

Astérisque

YVES COLIN DE VERDIÈRE

**Semi-classical measures and entropy [after Nalini
Anantharaman and Stéphane Nonnenmacher]**

Astérisque, tome 317 (2008), Séminaire Bourbaki,
exp. n° 978, p. 393-414

<http://www.numdam.org/item?id=AST_2008_317_393_0>

© Société mathématique de France, 2008, tous droits réservés.

L'accès aux archives de la collection « Astérisque » ([http://smf4.emath.fr/
Publications/Asterisque/](http://smf4.emath.fr/Publications/Asterisque/)) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>*

SEMI-CLASSICAL MEASURES AND ENTROPY
[after Nalini Anantharaman and Stéphane Nonnenmacher]

by Yves COLIN de VERDIÈRE

INTRODUCTION

This report is about recent progress on semi-classical localization of eigenfunctions for quantum systems whose classical limit is hyperbolic (Anosov systems); the main example is the Laplace operator on a compact Riemannian manifold with strictly negative curvature whose classical limit is the geodesic flow; the quantizations of hyperbolic cat maps, called “quantum cat maps”, are other nice examples. All this is part of the field called “quantum chaos”. The new results are:

- Examples of eigenfunctions for the cat maps with a strong localization (“scarring”) effect due to S. de Bièvre, F. Faure and S. Nonnenmacher [17, 16].
- Uniform distribution of Hecke eigenfunctions in the case of arithmetic Riemann surfaces by E. Lindenstrauss [26].
- General lower bounds on the entropy of semi-classical measures due to N. Anantharaman [1] and improved by N. Anantharaman–S. Nonnenmacher [3] and N. Anantharaman–H. Koch–S. Nonnenmacher [2]. This lower bound is sharp with respect to the cat maps examples.

We will mainly focus on this last result.

1. THE 2 BASIC EXAMPLES

1.1. Cat maps

We start with a matrix $A \in SL_2(\mathbb{Z})$ which is assumed to be hyperbolic: the eigenvalues λ_{\pm} of A satisfy $0 < |\lambda_-| < 1 < |\lambda_+|$. The action of A onto \mathbb{R}^2 defines a symplectic action U of A on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by considering action on points mod \mathbb{Z}^2 .

Such a map is a simple example of a chaotic map. It has been observed since a long time that such a map can be quantized: for each integer N , we consider the Hilbert space \mathcal{H}_N of dimension N of Schwartz distributions f which are periodic of period one and of which Fourier coefficients are periodic of period N : if $f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}$, we have, for all $k \in \mathbb{Z}$, $a_{k+N} = a_k$. Using the metaplectic representation applied to A , we get a natural unitary action \hat{U}_N onto the space \mathcal{H}_N . We are mainly interested in the eigenfunctions of \hat{U}_N . The semi-classical parameter is $\hbar = 1/N$ and the classical limit corresponds to large values of N . A good reference is [8].

1.2. The Laplace operators

On a smooth compact connected Riemannian manifold (X, g) without boundary, we consider the Laplace operator Δ given in local coordinates by

$$\Delta = -|g|^{-1} \partial_i g^{ij} |g| \partial_j$$

with $|g| = \det(g_{ij})$. The Laplace operator Δ is essentially self-adjoint on $L^2(X)$ with domain the smooth functions and has a compact resolvent. The spectrum is discrete and denoted by

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

with an orthonormal basis of eigenfunctions φ_k satisfying $\Delta \varphi_k = \lambda_k \varphi_k$. It is useful to introduce an effective Planck constant (the semi-classical small parameter) $\hbar := \lambda_k^{-\frac{1}{2}}$. We will rewrite the eigenfunction equation $\hbar^2 \Delta \varphi = \varphi$. The semi-classical limit $\hbar \rightarrow 0$ corresponds to the high frequency limit for the periodic solutions $u(x, t) = \exp(i\sqrt{\lambda_k}t)\varphi_k$ of the wave equation $u_{tt} + \Delta u = 0$. Instead of the wave evolution, we will use the Schrödinger evolution which is given by

$$\frac{\hbar}{i} u_t = -\frac{\hbar^2}{2} \Delta u ,$$

and introduce the unitary dynamics defined by the 1-parameter group

$$\hat{U}^t = \exp(-it\hbar\Delta/2), \quad t \in \mathbb{R}.$$

For the basic definitions, one can read [5].

1.3. The geodesic flow

If (X, g) is a Riemannian manifold and $v \in T_x X$ a tangent vector at the point $x \in X$, we define, for $t \in \mathbb{R}$, $G^t(x, v) = (y, w)$ as follows: if $\gamma(t)$ is the geodesic which satisfies $\gamma(0) = x$, $\dot{\gamma}(0) = v$, we put $y := \gamma(t)$ and $w := \dot{\gamma}(t)$. By using the identification of the tangent bundle with the cotangent bundle induced by the metric g (which is also the Legendre transform of the Lagrangian $\frac{1}{2}g_{ij}(x)v_i v_j$), we get a flow $(G^t)^*$ on T^*X which preserves the unit cotangent bundle denoted by Z . We denote by U^t the restriction of $(G^t)^*$ to Z . The *Liouville measure* dL on Z is the Riemannian

measure normalized as a probability measure. The Liouville measure dL is invariant by the geodesic flow.

2. CLASSICAL CHAOS

Good textbooks on the classical chaos are [21, 28, 10].

2.1. Classical Hamiltonian systems

We consider a closed phase space Z which is the torus $\mathbb{R}^2/\mathbb{Z}^2$ in the case of the cat map and the unit cotangent bundle in the case of the Laplace operator. On Z , we have the Liouville measure dL which is normalized as a probability measure. Moreover, we have a measure preserving smooth dynamics on Z which is the action of U in the cat map example and the geodesic flow in the Riemannian case. We will denote this action by U^t where t belongs to \mathbb{Z} or to \mathbb{R} .

2.2. Ergodicity

DEFINITION 2.1. — *The dynamical system (Z, U^t, dL) is ergodic if every measurable set which is invariant by U^t is of measure 0 or 1.*

As a consequence, we get the celebrated *Birkhoff ergodic Theorem*:

THEOREM 2.2. — *If (Z, U^t, dL) is ergodic, for every $f \in L^1(Z, dL)$ and almost every $z \in Z$:*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(U^t z) dt = \int_Z f dL .$$

The cat map is ergodic and the geodesic flow of every closed Riemannian manifold with < 0 sectional curvature is ergodic too.

2.3. Mixing

A much stronger property is the *mixing property* which says that we have a correlation decay:

DEFINITION 2.3. — *The dynamical system U^t is mixing if for every $f, g \in L^2(Z, dL)$ with $\int_Z f dL = 0$, we have*

$$\lim_{t \rightarrow \infty} \int_Z f(U^t(z))g(z)dL = 0 .$$

Cat maps as well as geodesic flows on manifolds with < 0 curvature are mixing. Mixing systems are ergodic.

2.4. Liapounov exponent

Chaotic systems are often presented as (deterministic) dynamical systems which are very sensitive to initial conditions.

DEFINITION 2.4. — *The global Liapounov exponent Λ_+ of the smooth dynamical system (Z, U^t) is defined as the lower bounds of the Λ 's for which the differential dU^t of the dynamics satisfies*

$$\|dU^t(z)\| = O(e^{\Lambda t}) ,$$

for $t \rightarrow +\infty$, uniformly w.r. to z .

For cat maps given by A , $\Lambda_+ = \log |\lambda_+|$. If X is a Riemannian manifold of sectional curvature -1 , $\Lambda_+ = 1$.

2.5. K-S entropy

Kolmogorov and Sinai start from the work of Shannon in information theory in order to introduce an entropy $h_{\text{KS}}(\mu)$ for a dynamical system with an invariant probability measure μ . The definition of the entropy uses partitions of the phase space and how they are refined by the dynamics:

DEFINITION 2.5. — *If $\mathcal{P} = \{\Omega_j | j = 1, \dots, N\}$ is a finite measurable partition of Z , we define the entropy $h(\mathcal{P}) := -\sum \mu(\Omega_j) \log \mu(\Omega_j)$.*

In terms of information theory, it is the average information you get by knowing in which of the Ω_j 's the point z lies. Let $\mathcal{P}^{\vee N}$ be the partition whose sets are

$$\Omega_{j_1, j_2, \dots, j_N} = \{z \in Z \text{ so that, for } l = 1, \dots, N+1, U^{l-1}(z) \in \Omega_{j_l}\} .$$

If we define $\mathcal{P}_1 \vee \mathcal{P}_2$ as the partition whose elements are the intersections of one element of the partition \mathcal{P}_1 and one element of the partition \mathcal{P}_2 , we get from the properties of the log function:

$$h(\mathcal{P}_1 \vee \mathcal{P}_2) \leq h(\mathcal{P}_1) + h(\mathcal{P}_2) .$$

Let us define $\mathcal{P}_1 = \mathcal{P}^{\vee n}$ and $\mathcal{P}_2 = U^{-n}(\mathcal{P}^{\vee m})$. Using the *invariance*⁽¹⁾ of μ by U , we get $h(\mathcal{P}_2) = h(\mathcal{P}^{\vee m})$. From $\mathcal{P}^{\vee(n+m)} = \mathcal{P}_1 \vee \mathcal{P}_2$, we get the *sub-additivity* of the sequence $N \rightarrow h(\mathcal{P}^{\vee N})$.

We define

$$h_{\text{KS}}(\mathcal{P}) := \lim_{N \rightarrow \infty} h(\mathcal{P}^{\vee N})/N ,$$

and $h_{\text{KS}}(\mu) = \sup_{\mathcal{P}} h_{\text{KS}}(\mathcal{P})$.

In the case of an hyperbolic dynamics, the entropy is reached by a partition whose all sets have small enough diameters.

⁽¹⁾ The invariance of μ is used in a crucial way here and, as we will see, it is one of the problem we have to solve when passing to the quantum case.

Useful remarks are:

- In the case of an hyperbolic dynamics, the entropy is reached by a partition whose all sets have small enough diameters.
- The entropy h_{KS} is an affine function on the convex set of invariant probability measures.
- The entropy is lower semi-continuous for the weak topology on the set of invariant probability measures.

A more intuitive definition was provided by the work of Brin and Katok. Let us choose some point $z \in Z$ and some $\epsilon > 0$. We define

$$d_t(z, z') = \sup_{0 \leq l \leq t} d(U^l(z), U^l(z')) .$$

THEOREM 2.6. — *If μ is a probability measure on Z which is invariant by U^t , the Kolmogorov-Sinaï entropy $h_{\text{KS}}(\mu)$ is given by*

$$h_{\text{KS}}(\mu) = \int_Z h_\mu(z) d\mu$$

with

$$h_\mu(z) = \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{|\log(\mu(\{z' | d_t(z, z') \leq \epsilon\})|)}{t} .$$

2.6. Hyperbolicity

Cat maps as well as geodesic flows on manifolds with < 0 curvature are *hyperbolic systems* in the sense of Anosov. They are the smooth dynamical systems which have the strongest chaotic properties. Let us give the definitions for flows:

DEFINITION 2.7. — *A smooth dynamical system (Z, U^t) generated by the vector field V is Anosov if there is a continuous splitting*

$$TZ = E_+ \oplus E_- \oplus \mathbb{R}V$$

so that, if dU^t is the differential of U^t , the splitting is preserved by dU^t , and, if dU_+^t (resp. dU_-^t) is the restriction of dU^t to E_+ (resp. E_-), there exist $C > 0$ and $k > 0$ so that:

$$\begin{aligned} \forall t \geq 0, \quad & \|dU_+^t\| \leq Ce^{-kt}, \\ \forall t \leq 0, \quad & \|dU_-^t\| \leq Ce^{kt}. \end{aligned}$$

The bundle E_+ (resp. E_-) is called the stable (resp. unstable) bundle.

Remark 2.8. — The stable and the unstable bundles are *integrable*. Each leaf is smooth: a stable leaf consists of points z which have asymptotic trajectories as $t \rightarrow +\infty$. However, in general, the stable bundle and the unstable bundle are *not smooth*, but only Hölder continuous.

We define then the unstable Jacobian $J_u(z)$ as the absolute value of the Jacobian determinant of $dU_-^1(z)$ w.r. to some Riemannian metric on Z . We have the following nice result which is a combination of results by Ruelle, Pesin [28] and Ledrappier-Young [25]:

THEOREM 2.9. — *If the dynamical system (Z, U^t) is Anosov and dL is an invariant absolutely continuous measure, for every invariant probability measure μ , we have:*

$$h_{\text{KS}}(\mu) \leq \int_Z \log(J_u(z))d\mu .$$

Moreover, with equality if and only if $\mu = dL$ ⁽²⁾.

3. TIME SCALES IN SEMI-CLASSICS

Good introductions to semi-classical analysis are [13, 14].

3.1. Ehrenfest time

Due to Heisenberg uncertainty principle, the wave packets in quantum mechanics cannot be localized into sets of “size”⁽³⁾ less than \hbar .

The Ehrenfest time is the time it takes for a cell of size \hbar to be expanded to the whole phase space, more precisely:

DEFINITION 3.1. — *The Ehrenfest time T_E is defined by*

$$T_E := \frac{|\log \hbar|}{\Lambda_+} .$$

Many estimates in semi-classics, which are well known for fixed finite time, can be extended uniformly to times which are of the order of a suitable fraction of T_E . For example Egorov Theorem [9] and the semi-classical trace formula [15].

⁽²⁾ The Jacobian $J_u(z)$ depends on the choice of a metric on Z , but the previous integral does not.

⁽³⁾ In fact Heisenberg principle would give a diameter of the order $\sqrt{\hbar}$, but it will only change the Ehrenfest time by a factor 2.

3.2. Heisenberg time

The Heisenberg time is the time needed to resolve the spectrum from the observation of a wave at some point $x_0 \in X$: we have $u(x_0, t) = \sum a_j \exp(-itE_j/\hbar)$ and we can get approximate values of the E_j 's only by knowing $u(x_0, t)$ on a window of time larger than the Heisenberg time.

This time is of the order of $\hbar/\delta E$ where δE is the (mean) spacing of eigenvalues. Using Weyl's law, δE is of the order \hbar^d where d is the dimension of the configuration space.

DEFINITION 3.2. — *The Heisenberg time is*

$$T_H := \frac{\hbar}{\delta E} .$$

This time is usually of the order of $\hbar^{-(d-1)}$ which is much larger than the Ehrenfest time.

Asymptotic calculations of the eigenmodes need a knowledge of the quantum dynamics until the Heisenberg time. It is possible to do that (at the moment) only for integrable systems for which the Ehrenfest time is $+\infty$. Gutzwiller type trace formulae are valid up to Ehrenfest times and are not quantization rules except for integrable systems for which they are equivalent, via the Bohr-Sommerfeld rules, to the Poisson summation formula.

4. THE SCHNIERMAN ERGODIC THEOREM

4.1. Quasi-modes

DEFINITION 4.1. — *If $f(\hbar)$ is a function satisfying $\lim_{\hbar \rightarrow 0} f(\hbar) = 0$, a sequence of L^2 normalized smooth functions φ_k is said to be an f -quasi-mode if $\|\hbar^2 \Delta \varphi_k - \varphi_k\|_2 = O(\hbar f(\hbar))$.*

If φ_k is an f -quasi-mode for the Laplace operator, $\exp(-it/\hbar)\varphi_k$ is a good approximation to $\hat{U}^t \varphi_k$ on a time interval of the order of $f(\hbar)^{-1}$.

4.2. Wigner measures and semi-classical measures

To any function $a \in C_o^\infty(T^*\mathbb{R}^d)$, we can associate a pseudo-differential operator which is given by:

$$\text{Op}_\hbar(a)u(x) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y|\xi|)/\hbar} a(x, \xi) u(y) |dy d\xi| .$$

We call such a recipe $a \rightarrow \text{Op}_\hbar(a)$ a *quantization*. Using partitions of unity, we can get a similar quantization on any closed manifold. In particular, if $a = a(x)$ is a function on X , $\text{Op}_\hbar(a)$ is the multiplication by a .

For a family of functions f_\hbar of L^2 norms $\equiv 1$, we define the *Wigner measures* as the Schwartz distributions defined on the manifold by

$$\int_{T^*X} adW_\hbar := \langle \text{Op}_\hbar(a)f_\hbar | f_\hbar \rangle .$$

They are also called the microlocal lifts of $|f_\hbar|^2|dx|$ because they project onto such measures by the canonical projection from T^*X onto X .

THEOREM 4.2. — *If f_\hbar is a sequence of $o(1)$ quasi-modes (see Definition 4.1), all weak limits (as Schwartz distributions) of dW_\hbar are probability measures on Z which are invariant by the geodesic flow.*

Remark 4.3. — It is possible to choose the quantization so that for any $a \geq 0$, $\text{Op}_\hbar(a)$ is a positive symmetric operator. The Wigner measures dW_\hbar depend on the chosen quantization, but the asymptotic behavior as $\hbar \rightarrow 0$ does not.

DEFINITION 4.4. — *Any such limit measure is called a semi-classical measure.*

Such measures were also introduced as a general tool in the study of partial differential equations by P. Gérard [18] and L. Tartar [35].

Remark 4.5. — If μ is the semi-classical measure of a sequence φ_{k_j} , the measures $|\varphi_{k_j}|^2|dx|$ on X converge to the projection of μ on X .

4.3. Localized eigenfunctions and scars

It has been well known since 40 years [4, 31], that it is possible to build f -quasi-modes, with $f(\hbar) = \hbar^N$, associated to any generic stable closed geodesic γ . The associated semi-classical measure is the average on γ . Typical eigenfunctions of integrable systems have semi-classical measures which are Lebesgue measures on Lagrangian tori. If $V(x)$ is a double well potential with a local maximum at $x = x_0$, the Dirac measure $\delta(x_0, 0)$ (the unstable equilibrium point) is also a semi-classical measure. An example, with a Laplace operator, of a sequence of eigenfunctions, for which the semi-classical measure is the average on an unstable closed geodesic, is described in [12].

Sequences of eigenfunctions can be very large at some places and can still have a uniform measure as a semi-classical measure: from the point of view of numerical calculations, it is impossible to see the difference. The numerical observations of such abnormally large eigenfunctions started with the work of S.W. McDonald & A.N. Kaufman [29, 30] in the case of the stadium billiard. They were called *scars*

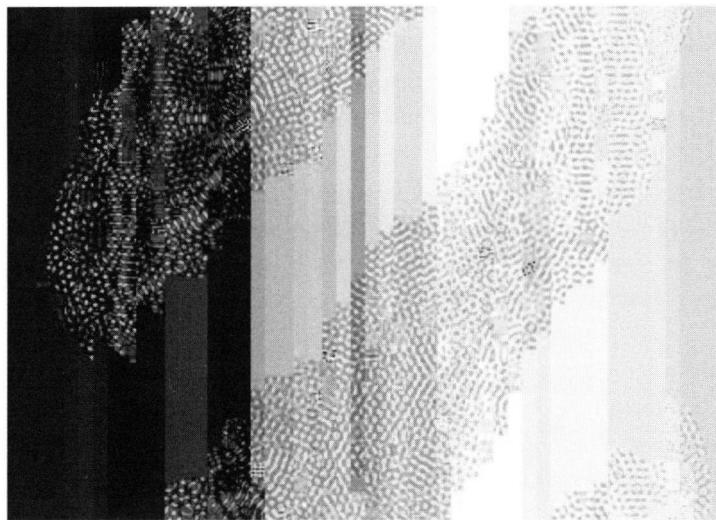


FIGURE 1. Scars for the stadium billiard: the intensity of some eigenfunctions is larger around some specific closed geodesics.

by E. Heller [20] which gave the following “definition”: *a quantum eigenstate of a classically chaotic system has a scar of a periodic orbit if its density on the classical invariant manifolds near the periodic orbit differs significantly from the classical expected density.* A typical problem related to scars is to get upper bounds of the L^∞ norms of the eigenfunctions. Some people called *strong scarring* the fact that the limit of the Wigner measures is not the Liouville measure.

4.4. The (micro-)local Weyl law

We consider some average of Wigner measures as follows:

$$dm := (2\pi\hbar)^{-d} \sum_{\hbar^2\lambda_k \leq 1} dW_{\varphi_k} .$$

The micro-local version of Weyl law, of which the local Weyl law (and hence the usual Weyl law) is a consequence if we integrate a function $a = a(x)$, is:

THEOREM 4.6. — *As $\hbar \rightarrow 0^+$, the measure dm converges weakly to the Liouville measure on the unit ball bundle B_1^*X .*

This result is an easy consequence of the functional calculus of pseudo-differential operators by looking at asymptotic of traces of $\Phi(\hbar^2\Delta)$.

4.5. The Schnirelman Theorem

The beginning of this story is the celebrated Schnirelman Theorem [34, 36, 11] and, for the case of manifold with boundary (a billiard), [19, 37]:

THEOREM 4.7. — *Let X be a closed Riemannian manifold whose geodesic flow is ergodic. Let (φ_k, λ_k) be an eigendecomposition of the Laplace operator. There exists a density one sub-sequence (λ_{k_j}) of the eigenvalues sequence⁽⁴⁾ so that the sequence $dW_{\varphi_{k_j}}$ weakly converges to the Liouville measure on the unit cotangent bundle.*

Since more than twenty years, the existence of atypical sub-sequences has been considered as an important problem. In particular, Rudnick and Sarnak [32] formulated the so-called *Quantum unique ergodicity conjecture* (QUE): there are no exceptional sub-sequences at least for the case of < 0 curvature.

4.6. Arithmetic case

Recently, E. Lindenstrauss [26] proved the QUE for a Hecke eigenbasis of *arithmetic* Riemann surfaces with constant curvature. His proof uses sophisticated results in ergodic theory of M. Ratner.

5. LOCALIZED STATES FOR THE CAT MAP

The only counter-example to QUE is for linear cat maps (see [7, 17, 16]). The basic fact is that the quantum cat map \hat{U}_N is a unitary periodic operator (i.e. there exists a non zero integer $T(N)$ so that $\hat{U}_N^{T(N)} = e^{iT(N)\alpha_n} \text{Id}$) in sharp contrast with the classical cat map which is chaotic! The smallest positive period $T_0(N)$ is the period of the permutation induced by the linear map A on $(\mathbb{Z}/N\mathbb{Z})^2$. The period $T_0(N)$ satisfies

$$2T_E = \frac{2|\log \hbar|}{\Lambda_+} \leq T_0(N) \leq 3N .$$

We will choose a sequence N_k so that the periods are close to $2T_E$. Let us denote $T_k := T_0(N_k)$. For such sequences, we have $T_H \sim T_E$.

⁽⁴⁾ The sub-sequence λ_{k_j} of the sequence λ_k is of density 1 if

$$\lim_{\lambda \rightarrow +\infty} \frac{\#\{j | \lambda_{k_j} \leq \lambda\}}{\#\{k | \lambda_k \leq \lambda\}} = 1 .$$

THEOREM 5.1. — *Let $\varphi \in \mathcal{H}_{N_k}$ be a coherent state located at the origin of the torus. The state*

$$\psi := \sum_{l=-T_k/2}^{T_k/2-1} e^{-il\alpha_N} \hat{U}_{N_k}^l \varphi$$

is an eigenstate of \hat{U}_{N_k} with eigenvalue $e^{i\alpha_N}$ and the associated semi-classical measure is $\mu = \frac{1}{2}(\delta(0) + dL)$. The entropy of μ is $\log(\lambda_+)/2$.

The idea of the proof is as follows: we split the state ψ into 2 parts: $\psi = \psi_{\text{loc}} + \psi_{\text{equi}}$, where $\psi_{\text{loc}} = \sum_{|l| \leq T_k/4} e^{-il\alpha_N} \hat{U}_{N_k}^l \varphi$ while ψ_{equi} is the remaining part of that sum. The state ψ_{loc} stays localized because all components involve times less than $T_E/2$, the part ψ_{equi} is equidistributed.

6. LOWER BOUNDS ON THE ENTROPY: THE A-N THEOREM

N. Anantharaman and S. Nonnenmacher in [3] and, with H. Koch, in [2] were improving a previous result of N. Anantharaman [1] as follows:

THEOREM 6.1. — *Let (X, g) be a smooth closed Riemannian manifold of dimension d with strictly negative sectional curvature. Let μ be any semi-classical measure (a weak limit of a sequence of Wigner measures) for an $o(|\log h|^{-1})$ -quasi-mode of the Laplace operator. We have the following lower bound for the entropy of μ :*

$$h_{\text{KS}}(\mu) \geq \int_Z \log J_u(z) d\mu - \frac{1}{2}(d-1)\Lambda_+ .$$

If the curvature is $\equiv -1$, it gives

$$h_{\text{KS}}(\mu) \geq \frac{d-1}{2} .$$

If the curvature varies a lot, the lower bound can be negative. In [1], it was proved that

THEOREM 6.2. — *If X is a closed Riemannian manifold with strictly negative curvature, then, for any semi-classical measure μ , the entropy $h_{\text{KS}}(\mu)$ is strictly positive.*

In particular, convex combinations of averages on closed geodesics are not semi-classical measures.

This cannot be obtained by local considerations around the closed geodesic as shown in the paper [12].

The analog of Theorem 6.1 for linear cat maps on the 2-torus is the lower bound

$$h_{\text{KS}}(\mu) \geq \frac{1}{2}\Lambda_+$$

which is a sharp bound w.r. to the example discussed in Section 5.

It is interesting to compare the previous results to the following one [24]:

THEOREM 6.3. — *Let X be a closed 2D Riemannian manifold with < 0 curvature and μ a probability measure on the unit cotangent bundle Z invariant by the geodesic flow for which $h_{\text{KS}}(\mu) > \frac{1}{2} \int_Z \log J_u(z) d\mu$; then the projection of μ onto X is absolutely continuous w.r. to the Lebesgue measure.*

7. ABOUT THE PROOF OF THE A-N THEOREM

We will not give the full proof, but only the key points avoiding the most technical parts for which we refer to the original papers [1, 3, 2]. Moreover, we will assume that φ_\hbar is an eigenfunction, not only a quasi-mode.

7.1. Heuristics

Let us start with a partition $\mathcal{P} = \{P_1, \dots, P_M\}$ of Z and a sequence φ_\hbar of eigenfunctions with a semi-classical measure μ on Z . Let p_j be the characteristic function of P_j . In order to get an estimate of the exponential decay of $C_n := \mu(p_{j_n} \circ U^{(n-1)} \dots p_{j_2} \circ U^1 \cdot p_{j_1})$ (and hence a lower bound of the entropy), we replace the partition of unity p_j by a smooth one and try to evaluate the quantum analog Q_n of C_n defined by

$$Q_n := \langle \hat{U}^{-(n-1)} \pi_{j_n} \hat{U}^{n-1} \circ \dots \circ \hat{U}^{-1} \pi_{j_2} \hat{U}^1 \circ \pi_{j_1} \varphi_\hbar | \varphi_\hbar \rangle ,$$

where the π_j 's are pseudo-differential operators of symbol p_j . Indeed, for fixed n , the expression Q_n converges to C_n as $\hbar \rightarrow 0$ due to the Egorov Theorem: $\hat{U}^{-j} \pi_j \hat{U}^j$ is a pseudo-differential operator of principal symbols $p_j \circ U^j$. N. Anantharaman already got a nice decay estimate for Q_n in [1]. The problem is that the decay estimates involve the expected classical exponential decay with an extra negative power of \hbar : the exponential decay of Q_n starts only for n of the order of $|\log \hbar|$, more precisely the Ehrenfest time T_E . But the Egorov Theorem is only valid for time of the order of $T_E/2!$ So we need to play with that: first, we introduce a quantum entropy and then, using the Egorov Theorem for a time $T_E/2$, we get a subadditivity estimate for it which allows to recover a nice estimate for a fixed time. We can then take the limit $\hbar \rightarrow 0$. The main 3 parts are:

- The *Quantum* part: abstract quantum entropy estimates (Section 7.2)
- The *Classical* part: decay estimates for Q_n (Sections 7.4, 7.5)
- The *Semi-Classical* part: subadditivity (Section 7.6).

7.2. Entropic uncertainty principle

The way to get a lower bound for the entropy from upper estimates is by an adaptation of the entropic uncertainty principle conjectured by Kraus in [23] and proved by Maassen and Uffink [27]. This principle states that, if a unitary matrix has “small” entries, then any of its eigenvectors must have a “large” Shannon entropy.

Let $(\mathcal{H}, \|\cdot\|)$ be a complex Hilbert space.

DEFINITION 7.1. — A quantum partition of unity is a family $\pi = (\pi_k)_{k=1,\dots,N}$ of linear operators $\pi_k : \mathcal{H} \rightarrow \mathcal{H}$ which satisfies

$$(1) \quad \sum_{k=1}^N \pi_k^\star \pi_k = Id .$$

In other words, for all $\psi \in \mathcal{H}$, we have

$$\|\psi\|^2 = \sum_{k=1}^N \|\psi_k\|^2 \quad \text{where we set } \psi_k = \pi_k \psi \quad \text{for } k = 1, \dots, N .$$

DEFINITION 7.2. — Let us give a family $\alpha = (\alpha_k)_{k=1,\dots,N}$ of positive real numbers; if $\|\psi\| = 1$, we define the entropy of ψ with respect to the partition π by:

$$h_\pi(\psi) = - \sum_k \|\psi_k\|^2 \log(\|\psi_k\|^2) ,$$

and the pressure w.r. the sequence α by:

$$p_{\pi,\alpha}(\psi) = - \sum_k \|\psi_k\|^2 \log(\alpha_k^2 \|\psi_k\|^2) .$$

THEOREM 7.3. — Let \mathcal{O} be a bounded operator and \hat{U} an isometry on \mathcal{H} and let us give 2 quantum partitions of unity $\pi = (\pi_k)_{1 \leq k \leq N}$ and $\tau = (\tau_j)_{1 \leq j \leq N}$ and 2 sequences of positive numbers $\alpha = (\alpha_k)$, $\beta = (\beta_j)$. Define $A = \max |\alpha_k|$ and $B = \max |\beta_j|$ and

$$c^{\pi,\alpha;\tau,\beta}(\hat{U}) := \max_{j,k} \alpha_j \beta_k \|\tau_j \hat{U} \pi_k^\star\| .$$

Then, for any normalized $\psi \in \mathcal{H}$ satisfying

$$\|(\text{Id} - \mathcal{O})\pi_k \psi\| \leq \epsilon ,$$

the pressures satisfy

$$p_{\tau,\beta}(\hat{U}\psi) + p_{\pi,\alpha}(\psi) \geq -2 \log(c^{\pi,\alpha;\tau,\beta}(\hat{U}) + NAB\epsilon) .$$

In particular, if ψ is an eigenvector of \hat{U} , we have

$$p_{\pi,\alpha}(\psi) + p_{\tau,\beta}(\psi) \geq -2 \log(c^{\pi,\alpha}(\hat{U}) + NAB\epsilon) .$$

Remark 7.4. — The result of [27] corresponds to the case where \mathcal{H} is an N -dimensional Hilbert space, $\alpha_j = \beta_k = 1$, and the operators $\pi_j = \tau_k$ are the orthogonal projectors on an orthonormal basis of \mathcal{H} . In this case, Theorem 7.3 reads

$$h_\pi(\hat{U}\psi) + h_\pi(\psi) \geq -2 \log c(\hat{U}),$$

where $c(\hat{U})$ is the supremum of all matrix elements of \hat{U} in the orthonormal basis associated to π .

The proof of Theorem 7.3 uses quite standard arguments of interpolation close to the Riesz-Thorin Theorem. It is given in Section 6 of [3].

7.3. Pseudo-differential partitions of unity

DEFINITION 7.5. — A semi-classical partition of unity *on the unit cotangent bundle $Z = T_1^*X$, associated to a finite open covering $(\Omega_l)_{2 \leq l \leq M}$ of Z , is a family of pseudo-differential operators $\pi_1, \dots, \pi_l, \dots, \pi_M$ which satisfies $\pi_l = \text{Op}_\hbar(q_l)$ with*

- $q_1 \equiv 0$ near Z and $q_1 \equiv 1$ outside a compact set;
- for $l > 1$, $q_l \in C_o^\infty(\Omega_l)$ (in fact, the q_l 's are symbols, i.e. they have a full asymptotic expansion into powers of \hbar), and

$$\sum_{l=1}^M \pi_l^* \pi_l = \text{Id} .$$

Remark 7.6. — The existence of such partitions of unity can be shown in two steps: first do it up to $O(\hbar^\infty)$, then find an explicit formula removing the $O(\hbar^\infty)$ part: if $\sum_{l=1}^M \tilde{\pi}_l^* \tilde{\pi}_l = \text{Id} + T$ with $T = O(\hbar^\infty)$, take $\pi_l = \tilde{\pi}_l(\text{Id} + T)^{-1/2}$.

We plan to apply Theorem 7.3 to the following objects:

- $\mathcal{H} = L^2(X)$;
- $N = M^n$;
- $\mathcal{O} = \chi_\hbar(h^2\Delta - 1)$ with $\chi_\hbar(E) = \chi_1(E/\hbar^{1-\delta})$ and $\chi_1 \in C_o^\infty(\mathbb{R})$ equal to 1 near 0;
- the following partition with $N = M^n$ elements:

DEFINITION 7.7. — For any sequence $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{1, \dots, M\}^n$, we define:

- for any operator A , $A(l) = \hat{U}^{-l} A \hat{U}^l$;
- the pseudo-differential operators

$$\Pi_{\vec{\epsilon}} := \pi_{\epsilon_n}(n-1) \pi_{\epsilon_{n-1}}(n-2) \cdots \pi_{\epsilon_1} ;$$

- the coarse-grained unstable Jacobian

$$J_u^{\vec{\epsilon}} := \prod_{l=0}^n \sup_{z \in \Omega_{\epsilon_l}} J_u(z) .$$

We will use the following quantum partitions of unity of \mathcal{H} :

$$\mathcal{P}^{\vee n} = \{\Pi_{\vec{\epsilon}}^* \mid |\vec{\epsilon}| = n\}$$

and

$$\mathcal{T}^{\vee n} = \{\Pi_{\vec{\epsilon}} \mid |\vec{\epsilon}| = n\},$$

and the weights:

$$\alpha_{\vec{\epsilon}} = \beta_{\vec{\epsilon}} = (J_u^{\vec{\epsilon}})^{\frac{1}{2}}.$$

7.4. Statement of the main estimate

We need the main estimate:

THEOREM 7.8. — *Let us assume that the pseudo-differential partition of unity $(\pi_l)_{1 \leq l \leq M}$ is given. Let us give some constant $C > 0$ and some $\delta > 0$ small enough. There exist a constant $c > 0$ independent of δ and a constant $C_\delta > 0$, so that, for any $n = |\vec{\epsilon}| \leq C|\log \hbar|$:*

- if X is a closed d -manifold with < 0 sectional curvature,

$$\|\Pi_{\vec{\epsilon}} \mathcal{O}\| \leq C_\delta \hbar^{-\frac{d-1}{2} - c\delta} (J_u^{\vec{\epsilon}})^{-\frac{1}{2}};$$

- for an hyperbolic quantum map on a $2d$ -torus, the same estimate holds with $(d-1)/2$ replaced by $d/2$.

The previous estimates will be useful, because $\Pi_{\vec{\epsilon}'} \hat{U}^n \Pi_{\vec{\epsilon}} = \hat{U}^n \Pi_{\vec{\epsilon}' + \vec{\epsilon}}$, in order to apply Theorem 7.3.

7.5. Proof of the main estimate

The proof of Theorem 7.8 is highly technical using a lot of careful estimates (19 pages in [3]!) and starts with the following identity:

$$\|\Pi_{\vec{\epsilon}}\| = \|\pi_{\epsilon_n} \hat{U} \pi_{\epsilon_{n-1}} \hat{U} \cdots \pi_{\epsilon_1}\|.$$

Let us give some ideas which make that we “believe” that such an estimate holds!

7.5.1. The linear hyperbolic map case. — In order to see the plausibility of such an estimate in the case of the linear cat map, I will show a similar one for the quite simple case where \hat{U} is the quantization of the linear map $U : T^*\mathbb{R} \rightarrow T^*\mathbb{R}$ given by $U(x, \xi) = (\lambda^{-1}x, \lambda\xi)$ with $\lambda > 1$.

$$\hat{U}f(x) = \lambda^{\frac{1}{2}} f(\lambda x).$$

Let us assume that $\text{Supp}(f) \subset [-1, +1]$ and $\text{Supp}(\hat{g}) \subset [-1, +1]$ where $\hat{g}(\xi) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} \exp(-ix\xi/\hbar) g(x) dx$. We want to get an estimate for

$$B(f, g) := \langle \hat{U}^n f | g \rangle = \lambda^{-n/2} \int f(x) g(x/\lambda^n) dx$$

in terms of the L^2 norms of f and g . Now we have the trivial inequality $\|g\|_{L^\infty} \leq C\hbar^{-1/2}\|\hat{g}\|_{L^2}$ and we can conclude

$$|B(f, g)| \leq C\lambda^{-n/2}\hbar^{-1/2}\|f\|_{L^2}\|g\|_{L^2}.$$

Note that in this rather trivial case the estimate holds without any restriction on n . We see also that there is a bad negative power of \hbar which cannot be removed!

7.5.2. The case of (non-linear) hyperbolic map. — A more geometric argument, in the case of an hyperbolic map, is as follows:

- decompose any semi-classical state of the form $f_1 = \pi_{\epsilon_1}(f)$ as a superposition of Lagrangian (WKB) states e_η associated to a smooth Lagrangian foliation of Ω_{ϵ_1} with leaves L_η transversal to the stable and the unstable foliations:

$$f_1(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(\eta) e_\eta(x) d\eta$$

where \hat{f} belongs to a bounded set of $C_o^\infty(\mathbb{R}^d)$ independently of \hbar ;

- let us consider the part L_η^n of L_u which satisfies, for $k = 1, \dots, n-1$, $U^k(L_\eta^n) \subset \Omega_{\epsilon_{k+1}}$. Then, if L_η^n is non empty for $n \rightarrow \infty$, there exists a point $z_0 \in \Omega_{\epsilon_1}$ so that $U^k(z_0) \in \Omega_{k-1}$ for all k and all such points are on the same stable leaf. As $n \rightarrow \infty$, the manifolds $U^{n-1}(L_\eta^n)$ smoothly converge to the intersection of the unstable manifold of $U^n(z_0)$ with Ω_{ϵ_n} , which is smooth;
- we can then get that the state $\Pi_{\tilde{\epsilon}}(e_\eta)$ is close to a Lagrangian state associated to an unstable leaf and symbol $\sim (J_u^n(z_0))^{-\frac{1}{2}}$;
- a nice estimate for $K(\eta, \eta') = \langle \Pi_{\tilde{\epsilon}}(e_\eta) | e_{\eta'} \rangle$ is provided from the fact that both functions are WKB states associated to transversal Lagrangian manifolds. We can use the symbolic calculus which gives the estimates $K(y, y') = O(\hbar^{-d/2}(J_u^n(z_0))^{-\frac{1}{2}})$.

7.5.3. The case of an Anosov flow. — The case of a Riemannian manifold presents new difficulties related to the localization near Z introduced with the operator \mathcal{O} : in order to get $(d-1)/2$, we need a kind of semi-classical reduction. We take $\mathcal{O} = P_{[1-\hbar^{1-b}, 1+\hbar^{1-b}]}$ where P_I is the spectral projector of $\hbar^2\Delta$ on the interval I .

7.6. Large time Egorov Theorem and sub-additivity

We have seen in Section 2.5 that the sub-additivity of $h(\mathcal{P}^N)$ is a consequence of the invariance of the measure μ . Here we have only an approximate invariance due to the Egorov Theorem.

The usual Egorov Theorem is:

THEOREM 7.9. — Let us give $a \in C_o^\infty(T^*X)$ and t fixed, then, if $A = \text{Op}_\hbar(a)$ and $A(t) = \hat{U}^{-t}A\hat{U}(t)$, the operator $A(t)$ is a pseudo-differential operator of principal symbol $a \circ U^t$.

In particular

$$\|A(t) - \text{Op}_\hbar(a \circ U^t)\|_{L^2 \rightarrow L^2} = O(\hbar) .$$

In order to prove the sub-additivity of quantum entropy, we will need the following weak (and easy) version of the main result of [9]:

THEOREM 7.10. — Let γ satisfy $0 < \gamma < 1$ and $a \in C_o^\infty(T^*X)$. We have, for $|t| \leq (1 - \gamma)T_E/2$:

$$\|\hat{U}^{-t}\text{Op}_\hbar(a)\hat{U}^t - \text{Op}_\hbar(a \circ U^t)\|_{L^2 \rightarrow L^2} = O(|t|\hbar^{(1+\gamma)/2})$$

and the:

COROLLARY 7.11. — For any $A = \text{Op}_\hbar(a)$, $B = \text{Op}_\hbar(b)$ with $a, b \in C_o^\infty(T^*X)$, we have, for $|t| \leq (1 - \gamma)T_E/2$:

$$\|[A(t), B]\| = O(\hbar^\gamma) .$$

COROLLARY 7.12. — For any $A = \text{Op}_\hbar(a)$ with $a \in C_o^\infty(T^*X)$, we have, for $|t| \leq (1 - \gamma)T_E$:

$$\|[A, A(t)]\| = O(\hbar^\gamma) .$$

This is because $\|[A, A(2t)]\| = \|[A(-t), A(t)]\|$.

For large times t , the function $a \circ U^t$ becomes less and less smooth due to the exponential divergence of trajectories. More precisely, we have

$$\|\partial_z^\alpha(a \circ U^t)\| = O(e^{\Lambda + |\alpha t|}) .$$

It implies that for $|t| \leq (1 - \gamma)T_E/2$, the function $a \circ U^t$ is in some symbol class Σ_ϵ with $\epsilon < \frac{1}{2}$ which is the limit for a nice pseudo-differential calculus. Here $b \in \Sigma_\epsilon$ means $\|\partial_z^\alpha b\| = O(\hbar^{-\epsilon|\alpha|})$.

We will apply the results of Section 7.2 to the quantum partition $\Pi_{\vec{\epsilon}}$ with all $\vec{\epsilon}$ of length n . We have the following approximate sub-additivity:

THEOREM 7.13. — Let us choose a family of normalized Laplace eigenfunctions $\Delta\varphi_\hbar = \hbar^{-2}\varphi_\hbar$. Let us denote by p_n the pressure of φ_\hbar associated to the partition $\mathcal{P}^{\vee n}$ and the weights $\alpha_{\vec{\epsilon}} = (J_u^{\vec{\epsilon}})^{\frac{1}{2}}$. We have, for any n_0 fixed and $n_0 + m \leq (1 - \delta')T_E$:

$$p_{n_0+m} \leq p_{n_0} + p_m + O_{n_0}(1) .$$

The previous theorem will give nice lower bounds of the pressure for fixed n_0 while the bound given in Theorem 7.8 is interesting only for n of the size of $|\log \hbar|$ due to the negative powers of \hbar .

7.7. The scheme of the proof

The proof of Theorem 6.1 involves the following steps:

7.7.1. *Applying the quantum uncertainty principle.* — We apply the quantum uncertainty principle (Theorem 7.3) to the following data:

- $\mathcal{H} := L^2(X)$, $N = M^n$ with $n \sim (1 - \delta')T_E$;
- the partitions $\mathcal{P}^{\vee n}$ and $\mathcal{T}^{\vee n}$ defined in Section 7.4 and the associated weights $\alpha_{\vec{\epsilon}}$; we will denote by p_n (resp. q_n) the corresponding pressures;
- the sequence of eigenfunctions φ_h satisfies $\hbar^2 \Delta \varphi_h = \varphi_h$ and has the semi-classical measure μ .

Using Theorem 7.8 in order to estimate the coefficients $c^{\mathcal{P}^{\vee n}, \alpha_n; \mathcal{T}^{\vee n}, \alpha_n}$, we get the following inequality:

$$\frac{p_n + q_n}{2} \geq - \left(\frac{d-1}{2} - c\delta \right) |\log \hbar| - O_\delta(1).$$

It is not possible to use this inequality for fixed n because $\log \hbar$ tends to $-\infty$ as $\hbar \rightarrow 0$. For $n \sim (1 - \delta')T_E$, the previous inequality gives:

$$(2) \quad \frac{p_n + q_n}{2n} \geq - \left(\frac{d-1}{2} - c\delta \right) \frac{\Lambda_+}{1 - \delta'} - O_\delta(1).$$

7.7.2. *Using sub-additivity.* — Before taking the semi-classical limit, we apply Theorem 7.13, in order to get the inequality (2) modulo $O(n_0^{-1})$ for $n = n_0$ fixed.

7.7.3. *Taking the semi-classical limit.* — We take now the semi-classical limit in inequality (2) using Egorov Theorem. Let us define $q_{\vec{\epsilon}} = q_{\epsilon_1} \cdot q_{\epsilon_2} \circ U \cdots \circ q_{\epsilon_{n_0}} \circ U^{n_0-1}$ and denote by μ the semi-classical measure of a sequence φ_h . We get

$$n_0^{-1} \left(- \sum_{|\vec{\epsilon}|=n_0} \mu(q_{\vec{\epsilon}}^2) \log \mu(q_{\vec{\epsilon}}^2) - \sum_{|\vec{\epsilon}|=n_0} \mu(q_{\vec{\epsilon}}^2) \log J_u^{\vec{\epsilon}} \right) \geq - \left(\frac{d-1}{2} - c\delta \right) \frac{\Lambda_+}{1 - \delta'} - O_\delta \left(\frac{1}{n_0} \right).$$

The second sum in the lefthandside can be simplified using the multiplicative property of $J_u^{\vec{\epsilon}}$ and the fact that μ is invariant by U . We get

$$n_0^{-1} \left(- \sum_{|\vec{\epsilon}|=n_0} \mu(q_{\vec{\epsilon}}^2) \log \mu(q_{\vec{\epsilon}}^2) \right) - \sum_{l=1}^M \mu(q_l^2) J_u^{\{l\}} \geq - \left(\frac{d-1}{2} - c\delta \right) \frac{\Lambda_+}{1 - \delta'} - O_\delta \left(\frac{1}{n_0} \right).$$

7.7.4. *Smoothing the initial partition.* — If the q_l 's were the characteristic functions of a partition of Z , we would have finished the proof. We start with a generating partition whose boundaries are of μ measure 0 and we can apply a smoothing argument.

8. EQUIPARTITION BY TIME EVOLUTIONS

Here, I will describe a very nice related result by R. Schubert [33]. Similar results for cat maps were already proved in [8]. Let us consider again the case of a d -dimensional closed Riemannian manifold X with < 0 curvature. Let us define $\varphi_0(x) = \hbar^{-d/2}$

$\chi((x - x_0)/\hbar) \eta(x)$ with $\chi \in C_o^\infty(\mathbb{R}^d \setminus \{0\})$, $\eta \in C_o^\infty(X)$, $\eta \equiv 1$ near x_0 .

THEOREM 8.1. — *If $\varphi(t)$ is the solution of the wave equation $\varphi_{tt} + \Delta\varphi = 0$ on X at time t with Cauchy data $\varphi(0) = \varphi_0$, $\varphi_t(0) = 0$, we have*

$$\left| \int_{T^*X} adW_{\varphi(t)} - \left(\int_{T^*X} adL \right) \|\varphi(0)\|_{L^2}^2 \right| = O(\hbar \exp(t\Lambda_+)) + o_{t \rightarrow \infty}(1).$$

This implies that for $0 \ll t \leq T_E$, the weak limit of the Wigner measure of $\varphi(t)$ is the Liouville measure times the square of the L^2 norm of φ_0 .

The proof of this result uses the large time Egorov Theorem (see Section 7.6) and the mixing property (+ a little bit of hyperbolicity).

REFERENCES

- [1] N. ANANTHARAMAN – “Entropy and localization of eigenfunctions”, to appear in *Annals of Math.*, 2008.
- [2] N. ANANTHARAMAN, H. KOCH & S. NONNENMACHER – “Entropy of eigenfunctions”, preprint arXiv:0704.1564, 2007.
- [3] N. ANANTHARAMAN & S. NONNENMACHER – “Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold”, *Ann. Inst. Fourier* **57** (2007), p. 2465–2523.

- [4] V. M. BABIČ & V. F. LAZUTKIN – “The eigenfunctions which are concentrated near a closed geodesic”, in *Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian)*, Izdat. Leningrad. Univ., Leningrad, 1967, p. 15–25.
- [5] M. BERGER, P. GAUDUCHON & E. MAZET – *Le spectre d'une variété riemannienne*, Lecture Notes in Mathematics, Vol. 194, Springer, 1971.
- [6] F. BONECHI & S. DE BIÈVRE – “Exponential mixing and $|\ln \hbar|$ time scales in quantized hyperbolic maps on the torus”, *Comm. Math. Phys.* **211** (2000), p. 659–686.
- [7] ———, “Controlling strong scarring for quantized ergodic toral automorphisms”, *Duke Math. J.* **117** (2003), p. 571–587.
- [8] A. BOUZOUINA & S. DE BIÈVRE – “Equipartition of the eigenfunctions of quantized ergodic maps on the torus”, *Comm. Math. Phys.* **178** (1996), p. 83–105.
- [9] A. BOUZOUINA & D. ROBERT – “Uniform semiclassical estimates for the propagation of quantum observables”, *Duke Math. J.* **111** (2002), p. 223–252.
- [10] M. BRIN & G. STUCK – *Introduction to dynamical systems*, Cambridge University Press, 2002.
- [11] Y. COLIN DE VERDIÈRE – “Ergodicité et fonctions propres du laplacien”, *Comm. Math. Phys.* **102** (1985), p. 497–502.
- [12] Y. COLIN DE VERDIÈRE & B. PARISSE – “Équilibre instable en régime semi-classique. I. Concentration microlocale”, *Comm. Partial Differential Equations* **19** (1994), p. 1535–1563.
- [13] M. DIMASSI & J. SJÖSTRAND – *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.
- [14] L. EVANS & M. ZWORSKI – “Lectures on semiclassical analysis (v. 3)”, <http://math.berkeley.edu/~zworski>, 2007.
- [15] F. FAURE – “Semi-classical formula beyond the Ehrenfest time in quantum chaos. (I) Trace formula”, *Ann. Inst. Fourier* **57** (2007), p. 2525–2599.
- [16] F. FAURE & S. NONNENMACHER – “On the maximal scarring for quantum cat map eigenstates”, *Comm. Math. Phys.* **245** (2004), p. 201–214.
- [17] F. FAURE, S. NONNENMACHER & S. DE BIÈVRE – “Scarred eigenstates for quantum cat maps of minimal periods”, *Comm. Math. Phys.* **239** (2003), p. 449–492.
- [18] P. GÉRARD – “Microlocal defect measures”, *Comm. Partial Differential Equations* **16** (1991), p. 1761–1794.
- [19] P. GÉRARD & É. LEICHTNAM – “Ergodic properties of eigenfunctions for the Dirichlet problem”, *Duke Math. J.* **71** (1993), p. 559–607.

- [20] E. J. HELLER – “Wavepacket dynamics and quantum chaology”, in *Chaos et physique quantique (Les Houches, 1989)*, North-Holland, 1991, p. 547–664.
- [21] A. KATOK & B. HASSELBLATT – *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, 1995.
- [22] D. KELMER – “Arithmetic quantum unique ergodicity for symplectic linear maps of the multidimensional torus”, preprint arXiv:math-ph/0510079v5.
- [23] K. KRAUS – “Complementary observables and uncertainty relations”, *Phys. Rev. D* (3) **35** (1987), p. 3070–3075.
- [24] F. LEDRAPPIER & E. LINDENSTRAUSS – “On the projections of measures invariant under the geodesic flow”, *Int. Math. Res. Not.* **9** (2003), p. 511–526.
- [25] F. LEDRAPPIER & L.-S. YOUNG – “The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin’s entropy formula”, *Ann. of Math.* (2) **122** (1985), p. 509–539.
- [26] E. LINDENSTRAUSS – “Invariant measures and arithmetic quantum unique ergodicity”, *Ann. of Math.* (2) **163** (2006), p. 165–219.
- [27] H. MAASSEN & J. B. M. UFFINK – “Generalized entropic uncertainty relations”, *Phys. Rev. Lett.* **60** (1988), p. 1103–1106.
- [28] R. MAÑÉ – *Ergodic theory and differentiable dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 8, Springer, 1987.
- [29] S. W. McDONALD & A. N. KAUFMAN – “Spectrum and eigenfunctions for a Hamiltonian with stochastic trajectories”, *Phys. Rev. Lett.* **42** (1979), p. 1189–1191.
- [30] ———, “Wave chaos in the stadium: statistical properties of short-wave solutions of the Helmholtz equation”, *Phys. Rev. A* (3) **37** (1988), p. 3067–3086.
- [31] J. V. RALSTON – “On the construction of quasimodes associated with stable periodic orbits”, *Comm. Math. Phys.* **51** (1976), p. 219–242.
- [32] Z. RUDNICK & P. SARNAK – “The behaviour of eigenstates of arithmetic hyperbolic manifolds”, *Comm. Math. Phys.* **161** (1994), p. 195–213.
- [33] R. SCHUBERT – “Semiclassical behaviour of expectation values in time evolved Lagrangian states for large times”, *Comm. Math. Phys.* **256** (2005), p. 239–254.
- [34] A. I. ŠNIREL’MAN – “Ergodic properties of eigenfunctions”, *Uspehi Mat. Nauk* **29** (1974), p. 181–182.
- [35] L. TARTAR – “ H -measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations”, *Proc. Roy. Soc. Edinburgh Sect. A* **115** (1990), p. 193–230.
- [36] S. ZELDITCH – “Uniform distribution of eigenfunctions on compact hyperbolic surfaces”, *Duke Math. J.* **55** (1987), p. 919–941.

- [37] S. ZELDITCH & M. ZWORSKI – “Ergodicity of eigenfunctions for ergodic billiards”, *Comm. Math. Phys.* **175** (1996), p. 673–682.

Yves COLIN de VERDIÈRE

Université de Grenoble I

Institut Fourier

UMR 5582 du CNRS

B.P. 74

F-38402 Saint-Martin-d’Hères Cedex

<http://www-fourier.ujf-grenoble.fr/~ycolver/>

E-mail : yves.colin-de-verdiere@ujf-grenoble.fr