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Astérisque, tome 298 (2005), p. 319-334

<http://www.numdam.org/item?id=AST_2005__298__319_0>

Astérisque
298, 2005, p. 319–334

$\mathcal{L}$-MODULES AND THE CONJECTURE OF RAPOPORT AND GORESKY-MACPHERSON

by

Leslie Saper

Abstract. — Consider the middle perversity intersection cohomology groups of various compactifications of a Hermitian locally symmetric space. Rapoport and independently Goresky and MacPherson have conjectured that these groups coincide for the reductive Borel-Serre compactification and the Baily-Borel-Satake compactification. This paper describes the theory of $\mathcal{L}$-modules and how it is used to solve the conjecture. More generally we consider a Satake compactification for which all real boundary components are equal-rank. Details will be given elsewhere [**26**]. As another application of $\mathcal{L}$-modules, we prove a vanishing theorem for the ordinary cohomology of a locally symmetric space. This answers a question raised by Tilouine.

Résumé **($\mathcal{L}$-modules et la Conjecture de Rapoport et Goresky-MacPherson).** — Considérons les groupes de cohomologie d'intersection (de perversité intermédiaire) de diverses compactifications d'un espace localement hermitien symétrique. Rapoport et, indépendamment, Goresky et MacPherson ont conjecturé que ces groupes coïncident pour la compactification de Borel-Serre réductive et la compactification de Baily-Borel-Satake. Cet article décrit la théorie des $\mathcal{L}$-modules et la façon dont elle peut s'employer pour résoudre la conjecture. Plus généralement, nous traitons une compactification de Satake pour laquelle toutes les composantes réelles à la frontière sont de « rang égal ». Les détails en seront disponibles ailleurs [**26**]. Comme application supplémentaire de la théorie des $\mathcal{L}$-modules, nous prouvons un théorème d'annulation sur le groupe de cohomologie ordinaire d'un espace localement symétrique. Ceci répond à une question soulevée par Tilouine.

2000 ***Mathematics Subject Classification.*** — 11F75, 14G35, 22E40, 22E45, 32S60, 55N33.
Key words and phrases. — Intersection cohomology, Shimura varieties, locally symmetric spaces, compactifications.

This research was supported in part by the National Science Foundation under grants DMS-8957216, DMS-9100383, and DMS-9870162, and by a grant from the Institut des Hautes Études Scientifiques. Early stages of this research were supported by The Duke Endowment, a Alfred P. Sloan Research Fellowship, and the Maximilian-Bickhoff-Stiftung during a visit to the Katholischen Universität Eichstätt. The author wishes to thank all these organizations for their hospitality and support.
The manuscript was prepared with the $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX macro system; diagrams were generated with the XY-pic package.

1. Introduction

In a letter to Borel in 1986 Rapoport made a conjecture (independently rediscovered by Goresky and MacPherson in 1988) regarding the equality of the intersection cohomology of two compactifications of a locally symmetric variety, the reductive Borel-Serre compactification and the Baily-Borel compactification. In this paper I describe the conjecture, introduce the theory of $\mathcal{L}$-modules which was developed to attack the conjecture, and explain the solution of the conjecture. The theory of $\mathcal{L}$-modules actually applies to the study of many other types of cohomology. As a simple illustration, I will answer at the end of this paper a question raised during the semester by Tilouine regarding the vanishing of the ordinary cohomology of a locally symmetric variety below the middle degree. Except in this final section, proofs are omitted; the details will appear in [**26**].

This paper is an expanded version of lectures I gave during the Automorphic Forms Semester (Spring 2000) at the Centre Émile Borel in Paris; I would like to thank the organizers for inviting me and providing a stimulating environment. During this research I benefited from discussions with numerous people whom I would like thank, in particular A. Borel, R. Bryant, M. Goresky, R. Hain, G. Harder, J.-P. Labesse, J. Tilouine, M. Rapoport, J. Rohlfs, J. Schwermer, and N. Wallach.

2. Compactifications

We consider a connected reductive algebraic group G defined over $\mathbb{Q}$ and its associated symmetric space $D = G(\mathbb{R})/KA_G$, where K is a maximal compact subgroup of $G(\mathbb{R})$ and A_G is the identity component of the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus in the center of G. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup which for simplicity here we assume to be neat. (Any arithmetic subgroup has a neat subgroup of finite index; the neatness hypothesis ensures that all arithmetic quotients in what follows will be smooth as opposed to V-manifolds or orbifolds.) The locally symmetric space $X = \Gamma\backslash D$ is in general not compact and we are interested in three compactifications (see Figure 1), belonging respectively to the topological, differential geometric, and (if D is Hermitian symmetric) complex analytic categories.

Let $\mathcal{P}$ (resp. $\mathcal{P}_1$) denote the partially ordered set of Γ-conjugacy classes of parabolic (resp. maximal parabolic) $\mathbb{Q}$-subgroups of G. For $P \in \mathcal{P}$, let L_P denote the Levi quotient P/N_P, where N_P is the unipotent radical of P. (When it is convenient we will identify L_P with a subgroup of P via an appropriate lift.) The *Borel-Serre compactification* [**4**] has strata $Y_P = \Gamma_P\backslash P(\mathbb{R})/K_P A_P$ indexed by $P \in \mathcal{P}$ (for $P = G$ we simply have $Y_G = X$). Here $\Gamma_P = \Gamma \cap P$, $K_P = K \cap P$, and A_P is the identity component of the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus in the center of L_P. The Borel-Serre compactification $\overline{X}$ is a manifold with corners, homotopically equivalent with X itself.

$$
\begin{array}{lcccc}
\text{Borel-Serre} & \overline{X} & = & \coprod_{P\in\mathcal{P}} Y_P, & Y_P = \Gamma_P\backslash P(\mathbb{R})/K_PA_P \\
 & \Big\downarrow & & \Big\downarrow \text{Collapse } \Gamma_{N_P}\backslash N_P(\mathbb{R}) \text{ fibers} & \\
\begin{array}{c}\text{Reductive}\\ \text{Borel-Serre}\end{array} & \widehat{X} & = & \coprod_{P\in\mathcal{P}} X_P, & X_P = \Gamma_{L_P}\backslash L_P(\mathbb{R})/K_PA_P \\
 & \pi\Big\downarrow & & \Big\downarrow \text{Project } X_P = X_{P,\ell}\times X_{P,h} \to X_{P,h} = X_{R,h} & \\
\begin{array}{c}\text{Baily-Borel}\\ \text{Satake}\end{array} & X^* & = & \coprod_{R\in\mathcal{P}_1} F_R, & F_R = X_{R,h}
\end{array}
$$

FIGURE 1

The arithmetic subgroup Γ induces arithmetic subgroups $\Gamma_{N_P} = \Gamma\cap N_P$ in N_P and $\Gamma_{L_P} = \Gamma_P/\Gamma_{N_P}$ in L_P. Let $D_P = L_P(\mathbb{R})/K_PA_P$ be the symmetric space associated to L_P and let $X_P = \Gamma_{L_P}\backslash D_P$ be its arithmetic quotient. Each stratum of $\overline{X}$ admits a fibration $Y_P \to X_P$ with fibers being compact nilmanifolds $\Gamma_{N_P}\backslash N_P(\mathbb{R})$. The union $\widehat{X} = \coprod_P X_P$ (with the quotient topology from the natural map $\overline{X} \to \widehat{X}$) is the *reductive Borel-Serre compactification*; it was introduced by Zucker [**34**]. The reductive Borel-Serre compactification is natural from a differential geometric standpoint since the locally symmetric metric on X degenerates precisely along these nilmanifolds near the boundary of $\overline{X}$.

Finally assume now that D is Hermitian symmetric. Then each D_P factors into a product $D_{P,\ell}\times D_{P,h}$, where $D_{P,h}$ is again Hermitian symmetric (see Figure 2). This induces a factorization (modulo a finite quotient) $X_P = X_{P,\ell}\times X_{P,h}$ of the arithmetic quotients and we consider the projection $X_P \to X_{P,h}$ onto the second factor. Now among the different $P\in\mathcal{P}$ that yield the same $X_{P,h}$, let $P^\dagger\in\mathcal{P}_1$ be the maximal one and set $F_{P^\dagger} = X_{P,h}$. Thus each stratum of $\widehat{X}$ has a projection $X_P \to F_{P^\dagger}$. The union $X^* = \coprod_{R\in\mathcal{P}_1} F_R$ (with the quotient topology from the map $\widehat{X}\to X^*$) is the *Baily-Borel-Satake compactification* X^*. Topologically X^* was constructed by Satake [**29**], [**30**] (though the description we have given is due to Zucker [**35**]); if Γ is contained in the group of biholomorphisms of D, the compactification X^* was given the structure of a normal projective algebraic variety by Baily and Borel [**2**].

The simplest example where all three compactifications are distinct is the Hilbert modular surface case. Here $G = R_{k/\mathbb{Q}}\,\mathrm{SL}(2)$ where k is a real quadratic extension. There is only one proper parabolic $\mathbb{Q}$-subgroup P up to $G(\mathbb{Q})$-conjugacy; Y_P is a torus bundle over $X_P = S^1$ and F_P is a point.

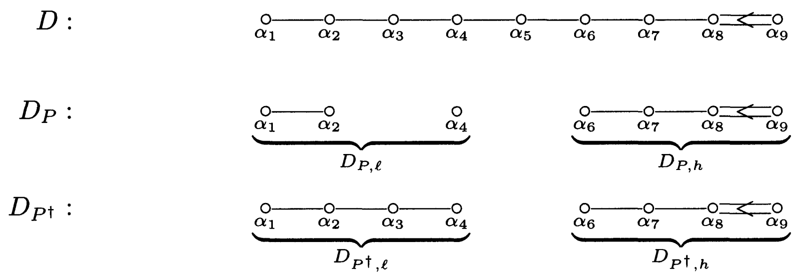


FIGURE 2. An example of $D_P = D_{P,\ell} \times D_{P,h}$ and $D_{P^\dagger} = D_{P^\dagger,\ell} \times D_{P^\dagger,h}$

3. The conjecture

Assume that D is Hermitian symmetric. Let $E \in \mathfrak{Mod}(G)$, the category of finite dimensional regular representations of G and let $\mathbb{E}$ denote the corresponding local system on X. Let $\mathcal{IC}(\widehat{X};\mathbb{E})$ and $\mathcal{IC}(X^*;\mathbb{E})$ denote middle perversity intersection cohomology sheaves[(1)] on $\widehat{X}$ and X^* respectively [**10**].

For example, $\mathcal{IC}(\widehat{X};\mathbb{E}) = \tau^{\leqslant p(\operatorname{codim} X_P)} j_{P*}\mathbb{E}$ if $\widehat{X}$ has only one singular stratum X_P; here j_{P*} denotes the derived direct image functor of the inclusion $j_P : \widehat{X} \smallsetminus X_P \hookrightarrow \widehat{X}$, $\operatorname{codim} X_P$ denotes the topological codimension, $p(k)$ is one of the middle perversities $\lfloor (k-1)/2 \rfloor$ or $\lfloor (k-2)/2 \rfloor$, and $\tau^{\leqslant p(k)}$ truncates link cohomology in degrees $> p(k)$. In general the pattern of pushforward/truncate is repeated over each singular stratum. Note that since $\widehat{X}$ may have odd codimension strata, $\mathcal{IC}(\widehat{X};\mathbb{E})$ depends on the choice of the middle perversity p; on the other hand, since X^* only has even codimension strata, $\mathcal{IC}(X^*;\mathbb{E})$ is independent of p.

***Main Theorem* (Rapoport's Conjecture).** — *Let X be an arithmetic quotient of a Hermitian symmetric space. Then $\pi_*\mathcal{IC}(\widehat{X};\mathbb{E}) \cong \mathcal{IC}(X^*;\mathbb{E})$. (That is, they are isomorphic in the derived category.)*

Following discussions with Kottwitz, Rapoport conjectured the theorem in a letter to Borel [**22**] and later provided motivation for it in an unpublished note [**23**]. Previously Zucker had noticed that the conjecture held for $G = \mathrm{Sp}(4)$, $\mathbb{E} = \mathbb{C}$. The conjecture was later rediscovered by Goresky and MacPherson and described in an unpublished preprint [**11**] in which they also announced the theorem for $G = \mathrm{Sp}(4)$, $\mathrm{Sp}(6)$, and (for $\mathbb{E} = \mathbb{C}$) $\mathrm{Sp}(8)$. The first published appearance of the conjecture was in a revised version of Rapoport's note [**24**] and included an appendix by Saper and Stern giving a proof of the theorem when $\mathbb{Q}$-rank $G = 1$.

(1) By a "sheaf" we will always mean a complex of sheaves representing an element of the derived category. A derived functor will be denoted by the same symbol as the original functor, thus we will write π_* instead of $R\pi_*$.

To see one reason why the conjecture might be useful in the theory of automorphic forms, note that the right hand side $\mathcal{IC}(X^*;\mathbb{E})$ is isomorphic to the L^2-cohomology sheaf $\mathcal{L}_{(2)}(X^*;\mathbb{E})$ by (the proof of) Zucker's conjecture [**17**], [**28**]. The trace of a Hecke operator on L^2-cohomology could then be studied topologically via the Lefschetz fixed point formula for $\mathcal{IC}(X^*;\mathbb{E})$. However the singularities of $\widehat{X}$ are simpler than those of X^* so a Lefschetz fixed point formula for $\mathcal{IC}(\widehat{X};\mathbb{E})$ should be easier to calculate. The conjecture says that this should give the same result. Also note that a Lefschetz fixed point formula for $\mathcal{IC}(\widehat{X};\mathbb{E})$ involves a sum over $\mathcal{P}$, while a Lefschetz fixed point formula for $\mathcal{IC}(X^*;\mathbb{E})$ involves a sum over $\mathcal{P}_1$. Thus it is more likely that the former can be directly related to the Arthur-Selberg trace formula for a Hecke operator on L^2-cohomology [**1**].

This program has been pursued by Goresky and MacPherson, but instead of $\mathcal{IC}(\widehat{X};\mathbb{E})$ they use the "middle weighted cohomology" $\mathcal{WC}(\widehat{X};\mathbb{E})$ in which cohomology classes in the link are truncated according to their *weight* as opposed to their degree. Thus weighted cohomology is an algebraic analogue of L^2-cohomology. Goresky and MacPherson prove (in joint work with Harder [**8**]) the analogue of the above theorem, $\pi_*\mathcal{WC}(\widehat{X};\mathbb{E}) \cong \mathcal{IC}(X^*;\mathbb{E})$, calculate the Lefschetz fixed point formula [**12**], and (in joint work with Kottwitz) show that it agrees with Arthur's trace formula for L^2-cohomology [**9**].

Nonetheless the original conjecture remains interesting for a number of reasons. First of all, intersection cohomology is a true topological invariant and the local cohomology of $\mathcal{IC}(\widehat{X};\mathbb{E})$ behaves better than that of $\mathcal{WC}(\widehat{X};\mathbb{E})$ when $\mathbb{E}$ varies. Secondly, the local property ("micro-purity") one needs to prove is much deeper for $\mathcal{IC}(\widehat{X};\mathbb{E})$ than for $\mathcal{WC}(\widehat{X};\mathbb{E})$ and should have applications elsewhere. And finally the method used to attack the conjecture, the theory of $\mathcal{L}$-modules, has application to other cohomology, in particular, weighted cohomology, L^2-cohomology, and ordinary cohomology.

In §§ 5–10 we will indicate how the Main Theorem follows from three theorems in the theory of $\mathcal{L}$-modules.

4. A generalization

This section is optional; we will indicate a more general context in which the Main Theorem holds. First we sketchily recall the general theory of Satake compactifications [**29**], [**30**], [**35**], [**6**]. By embedding D into a real projective space via a finite-dimensional representation σ of G and then taking the closure, Satake constructed a finite family of *Satake compactifications* ${}_{\mathbb{R}}D^*$ of D. Each of these is equipped with an action of $G(\mathbb{R})$ and is formed by adjoining to D certain *real boundary components*. Let D^* denote the union of D together with those real boundary components whose normalizer is defined over $\mathbb{Q}$; call these the *rational boundary components*. In the *geometrically rational* case (a condition satisfied for example if σ is $\mathbb{Q}$-rational[2]) one may equip D^* with a suitable topology so that $X^* = \Gamma\backslash D^*$ is a Hausdorff compactification of X; this is also called a *Satake compactification*. For D Hermitian symmetric, one of the Satake compactifications is (topologically equivalent to) the closure of the realization of D as a bounded symmetric domain and it is geometrically rational; the corresponding compactification of X is the Baily-Borel-Satake compactification.

Let ${}^0G = \bigcap_{\chi\in X_{\mathbb{Q}}(G)} \operatorname{Ker}\chi^2$ so that $G(\mathbb{R}) = {}^0G(\mathbb{R})A_G$ [**4**]. Suppose that $\operatorname{rank}{}^0G = \operatorname{rank}K$, that is, ${}^0G(\mathbb{R})$ has discrete series representations. This is equivalent to the assumption that the maximal $\mathbb{R}$-split torus in the center of G is also $\mathbb{Q}$-split and that the real points of G^{der} (the semisimple derived group) has discrete series representations. (We may also substitute here the adjoint group G^{ad} for G^{der}.) We say in this case that D is an *equal-rank symmetric space.* A Satake compactification ${}_{\mathbb{R}}D^*$ of D will be called a *real equal-rank Satake compactification* if all the real boundary components of ${}_{\mathbb{R}}D^*$ are also equal-rank symmetric spaces. The possible D that admit real equal-rank Satake compactifications are listed in [**36**]; they include the Hermitian symmetric cases but there are other infinite families as well. If such a ${}_{\mathbb{R}}D^*$ is geometrically rational[3] then the corresponding compactification X^* of X is also called a *real equal-rank Satake compactification*; note that we impose the equal-rank condition on all real boundary components even though only the rational boundary components contribute to X^*.

The generalization we alluded to above is that the Main Theorem holds for real equal-rank Satake compactifications. (Note that Borel conjectured that the analogue of the Zucker conjecture should remain true for such X^* and Saper and Stern (unpublished) observed that their proof could be adapted to this case.)

5. $\mathcal{L}$-modules

Now again let G be any connected reductive group over $\mathbb{Q}$ (with no Hermitian hypothesis). The "sheaf" $\mathcal{IC}(\widehat{X};\mathbb{E})$ is actually an object of $\mathbf{D}_{\mathcal{X}}(\widehat{X})$, the derived category of complexes of sheaves $\mathcal{S}$ on $\widehat{X}$ that are constructible. Here the constructibility of $\mathcal{S}$ means that if for all $P\in\mathcal{P}$ we let $i_P : X_P \hookrightarrow \widehat{X}$ denote the inclusion, then the local cohomology sheaf $H(i_P^*\mathcal{S}) = H(\mathcal{S}|_{X_P})$ is locally constant, or equivalently the cohomology sheaf $\mathbb{E}_P = H(i_P^!\mathcal{S})$ is locally constant on X_P. Thus by the correspondence between local systems and representations of the fundamental group one obtains a family of objects $E_P \in \mathbf{Gr}(\Gamma_{L_P})$, the category of graded Γ_{L_P}-modules, one for each $P\in\mathcal{P}$.

(2) Borel points out that in his 1962 Bruxelles conference paper "Ensembles fondamentaux pour les groupes arithmétiques" he proves geometric rationality only when σ is strongly $\mathbb{Q}$-rational. In [**27**] we prove geometric rationality for the general $\mathbb{Q}$-rational case.

(3) We show in [**27**] that this always holds except for certain explicitly described situations in $\mathbb{Q}$-rank 1 and 2 involving restriction of scalars.

Instead of $\mathcal{S}$ we wish to work with a combinatorial analogue in which $\mathbf{Gr}(\Gamma_{L_P})$ is replaced by $\mathbf{Gr}(L_P)$, the category of graded regular L_P-modules. This analogue is what we will call an *$\mathcal{L}$-module on $\widehat{X}$*. We will describe just what an $\mathcal{L}$-module is more precisely later, but first let us give some of the properties of the categories $\mathfrak{Mod}(\mathcal{L}_W)$ of $\mathcal{L}$-modules on W, where W is any locally closed union of strata of $\widehat{X}$:

(i) if $W = X_P$, then $\mathfrak{Mod}(\mathcal{L}_{X_P}) = \mathbf{C}(L_P)$, the category of complexes of regular L_P-modules;

(ii) for any inclusion $j : W \hookrightarrow W'$, there exist functors $j^*, j^! : \mathfrak{Mod}(\mathcal{L}_{W'}) \to \mathfrak{Mod}(\mathcal{L}_W)$ and $j_*, j_! : \mathfrak{Mod}(\mathcal{L}_W) \to \mathfrak{Mod}(\mathcal{L}_{W'})$, as well as a degree truncation functor $\tau^{\leqslant p} : \mathfrak{Mod}(\mathcal{L}_W) \to \mathfrak{Mod}(\mathcal{L}_W)$;

(iii) there is a realization functor $\mathcal{S}_W : \mathfrak{Mod}(\mathcal{L}_W) \to \mathbf{D}_{\mathcal{X}}(W)$ which commutes with the functors in (ii) and for which the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{Mod}(\mathcal{L}_{X_P}) \arrow[r] \arrow[d, shift left, "H(\cdot)"] & \mathbf{D}_{\mathcal{X}}(X_P) \arrow[d, shift left, "H(\cdot)"] \\
\mathbf{Gr}(L_P) \arrow[u, shift left] \arrow[r, "\mathrm{Res}"] & \mathbf{Gr}(\Gamma_{L_P}) \arrow[u, shift left]
\end{tikzcd} .$$

Note that one advantage of $\mathcal{L}$-modules over sheaves is that the left hand vertical arrows in (iii) are equivalences of categories, unlike those on the right; this is because $\mathfrak{Mod}(L_P)$ is a semisimple category.

So roughly speaking an $\mathcal{L}$-module is like a sheaf $\mathcal{S}$ with the "extra structure" that $\mathbb{E}_P = H(i_P^!\mathcal{S})$ is associated to a regular L_P-module, as opposed to merely a Γ_{L_P}-module. Condition (ii) implies that the usual operations on sheaves preserve this "extra structure". The following example shows this is reasonable. Let $\mathbb{E}$ be a local system on X associated to a regular representation E of G. The smooth part of the link bundle of a real codimension k stratum $X_P \subset \widehat{X}$ is the flat bundle with fiber $|\Delta_P|^\circ \times \Gamma_{N_P}\backslash N_P(\mathbb{R})$, where $|\Delta_P|^\circ$ is an open $(k-1)$-simplex and Γ_{L_P} acts via conjugation on the second factor [**8**, §8]. Thus $H(i_P^* i_{G*}\mathbb{E}) \cong \mathbb{H}(\Gamma_{N_P}\backslash N_P(\mathbb{R}); \mathbb{E})$, the local system associated to the Γ_{L_P}-module $H(\Gamma_{N_P}\backslash N_P(\mathbb{R}); \mathbb{E})$. However by van Est's theorem [**7**], $H(\Gamma_{N_P}\backslash N_P(\mathbb{R}); \mathbb{E})$ is isomorphic to the restriction of the regular L_P-module $H(\mathfrak{n}_P; E)$, where $\mathfrak{n}_P$ is the Lie algebra of $N_P(\mathbb{R})$.

In fact this also suggests how to precisely define $\mathcal{L}$-modules. Let $\mathcal{P}(W) \subseteq \mathcal{P}$ correspond to the strata of W. For $P \leqslant Q$ let $\mathfrak{n}_P^Q$ be the Lie algebra of $N_P(\mathbb{R})/N_Q(\mathbb{R})$. An *$\mathcal{L}$-module* $\mathcal{M} \in \mathfrak{Mod}(\mathcal{L}_W)$ is a family $(E_\cdot, f_{\cdot\cdot})$ consisting of objects $E_P \in \mathbf{Gr}(L_P)$ for every $P \in \mathcal{P}(W)$ and degree 1 morphisms $f_{PQ} : H(\mathfrak{n}_P^Q; E_Q) \xrightarrow{[1]} E_P$ for every $P \leqslant Q \in \mathcal{P}(W)$ such that

$$\sum_{P \leqslant Q \leqslant R} f_{PQ} \circ H(\mathfrak{n}_P^Q; f_{QR}) = 0$$

for all $P \leqslant R \in \mathcal{P}(W)$. The functors $i_P^!$ and i_P^* are given by

$$i_P^! \mathcal{M} = (E_P, f_{PP}) ,$$

$$i_P^* \mathcal{M} = \Big(\bigoplus_{P \leqslant R} H(\mathfrak{n}_P^R; E_R), \sum_{P \leqslant R \leqslant S} H(\mathfrak{n}_P^R; f_{RS}) \Big) .$$

We define the global cohomology $H(\widehat{X}; \mathcal{M})$ of an $\mathcal{L}$-module $\mathcal{M}$ to be the hypercohomology of its realization, $H(\widehat{X}; \mathcal{S}_{\widehat{X}}(\mathcal{M}))$. In general we will often write simply $\mathcal{M}$ for both the $\mathcal{L}$-module and its realization $\mathcal{S}_{\widehat{X}}(\mathcal{M})$; it should be clear what is meant from the context.

6. Examples of $\mathcal{L}$-modules

(i) Let $E \in \mathfrak{Mod}(G)$. Then the $\mathcal{L}$-module $i_{G*}E$ defined by $E_G = E$ and $E_P = 0$ for $P \neq G$ corresponds via $\mathcal{S}_{\widehat{X}}$ to $i_{G*}\mathbb{E}$ and its cohomology is the ordinary cohomology $H(X; \mathbb{E}) = H(\Gamma; E)$.

(ii) It follows immediately from the properties of $\mathcal{L}$-modules in the previous section that given $E \in \mathfrak{Mod}(G)$ there exists an $\mathcal{L}$-module $\mathcal{IC}(\widehat{X}; E)$ which maps under $\mathcal{S}_{\widehat{X}}$ to the intersection cohomology sheaf $\mathcal{IC}(\widehat{X}; \mathbb{E})$. For example, if $\mathcal{P} = \{G, P\}$ (that is, $\widehat{X}$ has only one singular stratum) and $p = p(\operatorname{codim} X_P)$, then

$$\mathcal{IC}(\widehat{X}; E) = \begin{pmatrix} E_G = E, \ E_P = (\tau^{>p} H(\mathfrak{n}_P; E))[-1], \\ f_{PG} : H(\mathfrak{n}_P; E) \to \tau^{>p} H(\mathfrak{n}_P; E) \end{pmatrix}$$

where $\tau^{>p} H(\mathfrak{n}_P; E) = \bigoplus_{i>p} H^i(\mathfrak{n}_P; E)[-i]$ and f_{PG} is the projection. Note that the truncation $\tau^{\leqslant p}$ of local cohomology at X_P has been implemented *externally* via a mapping cone; this is valid in view of the quasi-isomorphism $\tau^{\leqslant p} C \xrightarrow{\sim} \operatorname{Cone}(C \to \tau^{>p} C)[-1]$ for any complex C.

(iii) The weighted cohomology sheaf and the L^2-cohomology sheaf may also be lifted to $\mathcal{L}$-modules $\mathcal{WC}(\widehat{X}; E)$ and $\mathcal{L}_{(2)}(\widehat{X}; E)$; for the latter we must replace $\mathfrak{Mod}(L_P)$ by the category of *locally regular* L_P-modules to handle the potentially infinite dimensional local cohomologies.

7. Micro-support of $\mathcal{L}$-modules

The *support* of a sheaf $\mathcal{S}$ is the set of points x such that $H(\mathcal{S})_x \neq 0$. As is well-known the global cohomology of $\mathcal{S}$ vanishes if the support is empty (that is, the sheaf is quasi-isomorphic to 0). For an $\mathcal{L}$-module $\mathcal{M}$ we will state in the next section a more subtle vanishing result based on the *micro-support* of $\mathcal{M}$ which we now define; this is a rough analogue of the corresponding notion for sheaves [**13**].

Let $P \in \mathcal{P}$ and let $\mathfrak{Irr}(L_P)$ denote the set of irreducible regular L_P-modules. For $V \in \mathfrak{Irr}(L_P)$ let ξ_V be the character by which A_P acts on V. Let Δ_P be the simple roots of the adjoint action of A_P on $\mathfrak{n}_P$; the parabolic $\mathbb{Q}$-subgroups $Q \geqslant P$ are indexed by subsets Δ_P^Q of Δ_P. Define $P \leqslant Q_V \leqslant Q'_V \in \mathcal{P}$ by

$$\Delta_P^{Q_V} = \{\, \alpha \in \Delta_P \mid (\xi_V + \rho, \alpha) < 0 \,\} ,$$
$$\Delta_P^{Q'_V} = \{\, \alpha \in \Delta_P \mid (\xi_V + \rho, \alpha) \leqslant 0 \,\} ,$$

where ρ denotes one-half the sum of the positive roots of G and the inner product is induced by the Killing form of G. Let $M_P = {}^0L_P$ so that $L_P(\mathbb{R}) = M_P(\mathbb{R})A_P$. Let $V|_{M_P}$ denote the restriction of the representation V to M_P.

The *micro-support* $\mathrm{SS}(\mathcal{M})$ *of* $\mathcal{M}$ is the subset of $\coprod_{P\in\mathcal{P}} \mathfrak{Irr}(L_P)$ consisting of those $V \in \mathfrak{Irr}(L_P)$ satisfying

(i) $(V|_{M_P})^* \cong \overline{V|_{M_P}}$, and

(ii) there exists $Q_V \leqslant Q \leqslant Q'_V$ such that

$$H(i_P^* \hat{\imath}_Q^! \mathcal{M})_V \neq 0 . \tag{7.1}$$

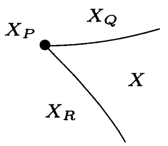
Here $\hat{\imath}$ $\hat{\imath}_Q : \widehat{X}_Q \hookrightarrow \widehat{X}$ is the inclusion of the closure of the stratum X_Q and the subscript V indicates the V-isotypical component. A simple example of the computation of micro-support will be given in § 11.

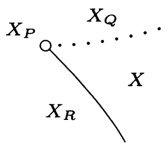
Condition (i) is equivalent to the existence of a nondegenerate sesquilinear form on V which is invariant under the action of M_P.

As for condition (ii), let $\hat{\jmath}_Q : \widehat{X} \smallsetminus \widehat{X}_Q \hookrightarrow \widehat{X}$ be the open inclusion. Note that we have a short exact sequence

$$0 \longrightarrow i_P^* \hat{\imath}_Q^! \mathcal{M} \longrightarrow i_P^* \mathcal{M} \longrightarrow i_P^* \hat{\jmath}_{Q*} \hat{\jmath}_Q^* \mathcal{M} \longrightarrow 0$$

and a corresponding long exact sequence. Topologically, this is the long exact sequence of the pair $(U, U \smallsetminus (U \cap \widehat{X}_Q))$ where U is a small neighborhood of a point of X_P. Thus condition (ii) means that





$$H(U; \mathcal{M})_V \longrightarrow H(U \smallsetminus (U \cap \widehat{X}_Q); \mathcal{M})_V$$

is not an isomorphism for some degree and for some Q between Q_V and Q'_V.

It is convenient to define the *essential micro-support* $\mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$ *of* $\mathcal{M}$ to be the subset consisting of those $V \in \mathrm{SS}(\mathcal{M})$ for which

$$\mathrm{Type}_V(\mathcal{M}) = \mathrm{Image}\big(H(i_P^* \hat{\imath}_{Q_V}^! \mathcal{M})_V \longrightarrow H(i_P^* \hat{\imath}_{Q'_V}^! \mathcal{M})_V\big)$$

is nonzero. The essential micro-support of $\mathcal{M}$ determines the micro-support (though not the actual parabolics Q that arise in condition (ii)). In fact the relation between $\mathrm{SS}(\mathcal{M})$ and $\mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$ is analogous to the relation between the strata of a nonreduced variety (possibly with embedded components) and the smooth open strata of the irreducible components: there exists a partial order $\preccurlyeq$ on $\coprod_{P\in\mathcal{P}} \mathfrak{Irr}(L_P)$ such that if $V \in \mathrm{SS}(\mathcal{M})$ then there exists $\widetilde{V} \in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$ with $V \preccurlyeq \widetilde{V}$, and if $\widetilde{V} \in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$ and $V \preccurlyeq \widetilde{V}$ then $V \in \mathrm{SS}(\mathcal{M})$.

8. A vanishing theorem for $\mathcal{L}$-modules

The justification for the definition of $\mathrm{SS}(\mathcal{M})$ is that it is an ingredient for a vanishing theorem for $H(\widehat{X};\mathcal{M})$. To state the theorem we need some more notation.

Let $V \in \mathfrak{Irr}(L_P)$ have highest weight $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ where $\mathfrak{h}$ is a fundamental (maximally compact) Cartan subalgebra for the Lie algebra $\mathfrak{l}_P$ of $L_P(\mathbb{R})$ equipped with a compatible ordering. Assume $(V|_{M_P})^* \cong \overline{V|_{M_P}}$ and define

$$\begin{aligned} L_P(\mu) &= \text{ the centralizer of } \mu \in \mathfrak{h}_{\mathbb{C}}^* \subset \mathfrak{l}_{P\mathbb{C}}^* \ , \\ &= \text{ the reductive subgroup of } L_P \text{ with roots } \{\, \gamma \in \Phi(\mathfrak{l}_{P\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid (\gamma,\mu) = 0 \,\} \ , \\ D_P(\mu) &= \text{ the associated symmetric space } L_P(\mu)(\mathbb{R})/(K_P \cap L_P(\mu))A_P \ . \end{aligned}$$

Choose a compatible ordering for which $\dim D_P(\mu)$ is maximized and let $D_P(V) = D_P(\mu)$. Suppose now that $V \in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$. Let $c(V;\mathcal{M}) \leqslant d(V;\mathcal{M})$ be the least and greatest degrees in which $\mathrm{Type}_V(\mathcal{M})$ is nonzero, and define

$$\begin{aligned} \widetilde{c}(V;\mathcal{M}) &= \tfrac{1}{2}(\dim D_P - \dim D_P(V)) + c(V;\mathcal{M}) \ , \\ \widetilde{d}(V;\mathcal{M}) &= \tfrac{1}{2}(\dim D_P + \dim D_P(V)) + d(V;\mathcal{M}) \ . \end{aligned}$$

Set

$$c(\mathcal{M}) = \inf_{V\in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})} \widetilde{c}(V;\mathcal{M}) \ , \qquad d(\mathcal{M}) = \sup_{V\in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})} \widetilde{d}(V;\mathcal{M}) \ .$$

(One can show that the same values are obtained if instead we consider all $V \in \mathrm{SS}(\mathcal{M})$ and let $c(V;\mathcal{M}) \leqslant d(V;\mathcal{M})$ be the least and greatest degrees in which (7.1) is nonzero (for any Q).)

Theorem 1. — *$H^i(\widehat{X};\mathcal{M}) = 0$ for $i \notin [c(\mathcal{M}), d(\mathcal{M})]$.*

Let us comment briefly on the proof which uses combinatorial Hodge-de Rham theory. The sheaf $\mathcal{S}_{\widehat{X}}(\mathcal{M})$ has an incarnation as a complex of fine sheaves whose global sections are "combinatorial" differential forms. That is, an element of $\Gamma(\widehat{X};\mathcal{S}^{\cdot}_{\widehat{X}}(\mathcal{M}))$ is a family $(\omega_P)_{P\in\mathcal{P}}$, where each ω_P is a special differential form on X_P with coefficients in $\mathbb{E}_P$. (For $P = G$, the *special differential forms* [**8**, (13.2)] on $X = X_G$ are those which near each boundary stratum Y_Q of the Borel-Serre compactification $\overline{X}$ are the pullback of an $N_Q(\mathbb{R})$-invariant form on Y_Q; they form a resolution of $\mathbb{E}_G$.) The differential is a sum of the usual de Rham exterior derivative (on each ω_P) together with operators based on the f_{PQ}.

To do harmonic theory we need a metric; unfortunately the locally symmetric metric on each X_P is not appropriate since it would introduce unwanted L^2-growth conditions on the differential forms. Instead the theory of *tilings* from [**25**] gives a natural piecewise analytic diffeomorphism of $\overline{X}$ onto a closed subdomain $\overline{X}_0$ of the interior X; the pullback of the locally symmetric metric under this map yields metrics on all X_P which extend to nondegenerate metrics on their boundary strata. Now a spectral analogue of the Mayer-Vietoris sequence as in [**28**] reduces the problem to a vanishing theorem for combinatorial L^2-cohomology near each stratum X_P. After unraveling the combinatorics one obtains contributions to the cohomology of the form $H_{(2)}(X_P;\mathbb{V}) \otimes \mathrm{Type}_V(\mathcal{M})$ for $V \in \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$; by Raghunathan's vanishing theorem [**20**], [**21**], [**28**] this is zero outside the degree range $[\widetilde{c}(V;\mathcal{M}), \widetilde{d}(V;\mathcal{M})]$. (The proof is actually more complicated since there are infinite dimensional contributions from $\mathrm{SS}(\mathcal{M}) \smallsetminus \mathrm{SS}_{\mathrm{ess}}(\mathcal{M})$ as well.)

9. Micro-purity of intersection cohomology

We will say an $\mathcal{L}$-module $\mathcal{M}$ on $\widehat{X}$ is *V-micro-pure* if $\mathrm{SS}_{\mathrm{ess}}(\mathcal{M}) = \{V\}$ with $\mathrm{Type}_V(\mathcal{M})$ concentrated in degree 0.

Theorem 2. — *Assume the irreducible components of the $\mathbb{Q}$-root system of G are of type A_n, B_n, C_n, BC_n, or G_2. Let $E \in \mathfrak{Irr}(G)$ satisfy $(E|_{{}^0G})^* \cong \overline{E|_{{}^0G}}$. Then $\mathcal{IC}(\widehat{X};E)$ is E-micro-pure.*

If D is a Hermitian symmetric space (or an equal-rank symmetric space admitting a real equal-rank Satake compactification as in § 4) G will have a $\mathbb{Q}$-root system of the indicated type and thus the theorem applies in the context of Rapoport and Goresky-MacPherson's conjecture. In fact it is quite possible that this restriction in the theorem may be removed; it is only required at one crucial stage in the proof.

What the theorem is asserting is that $V \notin \mathrm{SS}_{\mathrm{ess}}(\mathcal{IC}(\widehat{X};E))$ for $V \in \mathfrak{Irr}(L_P)$ with $P \neq G$. When P is a maximal parabolic we can give a brief indication of how this is proven; for definiteness we assume p is the upper middle perversity. In this case

$$H(\hat{\imath}_P^* \hat{\imath}_Q^! \mathcal{IC}(\widehat{X};E)) = \begin{cases} \tau^{\leqslant p} H(\mathfrak{n}_P;E) & \text{for } Q = G\ , \\ (\tau^{>p} H(\mathfrak{n}_P;E))[-1] & \text{for } Q = P\ , \end{cases} \tag{9.1}$$

where $p = \lfloor \frac{1}{2} \dim \mathfrak{n}_P \rfloor$. Let λ be the highest weight of E. By Kostant's theorem [**15**] an irreducible component V of $H(\mathfrak{n};E)$ has highest weight $w(\lambda+\rho)-\rho$ where

$$w \in W_P = \{w \in W \mid w^{-1}\gamma > 0 \text{ for all postive roots } \gamma \text{ of } \mathfrak{l}_{P\mathbb{C}}\}\ ,$$

the set of minimal length representatives of the Weyl group quotient $W_{L_P} \backslash W$. Furthermore V occurs in degree $\ell(w)$, the length of w, with multiplicity 1. Assume now that $V \in \mathrm{SS}_{\mathrm{ess}}(\mathcal{IC}(\widehat{X};E))$. Since the two cases in (9.1) above do not share a common component we must have $Q_V = Q'_V$, that is, $(\xi_V+\rho,\alpha) \neq 0$ for the unique $\alpha \in \Delta_P$. Furthermore (9.1) also shows that the possibilities $(\xi_V+\rho,\alpha) < 0$ and $(\xi_V+\rho,\alpha) > 0$ correspond respectively to $\ell(w) \leqslant \frac{1}{2}\dim \mathfrak{n}_P$ and $\ell(w) > \frac{1}{2}\dim \mathfrak{n}_P$. However the following lemma from [**26**] shows that in fact the opposite relation between weight and degree holds (the nonnegative term $\dim \mathfrak{n}_P(V)$ here may be ignored for now—it will be defined in § 11):

Lemma 3. — *Let* $V \in \mathfrak{Irr}(L_P)$ *have highest weight* $w(\lambda+\rho)-\rho$ *where* $w \in W_P$ *and* $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ *is dominant. Assume that* $(V|_{M_P})^* \cong \overline{V|_{M_P}}$.

(i) *If* $(\xi_V+\rho,\alpha) \leqslant 0$ *for all* $\alpha \in \Delta_P$, *then* $\ell(w) \geqslant \frac{1}{2}(\dim \mathfrak{n}_P + \dim \mathfrak{n}_P(V))$.

(ii) *If* $(\xi_V+\rho,\alpha) \geqslant 0$ *for all* $\alpha \in \Delta_P$, *then* $\ell(w) \leqslant \frac{1}{2}(\dim \mathfrak{n}_P - \dim \mathfrak{n}_P(V))$.

The only remaining possibility is that $\ell(w) = \frac{1}{2}\dim \mathfrak{n}_P$, but since $(\xi_V+\rho,\alpha) \neq 0$ and $(E|_{{}^0G})^* \cong \overline{E|_{{}^0G}}$ this is impossible by an argument based on [**3**]. By the way, Lemma 3 is basic to the proofs of Theorems 1, 4, and 5 as well and has its origin in a result of Casselman for $\mathbb{R}$-rank one [**5**].

When P is not a maximal parabolic the situation is far more complicated. The irreducible components of $H(i_P^*\mathcal{IC}(\widehat{X};E))$ are among those of $H(i_P^* i_{G*} i_G^* \mathcal{IC}(\widehat{X};E)) = H(\mathfrak{n}_P;E)$, but they may occur in various degrees and with multiplicity. Since we do not know a nonrecursive formula for $H(i_P^*\mathcal{IC}(\widehat{X};E))$ we must rely on the inductive definition. However condition (i) in the definition of micro-support is not preserved upon passing to a larger stratum. Specifically, let $P < R$ and suppose V is an irreducible component of $H(\mathfrak{n}_P;E) = H(\mathfrak{n}_P^R; H(\mathfrak{n}_R;E))$. It must lie within $H(\mathfrak{n}_P^R; V_R)$ for some irreducible component V_R of $H(\mathfrak{n}_R;E)$. The difficulty in using induction is that $(V|_{M_P})^* \cong \overline{V|_{M_P}}$ does not imply $(V_R|_{M_R})^* \cong \overline{V_R|_{M_R}}$.

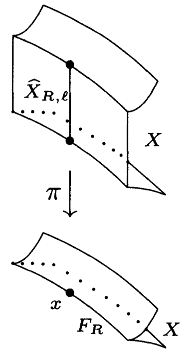
These difficulties do not apply to $\mathcal{WC}(\widehat{X};E)$ and in fact a fairly simple argument shows that Theorem 2 holds for $\mathcal{WC}(\widehat{X};E)$ without any hypothesis on the $\mathbb{Q}$-root system and for either middle weight profile. Indeed since $\mathcal{WC}(\widehat{X};E)$ is defined directly in terms of weight the relationship between weight and degree provided by Lemma 3 is not needed and hence the condition $(V|_{M_P})^* \cong \overline{V|_{M_P}}$ plays no role in the proof.

10. Functoriality of micro-support and proof of the Main Theorem

Let $\mathcal{M}$ be an $\mathcal{L}$-module which is E-micro-pure (for example, $\mathcal{M} = \mathcal{IC}(\widehat{X};E)$ by Theorem 2) and assume we are in the context of Rapoport and Goresky-MacPherson's conjecture, that is, D is Hermitian symmetric and $\pi : \widehat{X} \to X^*$ is the projection onto the Baily-Borel-Satake compactification. The desired equality $\pi_*\mathcal{M} = \mathcal{IC}(X^*;\mathbb{E})$ is equivalent to certain local vanishing and covanishing conditions on $\pi_*\mathcal{M}$ [**10**]. To state them, let $i_x : \{x\} \hookrightarrow X^*$ denote the inclusion of a point in a stratum $F_R \subset X^*$. Since every stratum of X^* has even codimension, $p(\operatorname{codim} F_R) = \frac{1}{2}\operatorname{codim} F_R - 1$. The local conditions that characterize intersection cohomology now can be expressed as

$$
(10.1)\qquad
\begin{aligned}
H^i(i_x^*\pi_*\mathcal{M}) &= 0 \qquad \text{for } x \in F_R,\ i \geqslant \tfrac{1}{2}\operatorname{codim} F_R, \text{ and}\\
H^i(i_x^!\pi_*\mathcal{M}) &= 0 \qquad \text{for } x \in F_R,\ i \leqslant \tfrac{1}{2}\operatorname{codim} F_R
\end{aligned}
$$

for every stratum $F_R \subset X^*$.



Recall that for every $P \in \mathcal{P}$ with $P^\dagger = R$ there is a factorization $X_P = X_{P,\ell} \times F_R$ and that $\pi|_{X_P}$ is simply projection onto the second factor. Thus $\pi^{-1}(x) = \coprod_{P^\dagger=R} X_{P,\ell} \times \{x\} = \widehat{X}_{R,\ell} \times \{x\}$ and we let $\hat{\imath}_{R,\ell} : \widehat{X}_{R,\ell} \cong \pi^{-1}(x) \hookrightarrow \widehat{X}$ be the inclusion. Since $H^i(i_x^*\pi_*\mathcal{M}) = H^i(\widehat{X}_{R,\ell}; \hat{\imath}^*_{R,\ell}\mathcal{M})$ and $H^i(i_x^!\pi_*\mathcal{M}) = H^i(\widehat{X}_{R,\ell}; \hat{\imath}^!_{R,\ell}\mathcal{M})$ we can use Theorem 1 to see these vanish for $i > d(\hat{\imath}^*_{R,\ell}\mathcal{M})$ and $i < c(\hat{\imath}^!_{R,\ell}\mathcal{M})$ respectively. Thus the following theorem implies that (10.1) holds (and hence completes the proof of the Main Theorem):

Theorem 4. — *Let $\mathcal{M}$ be an E-micro-pure $\mathcal{L}$-module and let F_R be a stratum of the Baily-Borel-Satake compactification X^*. Then*

$$d(\hat{\imath}^*_{R,\ell}\mathcal{M}) \leqslant \tfrac{1}{2}\operatorname{codim} F_R - 1 \qquad \textit{and} \qquad c(\hat{\imath}^!_{R,\ell}\mathcal{M}) \geqslant \tfrac{1}{2}\operatorname{codim} F_R + 1 \ .$$

The same result holds if D is an equal-rank symmetric space and X^* is a real equal-rank Satake compactification as in § 4. This theorem is actually a special case of a more general result on the functoriality of micro-support: for $\mathcal{M}$ an arbitrary $\mathcal{L}$-module and X^* a real equal-rank Satake compactification as in § 4, the theorem gives a bound on $\mathrm{SS}(\hat{\imath}^*_{R,\ell}\mathcal{M})$ and $\mathrm{SS}(\hat{\imath}^!_{R,\ell}\mathcal{M})$ in terms of $\mathrm{SS}(\mathcal{M})$.

Since $\mathcal{WC}(\widehat{X}; E)$ is also E-micro-pure, the same argument yields a new proof of the main result of [**8**] (and in fact a generalization to real equal-rank Satake compactifications).

11. Example/application: ordinary cohomology

As another application of $\mathcal{L}$-modules we consider the ordinary cohomology $H(X; \mathbb{E})$ or $H(\Gamma; E)$ with coefficients in $E \in \mathfrak{Irr}(G)$. This is the cohomology $H(\widehat{X}; \mathcal{M})$ for the $\mathcal{L}$-module $\mathcal{M} = i_{G*}E$ which has $E_G = E$ and $E_P = 0$ for $P \neq G$ (see § 6(i)).

We calculate the micro-support of $i_{G*}E$. Since $i^!_Q i_{G*}E = E_Q$ we see that

$$H(i_P^* \hat{\imath}^!_Q i_{G*}E) = \begin{cases} H(\mathfrak{n}_P; E) & \text{for } Q = G\ , \\ 0 & \text{for } Q \neq G\ . \end{cases}$$

Thus for $V \in \mathfrak{Irr}(L_P)$ to be in $\mathrm{SS}(i_{G*}E)$ it must be an irreducible component of $H(\mathfrak{n}_P; E)$ satisfying $(V|_{M_P})^* \cong \overline{V|_{M_P}}$ and $(\xi_V + \rho, \alpha) \leqslant 0$ for all $\alpha \in \Delta_P$ (since $Q = G$ implies $Q'_V = G$). The essential micro-support will consist of such V satisfying in addition the strict inequalities $(\xi_V + \rho, \alpha) < 0$.

Let λ be the highest weight of E. As in § 9, the irreducible components of $H(\mathfrak{n}_P; E)$ are the modules $V_{w(\lambda+\rho)-\rho} \in \mathfrak{Irr}(L_P)$ with highest weight $w(\lambda + \rho) - \rho$ for $w \in W_P$. Let $\tau_P : \mathfrak{h}^*_{\mathbb{C}} \to \mathfrak{h}^*_{\mathbb{C}}$ transform the highest weight of a representation of L_P into the highest weight of its complex conjugate contragredient; we assume that $\mathfrak{h} = \mathfrak{b}_P + \mathfrak{a}_P =$

$\mathfrak{b}_{P,\mathfrak{k}} + \mathfrak{b}_{P,\mathfrak{p}} + \mathfrak{a}_P$ is a fundamental Cartan subalgebra of $\mathfrak{l}_P$ equipped with a compatible order so that τ_P is simply the Cartan involution [**3**]. We can now reexpress our calculation as

$$\mathrm{SS}_{\mathrm{ess}}(i_{G*}E) = \coprod_P \{ V_{w(\lambda+\rho)-\rho} \mid w \in W_P,\ (w(\lambda+\rho),\alpha) < 0 \text{ for all } \alpha \in \Delta_P, \\ \text{and } \tau_P(w(\lambda+\rho)|_{\mathfrak{b}_P}) = w(\lambda+\rho)|_{\mathfrak{b}_P} \} \ .$$

(In the last equation we have used the fact that $\tau_P(\rho|_{\mathfrak{b}_P}) = \rho|_{\mathfrak{b}_P}$.) Furthermore since $V = V_{w(\lambda+\rho)-\rho}$ occurs in $H(i_P^* \hat{\imath}_Q^! i_{G*}E)$ in degree $\ell(w)$ we see that

$$\widetilde{c}(V; i_{G*}E) = \tfrac{1}{2}(\dim D_P - D_P(V)) + \ell(w) \ . \tag{11.1}$$

We use Lemma 3 to estimate $\ell(w)$, however now we need the term $\dim \mathfrak{n}_P(V)$. To define it, recall we have defined $L_P(\mu) \subseteq L_P$ in § 8 to have roots $\gamma \perp \mu = w(\lambda+\rho)-\rho$. Since $(w(\lambda+\rho)-\rho)|_{\mathfrak{b}_P}$ is invariant under τ_P, the roots of $L_P(\mu)$ are stable under τ_P. Thus given an $L_P(\mu)$-irreducible submodule of $\mathfrak{n}_{P\mathbb{C}}$, the transform by $-\tau_P$ of its weights are the weights of another $L_P(\mu)$-irreducible submodule of $\mathfrak{n}_{P\mathbb{C}}$. Define $\mathfrak{n}_P(\mu)$ to be the sum of the $L_P(\mu)$-irreducible submodules of $\mathfrak{n}_P$ whose weights are stable under $-\tau_P$. Choose a compatible ordering for which $\dim \mathfrak{n}_P(\mu)$ is maximized and let $\mathfrak{n}_P(V) = \mathfrak{n}_P(\mu)$. Note that $\mathfrak{n}_P(V)$ contains the root spaces of the positive $(-\tau_P)$-invariant roots, that is, the real roots.

We now make two assumptions: that D is Hermitian symmetric, or more generally equal-rank, and that E has regular highest weight λ. By the first assumption the Lie algebra of ${}^0G(\mathbb{R})$ also possesses a compact Cartan subalgebra and therefore by the Kostant-Sugiura theory of conjugacy classes of Cartan subalgebras [**14**], [**31**], [**32**] there must exist at least $\dim \mathfrak{b}_{P,\mathfrak{p}} + \dim \mathfrak{a}_P - \dim \mathfrak{a}_G$ orthogonal real roots. Thus

$$\dim \mathfrak{n}_P(V) \geqslant \dim \mathfrak{b}_{P,\mathfrak{p}} + \dim \mathfrak{a}_P - \dim \mathfrak{a}_G \ . \tag{11.2}$$

On the other hand, note that if $\gamma^\vee = 2\gamma/(\gamma,\gamma)$ then $(\rho,\gamma^\vee) = 1$ if and only if γ is simple. Consequently for γ a simple root of L_P in any compatible ordering we have

$$\gamma \text{ is a root of } L_P(\mu) \iff (w(\lambda+\rho),\gamma^\vee) = (\rho,\gamma^\vee) \iff (\lambda+\rho, w^{-1}\gamma^\vee) = 1 \\ \iff (\lambda, w^{-1}\gamma) = 0 \text{ and } w^{-1}\gamma \text{ is simple.}$$

Thus the second assumption implies that $L_P(\mu) = H$, the Cartan subgroup, and hence

$$\dim D_P(V) = \dim \mathfrak{b}_{P,\mathfrak{p}} \ . \tag{11.3}$$

Lemma 3(i) and equations (11.1)–(11.3) yield the estimate $\widetilde{c}(V; i_{G*}E) \geqslant \frac{1}{2}(\dim D_P + \dim \mathfrak{a}_P + \dim \mathfrak{n}_P - \dim \mathfrak{a}_G) = \frac{1}{2}\dim X$. Thus Theorem 1 implies

Theorem 5. — *If X is an arithmetic quotient of a Hermitian or equal-rank symmetric space and E has regular highest weight then $H^i(X;\mathbb{E}) = 0$ for $i < \frac{1}{2}\dim X$.*

This resolves a question posed by Tilouine during the Automorphic Forms Semester. For the case $G = R_{k/\mathbb{Q}}\,\mathrm{GSp}(4)$ where k is a totally real number field the theorem is proven in [**33**] using results of Franke. For applications of the theorem see [**18**], [**19**]. While this paper was being prepared we heard that Li and Schwermer also had a proof of the theorem.[(4)]

A vanishing range for the case where E does not have regular highest weight may be obtained by replacing (11.2) and (11.3) by the more subtle estimate on $\dim \mathfrak{n}_P(V)$ given in [**26**].

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[(4)] Added Oct. 2003: See [**16**]. The methods are completely different. They show vanishing in the range $i < \frac{1}{2}(\dim X - (\operatorname{rank}{}^0G - \operatorname{rank} K))$ without assuming D is equal-rank. This stregthened theorem also follows from the methods of the present paper: if D is not equal-rank, equation (11.2) remains true provided we subtract $\operatorname{rank}{}^0G - \operatorname{rank} K$ from the right-hand side.

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