\mathcal{O}(n, -n)$, $\tilde{C} \cap D = \varnothing$, and the twistor space Z is actually equal to $P(E_n) \setminus D$.

Given that the metric has an orbifold singularity around a copy of $\mathbf{R}\mathrm{P}^2$, there must be a singular compactification of this twistor space by adding in a 2-sphere. We shall see this next by using algebraic geometry instead of differential geometry, by showing that D can be blown down to a rational curve.

By the definition of D, the kernel of ev is naturally isomorphic on D to the tautological bundle H^{-1} and so from (6) $H^{-1} \cong \Lambda^2 p^*E_n(-n, 0)$ but $c_1(E_n) = (n-1)x$ thus $c_1(p^*E_n) = (n-1, n-1)$ and hence $H^{-1} \cong \mathcal{O}(-1, n-1)$. The cohomology class of D is of the form $ah + bx$ where $h = c_1(H)$, and as D intersects a generic fibre of $P(E_n)$ in two points, $a = 2$. Since $H \cong \mathcal{O}(1, -n+1)$ we have $h^2[D] = -2(n-1)$ and using $h^2 = c_1(E_n)h - c_2(E)$ we find $b = n$. Hence D is a divisor of the line bundle $H^2 \otimes p^*\mathcal{O}(n)$. Its normal bundle is therefore

$$H^2 \otimes p^*\mathcal{O}(n)|_D = \mathcal{O}(2, -2n+2) \otimes \mathcal{O}(n, n) = \mathcal{O}(n+2, -n+2).$$

For $n > 2$ the second degree is negative and we can therefore blow down the second $\mathbf{C}\mathrm{P}^1$ factor in $D \cong \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$. For $n = 3$ the resulting manifold is smooth, but for $n > 3$ we have an orbifold singularity along a rational curve, locally modelled on a quotient of $\mathbf{C}^3$ by $\mathbf{Z}/(n-2)$ acting as $(z_1, z_2, z_3) \mapsto (z_1, \omega z_2, \omega z_3)$.

Remark 4.3. — In the case $n = 3$, the smooth blow-down is just $\mathbf{C}\mathrm{P}^3$. Take a rational normal curve $C \subset \mathbf{C}\mathrm{P}^3$ (a twisted cubic). It is well-known that through a generic point in $\mathbf{C}\mathrm{P}^3$ there passes a unique secant to C. The secant intersects C in two points and so this defines a rational map from $\mathbf{C}\mathrm{P}^3$ to the symmetric product $S^2C = \mathbf{C}\mathrm{P}^2$; the fibre is the secant itself. Blowing up C gives the projective bundle $P(E_3)$. Another way of looking at this is to view the family of secants as a map from S^2C to the Grassmannian of lines in $\mathbf{C}\mathrm{P}^3$ (its image is actually a Veronese surface in the Plücker embedding). The projective bundle is then the pull-back of the tautological bundle on the Grassmannian.

4.3. The contact form. — In the approach of [**14**], the contact form was defined using the Maurer-Cartan form of SO(3), but there is another way which makes contact with the symplectic geometry of rational maps discussed earlier, and gives an alternative viewpoint on the twistor space.

Consider an affine coordinate z on $\mathbf{C} \subset P(V)$ and a trivialization $dz^{-n/2}$ of $\mathcal{O}(n)$. Then for $(z_1, z_2) \in \mathbf{C} \times \mathbf{C} \subset P(V) \times P(V)$ with $z_1 \neq z_2$, the fibre of the bundle E_n defined by the direct image sheaf of $\mathcal{O}(n, 0)$ consists of a linear combination of $dz^{-n/2}$ at z_1 and z_2, and we can take local coordinates w_1, w_2 relative to this basis. On the complement of the divisor D we have w_1 and w_2 non-zero. Define a one-form by

$$\varphi = (z_1 - z_2)\left(\frac{dw_1}{w_1} - \frac{dw_2}{w_2}\right) + n(dz_1 + dz_2). \tag{7}$$

Note first that this is invariant under the exchange of z_1 and z_2 together with w_1 and w_2. Also, φ is homogeneous of degree 0 in the w_i and annihilates the Euler vector field

$$W = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}$$

and so descends to the projective bundle $P(E_n)$.

As in Section 2, we can associate to the data z_1, z_2, w_2, w_2 a degree 2 rational map

$$S(z) = \frac{a_0 + a_1 z}{b_0 + b_1 z + z^2}$$

where the denominator is $(z - z_1)(z - z_2)$ and the numerator is the unique linear polynomial which takes the value w_1 at z_1 and w_2 at z_2. That part of the twistor space Z which lies over the open set of $P(S^2V)$ consisting of quadratic polynomials with finite roots can then be interpreted as the quotient of the space of rational maps R_2 by scalar multiplication. The form φ extends too, for it may be written as $\varphi = i_U\omega + ndx$ where $x = z_1 + z_2$, ω is the symplectic form described in Section 2.1 and U is the vector field

$$(z_1 - z_2)\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right).$$

Using $x = z_1 + z_2$ and $y = z_1 z_2$ as local affine coordinates on $P(S^2V)$, we can write

$$U = (4y - x^2)\frac{\partial}{\partial y}$$

which is thus well-defined on the open set of $P(S^2V)$. It follows that φ is well-defined even where $z_1 = z_2$. Moreover,

$$\varphi \wedge d\varphi = -2ndz_1 \wedge dz_2 \wedge \left(\frac{dw_1}{w_1} - \frac{dw_2}{w_2}\right) = -2ni_W\omega \wedge \omega$$

and since ω is symplectic this defines a contact structure.

Now consider the action of a Möbius transformation $f(z) = (az+b)/(cz+d)$. It acts on $w \in \mathcal{O}(n,0)$ over z by $w \mapsto w(cz+d)^n$. A short calculation gives

$$f^*\varphi = \frac{1}{(cz_1+d)(cz_2+d)}\varphi.$$

It follows that we can use the same description replacing ∞ by $f(\infty)$. In particular, taking the three points $0, 1, \infty \in P(V)$, we cover $P(S^2V)$ by three corresponding affine open sets consisting of quadratic polynomials for which 0, or 1 or ∞ is not a root. It follows that φ extends as a line-bundle valued form over the whole of Z.

Our conclusion is that the twistor space has an alternative description: it is covered by open sets each of which is isomorphic to the quotient of the space R_2 of rational maps by scalar multiplication, where the identification preserves the contact structure (7).

4.4. SU(3) **monopoles.** — We now approach the question of what the moduli space for a hyperbolic SU(3) monopole with minimal symmetry breaking looks like. The Euclidean case was dealt with by Dancer [**7**]. There is again a spectral curve involved, which for charge 2 is an elliptic curve, but this time it is unconstrained: instead we have a choice of data, which is a pair of sections of $\mathcal{O}(\ell+1, -\ell)$ which are linearly independent at each point. This is a line bundle of degree 2 on an elliptic curve and so has a two-dimensional space of sections. When we centre the monopole we have five real degrees of freedom for the conic and an SU(2) gauge action which acts on the pair of sections, giving an 8-dimensional moduli space.

We shall try and defined a quaternionic Kähler metric from a twistor space. As we have seen before, the direct image sheaves of $\mathcal{O}(\ell+1, -\ell)$ and $\mathcal{O}(2\ell+1, 0)$ have the same projective bundle, so a pair of sections of $\mathcal{O}(\ell+1, -\ell)$ over the conic $C(S)$ defines a section of $P(E_n \otimes \mathbf{C}^2)$, a 5-dimensional complex manifold, where $n = 2\ell+1$. These sections will be the twistor lines.

Remark 4.4. — The linear independence condition means that the twistor lines lie in an open set of $P(E_n \otimes \mathbf{C}^2)$, which gives an alternative description of Z as the principal $\mathrm{PSL}(2, \mathbf{C})$ frame bundle of $P(E_n)$.

We now have to introduce a contact form and here we take the lead from the Euclidean case treated by Dancer. In [**8**] he defines a symplectic form on the space of rational maps $z \mapsto [f_1(z), f_2(z), f_3(z)]$ of degree 2 from $\mathbf{CP}^1$ to $\mathbf{CP}^2$ which take ∞ to $[0,0,1]$, so that f_1, f_2 are linear and $f_3 = (z-z_1)(z-z_2)$. If $f_1(z_i) = p_i, f_2(z_i) = q_i$ and $z_1 \neq z_2$ then this form is, up to a constant,

$$\omega = -\frac{1}{z_1 - z_2} dz_1 \wedge dz_2 + dz_1 \wedge \theta_1 + dz_2 \wedge \theta_2 + (z_1 - z_2) d\theta_1$$

where

$$\theta_1 = \frac{q_1 dp_2 - p_1 dq_2}{p_1 q_2 - p_2 q_1} \qquad \theta_2 = \frac{p_2 dq_1 - q_2 dp_1}{p_1 q_2 - p_2 q_1}.$$

Following the rational map description of $P(E_n)\setminus D$, we can apply a similar argument here and define a one-form in the open set $z_1, z_2 \neq \infty$ by

$$\varphi = (z_1 - z_2)(\theta_1 - \theta_2) + nd(z_1 + z_2) \tag{8}$$

As in the previous case, the form φ may be written $\varphi = i_U\omega + ndx$ and the fact that ω is symplectic shows that this is a contact form on the quotient by the scalars acting on the rational maps. It has the same transformation properties as the contact form in the SU(2) case, and so extends.

The twistor lines again cover $C(S) \subset P(S^2V)$. We know that S has two sections s_1, s_2 of $\mathcal{O}(\ell+1, -\ell)$. In local coordinates $(z_1, z_2, p_1, p_2, q_1, q_2)$, p_1, p_2 are the values of s_1 at z_1, z_2 and q_1, q_2 the values of s_2, so the two sections give, in the rational map picture, the two numerators.

There are now two group actions – a geometrical action of SO(3), the hyperbolic isometries fixing the centre, and a gauge action by PSU(2) which changes the basis of sections of $\mathcal{O}(\ell+1, -\ell)$ (part of the principal bundle action according to Remark 4.4).

Given the twistor space we need to find the rational curves more explicitly, but there is a very concrete way of doing this in the case where $n = 2\ell+1$ is an odd integer. The spectral curve is an elliptic curve $S \subset P(V) \times P(V)$, a divisor of $\mathcal{O}(2,2)$, and projects to a conic $C(S) \subset P(S^2V)$. Choose a point $P_0 = (x_0, y_0) \in S \subset P(V) \times P(V)$ and take the line $\{x_0\} \times P(V)$ through this point. It intersects S in a point $Q_0 = (x_0, y_1)$ and so the divisor class $P_0 + Q_0 \sim \mathcal{O}(1,0)$. Now take the line $P(V) \times \{y_1\}$ which passes through Q_0 and intersects S again in $P_1 = (x_1, y_1)$, and continue. We have the divisor classes

$$P_0 + Q_0 \sim \mathcal{O}(1,0) \quad Q_0 + P_1 \sim \mathcal{O}(0,1) \quad P_1 + Q_1 \sim \mathcal{O}(1,0)\dots$$

from which we get

$$P_0 + Q_0 + \cdots + P_\ell + Q_\ell \sim \mathcal{O}(\ell+1, 0)$$
$$Q_0 + P_1 + Q_1 + P_2 + \cdots + Q_{\ell-1} + P_\ell \sim \mathcal{O}(0, \ell)$$

and so

$$P_0 + Q_\ell \sim \mathcal{O}(\ell+1, -\ell).$$

Hence $P_\ell + Q_0$ is the zero set of a section of this bundle on S.

Down in $P(S^2V)$ we start at the image $X_0 \in C(S)$ of P_0, draw a tangent to the diagonal conic Δ to meet $C(S)$ at X_1, and continue. The Poncelet problem of the "in-and-circumscribed polygon" to two conics is the closure condition for this process and was the basis of the explicit formulas in [**13**],[**14**].

The question we ask ourselves now is whether this twistor data generates an orbifold quaternionic Kähler metric in eight dimensions. The twistor space has an open orbit under the complexified action of $\mathrm{SO}(3, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C})$ but there seems no obvious way of equivariantly blowing down any lower dimensional orbits. In fact we shall see that there is no orbifold compactification in this case. We don't need to calculate the whole metric, just the induced metric on a certain totally geodesic submanifold.

4.5. Axially symmetric monopoles. — For each charge and mass there is, for the group $\mathrm{SU}(2)$, a unique monopole which is symmetric about a given axis. For $\mathrm{SU}(3)$ this is no longer the case and we shall compute the metric restricted to a surface of revolution which represents all such axially symmetric monopoles.

An axially symmetric spectral curve is of the form $(w - \mu z)(w - \mu^{-1} z) = 0$ and this defines the rational normal curve $z \mapsto w^2 - (\mu + \mu^{-1}) wz + z^2$ in the space of quadratic polynomials in w. The parameter μ is real or complex depending on whether $\mu + \mu^{-1} - 2$ is positive or negative.

A section of $\mathcal{O}(\ell + 1, -\ell)$ constructed as above and with $(x_0, y_0) = (\mu, 1)$ is given by

$$\frac{(w - \mu)(w - \mu^3) \dots (w - \mu^{(2\ell+1)})}{(z - \mu^2)(z - \mu^4) \dots (z - \mu^{2\ell})}$$

in the local trivialization $dw^{-(\ell+1)/2} dz^{-\ell/2}$. On the branch $w = \mu z$ it has the form $\mu^{\ell+1}(z - 1)$ and on the branch $w = \mu^{-1} z$ is $\mu^{-(\ell+1)}(z - \mu^{2\ell+2})$. This is a section of a line bundle of degree 2 on a (degenerate) elliptic curve which therefore has two linearly independent sections. Changing the initial point (x_0, y_0) it is clear that this space is spanned by the two sections s_1 and s_2 where $s_1 = 1$ on each branch and $s_2 = \mu^{\ell+1} z$ on the first branch and $s_2 = \mu^{-(\ell+1)} z$ on the second.

The geometrical S^1-action $z \mapsto \lambda z$ acts as $(s_1, \lambda s_2)$ and the gauge action is $(s_1, s_2) \mapsto (\lambda^{-1/2} s_1, \lambda^{1/2} s_2)$, so coupling the two multiplies (s_1, s_2) by $\lambda^{1/2}$. This lifting of the geometric circle action means that we can consider the metric on the fixed point set, which is a totally geodesic surface of revolution.

To do the calculation we need to use coordinates for this data on a varying curve: we set $\mu = e^{2t}$ (where t is real in the first instance) and on the first branch $w = e^t u, z = e^{-t} u$ and on the second $z = e^t u, w = e^{-t} u$. Then u is a rational parametrization of the plane conic defined by the spectral curve: in fact u is an affine parameter on the diagonal conic Δ and we have transformed it by the hyperbolic isometry $\mathrm{diag}(e^t, e^{-t})$. The real structure is given by $u \mapsto -1/\bar{u}$. This provides a uniform parametrization of our family of conics.

The twistor line for an axisymmetric monopole is then given, with $n = 2\ell + 1$, by:

$$z_1 = e^t u \quad z_2 = e^{-t} u \quad p_1 = 1 \quad p_2 = 1 \quad q_1 = e^{-i\phi - nt} u \quad q_2 = e^{-i\phi + nt} u.$$

(recall that p_i and q_i are the values of s_1 and s_2 at z_i). Differentiating with respect to u, the tangent to the line is spanned by

$$X = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}.$$

Differentiating with respect to t and ϕ, an infinitesimal variation of the twistor line is given by:

$$Y = \dot{t}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right) - (i\dot{\phi} + n\dot{t})q_1\frac{\partial}{\partial q_1} - (i\dot{\phi} - n\dot{t})q_2\frac{\partial}{\partial q_2}.$$

At this stage we put into effect the description of the metric in Section 4.1. The tangent bundle of the rational curve C is defined by X and we must use the contact form to embed the normal bundle N in TZ. Thus Y is a section of TZ over C and

$$Y - \frac{\varphi(Y)}{\varphi(X)}X$$

is a section of the normal bundle.

We evaluate the contact form on the vectors X and Y to obtain

$$\varphi(X) = 2u(n\cosh t - \sinh t\coth nt) \qquad \varphi(Y) = 2i\dot{\phi}\sinh t\coth nt.$$

When $\dot{t} = 0, \dot{\phi} = 1$ the section of the normal bundle is then

$$Y_0 = -2i\frac{1}{\varphi(X)}\left[\sinh t\coth nt\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right) + n\cosh t\left(q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}\right)\right]$$

and when $\dot{\phi} = 0, \dot{t} = 1$

$$Y_1 = z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} - n\left(q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}\right).$$

On our two-dimensional submanifold, whose tangent space is spanned by $\partial/\partial t$ and $\partial/\partial\phi$, the area form is $d\varphi(Y_1, Y_0)$:

$$n\frac{n\sinh t\cosh t\,\mathrm{cosech}^2\, nt - \coth nt}{n\cosh t - \sinh t\coth nt}d\phi\wedge dt. \tag{9}$$

To obtain the metric, we also need the conformal structure which, from Remark 4.2 we can derive by considering complex variations of the twistor line which preserve a point. Our twistor lines are $\mathbf{C}^*$-invariant so we have to consider variations preserving a fixed point of the action. Unfortunately, this is where our local coordinates break down – we have to consider a description of the vector bundle E_n at a branch point of the covering $p : P(V)\times P(V) \to P(S^2V)$ restricted to the conic. The map p is the quotient by permuting the factors, so we can describe it as $p(w,z) = [1, w+z, wz] \in \mathbf{C}\mathrm{P}^2$. The equation of the spectral curve is $w^2 - (\mu+\mu^{-1})wz + z^2 = 0$ and clearly any local function $f(w,z)$ can be written as $g(z) + wh(z)$ modulo the equation of the curve, or more conveniently as

$$f_0(w+z) + (w-z)f_1(w+z).$$

The direct image sheaf of $\mathcal{O}$ is then generated by 1 and $w - z$ over the functions on the conic.

We apply this now to our two sections s_1, s_2 of $\mathcal{O}(\ell + 1, -\ell)$. The first is equal to 1 on both branches and gives $f_1 = 0, f_0 = 1$. The second is $\mu^{\ell+1} z$ if $w = \mu z$ and $\mu^{-(\ell+1)} z$ if $w = \mu^{-1} z$. This gives

$$f_0(u) = \frac{1}{2}\frac{\cosh(2\ell+1)t}{\cosh t}u \qquad f_1(u) = \frac{1}{2}\frac{\sinh(2\ell+1)t}{\sinh t}.$$

Thus a variation of the twistor line which keeps the point $u = 0$ fixed is obtained by fixing (again with $n = 2\ell + 1$)

$$e^{-i\phi}\frac{\sinh nt}{\sinh t}.$$

It follows that the conformal structure is defined by

$$d\phi^2 + (n\coth nt - \coth t)^2 dt^2.$$

From (9), we conclude:

Proposition 4.5. — *The metric g restricted to the space of axially symmetric* SU(3) *monopoles is:*

$$n\frac{\coth nt - n\sinh t\cosh t\,\mathrm{cosech}^2\, nt}{(n\cosh t - \sinh t\coth nt)(n\coth nt - \coth t)}(d\phi^2 + (n\coth nt - \coth t)^2 dt^2).$$

When, in the equation of the spectral curve, $\mu + \mu^{-1} < 2$ then t becomes imaginary and we must replace the hyperbolic functions by the corresponding trigonometrical ones. Note that, near $\mu = 1$ (or $t = 0$), the metric is still regular and behaves like

$$\frac{2n}{n^2-1}(d\phi^2 + \left(\frac{n^2-1}{3}\right)^2 t^2 dt^2).$$

Fix n and consider the limit of the metric as $t \to \infty$. We find that the metric approximates

$$\frac{ne^{-t}}{(n-1)^2}(d\phi^2 + (n-1)^2 dt^2)$$

and putting $r^2 = e^{-t}$ this gives

$$\frac{n}{(n-1)^2}(r^2 d\phi^2 + 4(n-1)^2 dr^2)$$

which has an orbifold singularity: a quotient by $\mathbf{Z}/2(n-1)$. Here the spectral curve exactly corresponds to a pair of points, so the region is analogous to the orbifold singularity in the SU(2) case.

At the other extreme, consider the metric on the trigonometric branch:

$$n\frac{\cot nt - n\sin t\cos t\,\mathrm{cosec}^2\, nt}{(n\cos t - \sin t\cot nt)(n\cot nt - \cot t)}(d\phi^2 + (n\cot nt - \cot t)^2 dt^2)$$

as $t = \pi/n - u$ for u small. Then we obtain

$$n \cos \frac{\pi}{n}(d\phi^2 + u^{-2}du^2)$$

which is asymptotic to a cylinder.

Now suppose the moduli space had an equivariant orbifold compactification. Then the fixed point set of the circle action would extend to a compact orbifold and in particular would have finite area. But the cylinder has infinite area and so an orbifold compactification is impossible.

Remark 4.6. — Dancer explicitly wrote down the metric in the Euclidean case. If we fix t and ϕ and put $r = nt$, then the metric ng as $n \to \infty$ has a limit which is

$$\frac{r(\coth r - r \operatorname{cosech}^2 r)}{r \coth r - 1}\left(d\phi^2 + \left(\coth r - \frac{1}{r}\right)^2 dr^2\right).$$

This is precisely Dancer's metric (see [**7**] Theorem 5.1, or put $f_1 = -D \coth 3D, f_2 = f_3 = -D \operatorname{cosech} 3D$ in the formula in Section 2.) Thus in the infinite mass limit, or as the curvature of hyperbolic space tends to zero, our metric approaches the known Euclidean monopole metric. In the Euclidean case, the metric is asymptotically cylindrical where ours has an orbifold singularity, and asymptotically conical (with vertex angle $\pi/3$) where ours is cylindrical.

5. New metrics for old

The relationship between these metrics and their physical origins in the study of monopoles on hyperbolic space is not at all clear. We have proceeded by analogy and used spectral data rather than the fields themselves to provide a route to the metric. On the other hand they provide us also with a means for constructing other solutions to Einstein's equations. As the reader may find in [**3**], when a quaternionic Kähler manifold has positive scalar curvature, its twistor space has a natural Kähler-Einstein metric. Thus the singular spaces obtained in Section 4.2 are Fano varieties with explicit Kähler-Einstein metrics. But one can go further – the principal SO(3) bundle of the rank three bundle $\mathcal{Q}$ of imaginary quaternions on a quaternionic Kähler manifold also has a natural Einstein metric. This is a 3-Sasakian metric (which also means that by rescaling the SO(3) orbits one can find yet another Einstein metric). The $4n+3$-dimensional 3-Sasakian manifold is a principal S^1-bundle over the twistor space. One should read about these in the recently published book of Boyer and Galicki [**4**], in many respects a worthy successor to [**3**]. (The authors of that book note that "3-Sasakian manifolds are never mentioned in Besse" which is quite true, though had Arthur Besse known Bär's result that the metric cone on a 3-Sasakian manifold is hyperkähler he would almost certainly have taken them more seriously). For our orbifold examples, the 3-Sasakian manifold is actually smooth: the circle action is semi-free and has finite isotropy subgroup over the singular points of the twistor space. What is perhaps more interesting is that these 7-manifolds – as differentiable manifolds with a cohomogeneity one group action – have occurred in a completely different context, that of manifolds of positive curvature **[11]**,**[24]**. A series of manifolds P_k and Q_k were found to be candidates for having metrics of positive sectional curvature. These manifolds are the (2-fold) universal covers of the 3-Sasakian manifolds associated to the moduli spaces of hyperbolic charge 2 monopoles of mass $(2k-1)/2$ and k respectively. Quite recently, Dearricott (unpublished) and, independently, Ziller **[24]** have shown that P_2 does indeed admit a positively curved metric.

From **[9]**, the sectional curvature of the 3-Sasakian metric on the 7-manifold will be positive if the sectional curvature of the 4-manifold is positive. For the Einstein metrics described above this is true when the two monopoles are well separated but not when they are close to an axially symmetric one. Indeed, the scattering of Euclidean monopoles described in **[2]** involves some negative curvature behaviour which seems likely to persist in the hyperbolic case. The positively curved examples on P_2 are constructed by concretely deforming the 3-Sasakian metric.

There may however be other self-dual Einstein structures on the 4-dimensional spaces. Indeed, one of the spin-offs of Dancer's work on SU(3) monopoles was a hyperkähler deformation of the metric (1), obtained as a hyperkähler quotient of the Euclidean SU(3) moduli space. In the hyperbolic SU(3) case described in Section 4.4 we have an action of the rank two group $\mathrm{SO}(3)\times\mathrm{PSU}(2)$ and so we could attempt to take a quotient by a circle subgroup. Note that if the circle is in the gauge action PSU(2), then it has a commuting SO(3) action which descends to the quaternionic Kähler quotient, so already we know that this particular quotient is an SO(3)-invariant self-dual Einstein manifold. In fact this quotient is the SU(2) moduli space. To see this, recall **[4]** that from the twistor point of view quaternionic Kähler reduction proceeds by evaluating the contact form φ on the vector fields generated by the group action to get a section of $\mathfrak{g}^*\otimes K^{-1/n}$. The twistor space of the reduction is the quotient of the zero-set of this by the complexified group action. In our case the gauge circle action is $(p_1,p_2,q_1,q_2)\mapsto(p_1,p_2,e^{i\theta}q_1,e^{i\theta}q_2)$ which generates the vector field

$$X=q_1\frac{\partial}{\partial q_1}+q_2\frac{\partial}{\partial q_2}.$$

Evaluating the contact form (8) gives

$$\varphi(X)=(z_1-z_2)\left(\frac{p_2q_1+p_1q_2}{p_2q_1-p_1q_2}\right)$$

and the zero set of this is $p_2q_1+p_1q_2=0$. As a subset of $P(E_n\otimes\mathbf{C}^2)$ the equation $p_2q_1-p_1q_2=0$ is the quadric $P(E_n)\times\mathbf{CP}^1$ and the complement is the SU(3) twistor space. The quotient twistor space is defined by $p_2q_1 + p_1q_2 = 0$ modulo $(p_1, p_2, q_1, q_2) \mapsto (\lambda p_1, \lambda p_2, \mu q_1, \mu q_2)$. The projection to $[p_1, p_2] \in P(E_n)$ maps this isomorphically to the complement of $p_1 = 0$ and $p_2 = 0$ which is $P(E_n) \setminus D$, the twistor space for the SU(2) moduli space.

A more general circle subgroup of $\mathrm{SO}(3) \times \mathrm{PSU}(2)$ will yield a quotient with only a circle action, but whether it is an orbifold metric or not requires further investigation which we have no time to pursue here.

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