ARTUR O. LOPES
PHILIPPE THIEULLLEN

Sub-actions for Anosov diffeomorphisms

Astérisque, tome 287 (2003), p. 135-146

<http://www.numdam.org/item?id=AST_2003__287__135_0>
SUB-ACTIONS FOR ANOSOV DIFFEOMORPHISMS

by

Artur O. Lopes & Philippe Thieullen

Abstract. — We show a positive Livsic type theorem for $C^2$ Anosov diffeomorphisms $f$ on a compact boundaryless manifold $M$ and Hölder observables $A$. Given $A : M \to \mathbb{R}$, $\alpha$-Hölder, we show there exist $V : M \to \mathbb{R}$, $\beta$-Hölder, $\beta < \alpha$, and a probability measure $\mu$, $f$-invariant such that

$$A \leq V \circ f - V + \int A \, d\mu.$$  

We apply this inequality to prove the existence of an open set $\mathcal{G}_\beta$ of $\beta$-Hölder functions, $\beta$ small, which admit a unique maximizing measure supported on a periodic orbit. Moreover the closure of $\mathcal{G}_\beta$, in the $\beta$-Hölder topology, contains all $\alpha$-Hölder functions, $\alpha$ close to one.

1. Introduction

We consider a compact riemannian manifold $M$ of dimension $d \geq 2$ without boundary and a $C^2$ transitive Anosov diffeomorphism $f : M \to M$. The tangent bundle $TM$ admits a continuous $Tf$-invariant splitting $TM = E^u \oplus E^s$ of expanding and contracting tangent vectors. We assume $M$ is equipped with a riemannian metric and there exists a constant $C(M)$, depending only on $M$ and the metric and constants $\alpha < \lambda_0 \frac{\alpha^2}{\lambda_0} < \lambda_1 < 1 < \lambda_u < \Lambda_u$

such that for all $n \in \mathbb{Z}$

$$\begin{align*}
C(M)^{-1} \Lambda_u^n &\leq \|T_x f^n \cdot v\| \leq C(M) \Lambda_u^n \quad \text{for all } v \in E^u_x, \\
C(M)^{-1} \Lambda_s^n &\leq \|T_x f^n \cdot v\| \leq C(M) \Lambda_s^n \quad \text{for all } v \in E^s_x.
\end{align*}$$

2000 Mathematics Subject Classification. — 37D20.

Key words and phrases. — Anosov diffeomorphisms, minimizing measures.

A.L.: Partially supported by PRONEX-CNPq - Sistemas Dinâmicas and Institute of Millenium - IMPA.
Ph.T.: Partially supported by CNRS URA 1169.
Livsic theorem [5] asserts that, if $A: M \to M$ is a given Hölder function and satisfies $\int A \, d\mu = 0$ for all $f$-invariant probability measure $\mu$, then $A$ is equal to a coboundary $V$ (which is Hölder too), that is:

$$A = V \circ f - V. $$

What happens if we only assume $\int A \, d\mu \geq 0$ for all $f$-invariant probability measure $\mu$? We denote by $\mathcal{M}(f)$, the set of $f$-invariant probability measures and $m(A, f) = \sup \{ \int A \, d\mu \mid \mu \in \mathcal{M}(f) \}$.

For a $\beta$-Hölder function $V$

$$\text{Höld}_\beta(V) = \sup_{0<d(x,y)} \left\{ \frac{|V(x) - V(y)|}{d(x,y)^\beta} \right\}.$$

We prove the following:

**Theorem 1.** — Let $f: M \to M$ be a $C^2$ transitive Anosov diffeomorphism on a compact manifold $M$ without boundary. For any given $\alpha$-Hölder function $A: M \to \mathbb{R}$, there exists a $\beta$-Hölder function $V: M \to \mathbb{R}$, that we call sub-action, such that:

$$A = V \circ f - V + m(A, f),$$

and

$$\beta = \alpha \frac{\ln(1/\lambda_a)}{\ln(\Lambda_u/\Lambda_s)}, \quad \text{Höld}_\beta(V) \leq \frac{C(M)}{\min(1 - \lambda_a^\alpha, 1 - \lambda_s^\alpha)^2} \text{Höld}_\alpha(A)$$

where $C(M)$ is some constant depending only on $M$ and the metric.

By analogy with Hamiltonian mechanics and the way we define $V$ from $A$, we may interpret $A$ as a lagrangian and $V$ as a sub-action. This result extends a similar one we obtained in [4] for expanding maps of the circle (see [2] [6] for related results). The same techniques of [4] also apply for the one-directional shift as it is mentioned in [4].

The proof we give here is for bijective smooth systems, and we obtain $V$ continuous in all $M$. Our result can not be derived (via Markov partition) directly from an analogous result for the bi-directional shift.

**Corollary 2.** — The hypothesis are the same as in theorem 1. The following statements are equivalent:

(i) $A \geq V \circ f - V$ for some bounded measurable function $V$,

(ii) $\int A \, d\mu \geq 0$ for all $f$-invariant probability measure $\mu$,

(iii) $\sum_{k=0}^{p-1} A \circ f^k(x) \geq 0$ for all $p \geq 1$ and point $x$ periodic of period $p$,

(iv) $A \geq V \circ f - V$ for some Hölder function $V$.

The proof of that corollary is straightforward and uses (for (iii) $\Rightarrow$ (ii)) the fact that the convex hull of periodic measures is dense in the set of all $f$-invariant probability measures for topological dynamical systems satisfying the shadowing lemma (see Lemma 5). F. Labourie suggested to us the following corollary:
Corollary 3. — The hypothesis are the same as in theorem 1. If $A$ satisfies $\int A d\mu \geq 0$ for all $\mu \in \mathcal{M}(f)$ and $\sum_{k=0}^{p-1} A \circ f^k(x) > 0$ for at least one periodic orbit $x$ of period $p$ then $\int A d\lambda > 0$ for all probability measure $\lambda$ giving positive mass to any open set.

Again the proof is straightforward: $R = A - V \circ f + V \geq 0$ for some continuous $V$ and $\int R d\lambda = 0$ for such a measure $\lambda$ implies $R = 0$ everywhere and in particular $\sum_{k=0}^{p-1} A \circ f(x) = 0$ for all periodic orbit $x$.

Any measure $\mu$ satisfying $\int A d\mu = m(A, f)$ is called a maximizing measure and since $A$ is continuous, such a measure always exists. It is then natural to ask the following two questions: For which $A$, the set of maximizing measures is reduced to a single measure? In the case there exists a unique maximizing measure, to what kind of compact set, the support of this measure looks like?

The following theorem gives a partial answer for “generic” functions $A$.

Theorem 4. — Let $f : M \to M$ be a $C^2$ transitive Anosov diffeomorphism and $\beta < \ln(1/\lambda_\alpha)/\ln(\Lambda_u/\Lambda_s)$. Then there exists an open set $\mathcal{G}_\beta$ of $\beta$-Hölder functions (open in the $C^\beta$-topology) such that:

(i) any $A$ in $\mathcal{G}_\beta$ admits a unique maximizing measure $\mu_A$;

(ii) the support of $\mu_A$ is equal to a periodic orbit and is locally constant with respect to $A \in \mathcal{G}_\beta$;

(iii) any $\alpha$-Hölder function with $\alpha > \beta \ln(\Lambda_u/\Lambda_s)/\ln(1/\lambda_\alpha)$ is contained in the closure of $\mathcal{G}_\beta$ (the closure is taken with respect to the $C^\beta$-topology).

The proof of Theorem 4 is a simplification of what we gave in [4] in the one-dimensional setting. The existence of sub-actions is in both cases the main ingredient of the proof.

Now we will concentrate in one of our main results, namely, Theorem 1; the basic idea is the following: given a finite covering of $M$ by open sets $\{U_1, \ldots, U_i\}$ with sufficiently small diameter, we construct a Markov covering (and not a Markov partition) $\{R_1, \ldots, R_i\}$ of rectangles: each $R_t$ contains $U_t$ and satisfies

$$x \in U_t \cap f^{-1}(U_j) \implies f(W^s(x, R_t)) \subset W^s(f(x), R_j),$$

where $W^s(x, R_t)$ denotes the local stable leaf through $x$ restricted to $R_t$. We then associate to each $R_t$ a local sub-action $V_i$, defined on $R_t$ by:

$$V_i(x) = \sup \{ S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geq 0, y \in W^s(x, R_t) \}$$

where $\Delta^s(y, x)$ is a kind of cocycle along the stable leaf $W^s(x)$:

$$\Delta^s(y, x) = \sum_{n \geq 0} (A \circ f^n(y) - A \circ f^n(x)), $$

and where $S_n(A - m) = \sum_{k=0}^{n-1} (A - m) \circ f^k$.

This family $\{V_1, \ldots, V_t\}$ of local sub-actions satisfies the inequality:

$$x \in U_t \cap f^{-1}(U_j) \implies V_i(x) + A(x) - m \leq V_j \circ f(x)$$
and enable us to construct a global sub-action $V$:

$$V(x) = \sum_{i=1}^{l} \theta_i(x) V_i(x)$$

where $\{\theta_1, \ldots, \theta_l\}$ is a smooth partition of unity associated to the covering $\{U_1, \ldots, U_l\}$. The main difficulty is to prove that each $V_i$ is Hölder on $R_i$.

### 2. Existence of sub-actions

We continue our description of the dynamics of transitive Anosov diffeomorphisms (for details information, see Bowen’s monography [3]). All the results we are going to use depend on a small constant of expansiveness $\varepsilon^* > 0$ (by definition this constant says that any pseudo-orbit can be followed by true orbits) depending on $f$ and $M$ in the following way:

$$\varepsilon^* = C(M)^{-1} \min \left( \frac{\lambda_n - 1}{\|D^2f\|_\infty}, \frac{1 - \lambda_n}{\|D^2f\|_\infty} \right)$$

where $C(M) \geq 1$ is a constant depending only on $M$ and the riemannian metric. At each point $x$, one can define its local stable manifold $W^s_\varepsilon(x)$ for every $\varepsilon < \varepsilon^*$:

$$W^s_\varepsilon(x) = \{ y \in M \mid d(f^n(x), f^n(y)) \leq \varepsilon \ \forall \ n \geq 0 \}$$

which are $C^2$ embedded closed disks of dimension $d^s = \dim E^s_x$ and tangent to $E^s_x$. In the same manner, $W^u_\varepsilon(x)$ is defined replacing $f$ by $f^{-1}$. If two points $x, y$ are close enough, $d(x, y) < \delta$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ have a unique point in common, called $[x, y]$:

$$[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = W^s_\varepsilon(x) \cap W^u_\varepsilon(y),$$

where $\varepsilon = K^* \delta$ and $K^*$ is again a large constant depending on $M$ and $f$:

$$K^* = \frac{C(M)}{\min (1 - \lambda_n^{-1}, 1 - \lambda_n)}.$$  

This estimate is in fact a particular case of Bowen’s shadowing lemma:

**Lemma 5 (Bowen).** — If $\delta$ is small enough, $\delta < \varepsilon^*/K^*$, if $(x_n)_{n \in \mathbb{Z}}$ is a bi-infinite $\delta$-pseudo-orbit, that is, $d(f^n(x_n), f^n(x_{n+1})) < \delta$ for all $n \in \mathbb{Z}$, then there exists a unique true orbit $\{f^n(x)\}_{n \in \mathbb{Z}}$ which $\varepsilon$-shadow $(x_n)_{n \in \mathbb{Z}}$. that is $d(f^n(x), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$ with $\varepsilon = K^* \delta$.

This lemma (see [3]) for proof) is the main ingredient for constructing (dynamical) rectangles. A rectangle $R$ is a closed set of diameter less than $\varepsilon^*/K^*$ satisfying:

$$x, y \in R \implies [x, y] \in R.$$

We will not use the notion of proper rectangles but will use instead the notion of Markov covering.
Definition 6. — Let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be a covering of $M$ by open sets of diameter less than $\varepsilon^*/(K^*)^2$. We call a Markov covering associated to $\mathcal{U}$, a finite set $\mathcal{R} = \{R_1, \ldots, R_l\}$ of rectangles of diameter less than $\varepsilon^*/K^*$ satisfying:

- $U_i \subset R_i$
- $x \in U_i \cap f^{-1}(U_j) \implies f(W^s(x, R_i)) \subset W^s(f(x), R_j)$
- $y \in f(U_i) \cap U_j \implies f^{-1}(W^u(y, R_j)) \subset W^u(f^{-1}(y), R_i)$
- $\forall j, \exists i, f(U_i) \cap U_j \neq \emptyset$

where $W^s(x, R_i) = W^s_\varepsilon(x) \cap R_i$ and $W^u(y, R_j) = W^u_\varepsilon(y) \cap R_j$.

An easy consequence of the shadowing lemma shows there always exist such Markov coverings:

Proposition 7. — For every covering $\mathcal{U}$ of $M$ by open sets such that the diameter of each $U_i$ is less than $\varepsilon^*/(K^*)^2$, there exists a Markov covering $\mathcal{R}$ by rectangles of diameter less than $\varepsilon^*/K^*$.

Proof. — Given $\mathcal{U} = \{U_1, \ldots, U_l\}$ such a covering, we define the following compact space of $\varepsilon^*/(K^*)^2$ pseudo-orbits:

$$\Sigma = \{\omega = (\ldots, \omega_{-2}, \omega_{-1} | \omega_0, \omega_1, \ldots) \text{ s.t. } U_{\omega_n} \cap f^{-1}(U_{\omega_{n+1}}) \neq \emptyset\}.$$  

Here $\omega$ is a sequence of indices in $\{1, \ldots, l\}$ and $\Sigma$ is a subshift of finite type where $i \rightarrow j$ is a possible transition iff $U_i \cap f^{-1}(U_j)$ is not empty. Given such $\omega \in \Sigma$, we choose for all $n \in \mathbb{Z}$, $x_n \in U_{\omega_n}$ so that $f(x_n) \in U_{\omega_{n+1}}$. Then $(x_n)_{n \in \mathbb{Z}}$ is an $\varepsilon^*/(K^*)^2$ pseudo-orbit which corresponds to a unique true orbit $(f^n(x))_{n \in \mathbb{Z}}$ satisfying:

$$d(f^n(x), U_{\omega_n}) < \varepsilon^*/K^* \quad \forall n \in \mathbb{Z}.$$  

Since $\varepsilon^*$ is a constant of expansiveness, there can exists at most one point $x$ satisfying the previous inequality for all $n$. We call that point $\pi(\omega)$ and notice that the map $\pi : \Sigma \rightarrow M$ is surjective (for $\mathcal{U}$ is a covering), commutes with the left shift $\sigma$, $f \circ \pi = \pi \circ \sigma$, is continuous by expansiveness (in fact Hölder if $\Sigma$ is equipped with the standard metric). Also notice that $\pi$ may not be finite-to-one. We first construct a Markov cover on $\Sigma$ as usual by the bracketing

$$[\omega, \omega'] = (\ldots, \omega'_{-2}, \omega'_{-1} | \omega_0, \omega_1, \ldots)$$

where $\omega = (\omega_n)_{n \in \mathbb{Z}}$, $\omega' = (\omega'_n)_{n \in \mathbb{Z}}$ and $\omega' = \omega_0$. By uniqueness in the construction of $\pi(\omega)$, we get

$$\pi(\omega, \omega') = [\pi(\omega), \pi(\omega')]$$

$\pi([i]) = R_i$ is a rectangle of $M$ containing $U_i$

$$\pi(W^s(\omega, [i])) = W^s(\pi(\omega), R_i) \quad \text{whenever } \omega_0 = i$$
where $[i], i = 1, \ldots, l$, is the cylinder $\{ \omega \in \Sigma | \omega_0 = i \}$ and \( W^s(\omega, [i]) \) is the symbolic stable set $\{ \omega' \in \Sigma | \omega'_n = \omega_n \ \forall \ n \geq 0 \}$. (For the proof of the last equality, we just notice: if $x = \pi(\omega)$, $y \in W^s(x, R_i)$ and $y = \pi(\omega')$ then $\pi([\omega \omega']) = y$ and $[\omega, \omega'] \in W^s(\omega, [i])$.)
To finish the proof we only show
\[
x \in U_i \cap f^{-1}(U_j) \implies f(W^s(x, R_i)) \subset W^s(f(x), R_j).
\]
Indeed, $x = \pi(\omega)$ for some $\omega = (\cdots, \omega_{-1} | i, j, \omega_2, \cdots)$ and
\[
\sigma(W^s(\omega, [i]) \subset W^s(\sigma(\omega), [j]).
\]
To conclude, we apply $\pi$ on both sides. 

**Definition 8.** — Let $\mathcal{R} = \{R_1, \ldots, R_l\}$ be a Markov covering of $M$ associated to some open covering $\mathcal{U} = \{U_1, \ldots, U_l\}$. We define a local sub-action by
\[
V_i(x) = \sup \{ S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \ | \ n \geq 0, \ y \in W^s(x, R_i)\}
\]
for $x \in U_i$, and where $S_nB = \sum_{k=0}^{n-1} B \circ f^k$, $\Delta^s(y, x) = \sum_{k \geq 0}(A \circ f^k(y) - A \circ f^k(x))$ and the supremum is taken over all $n \geq 0$ and points $y \in W^s(x, R_i)$.

Before showing $V_i$ is a (finite!) Hölder function on each $R_i$, let’s conclude the proof of Theorem 1:

**Proof of Theorem 1.** — Let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be an open covering of $M$, $\{R_1, \ldots, R_l\}$ a Markov covering associated to $\mathcal{U}$ and $\{\theta_1, \ldots, \theta_l\}$ a partition of unity adapted to $\mathcal{U}$. Let $\{V_1, \ldots, V_l\}$ constructed as above and
\[
V = \sum_i \theta_i V_i.
\]
Suppose we have proved that $x \in U_i \cap f^{-1}(U_j)$ implies
\[
V_i(x) + (A - m)(x) \leq V_j \circ f(x).
\]
Multiplying this inequality by $\theta_i(x)\theta_j \circ f(x)$ and summing over $i$ and $j$ (whether or not $i \to j$ is a possible transition), we get
\[
V(x) + (A - m)(x) \leq V \circ f(x) \quad (\forall \ x \in M).
\]
We now prove the local sub-cohomological equation: if $x \in U_i \cap f^{-1}(U_j)$ and $y \in W^s(x, R_i)$, then $f(y) \in W^s(f(x), R_j)$ and
\[
S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) + (A - m)(x)
= S_{n+1}(A - m) \circ f^{-(n+1)} \circ f(y) + \Delta^s(f(y), f(x)) \leq V_j \circ f(x).
\]
Taking the supremum over all $n \geq 0$ and all $y \in W^s(x, R_i)$, we get indeed
\[
V_i(x) + (A - m)(x) \leq V_j \circ f(x).
\]
That finishes the proof of theorem 1.
We now come to our main technical lemma. We notice that, even in the case where $A$ is Lipschitz, we only obtain a Hölder sub-action.

**Lemma 9.** — If $A$ is $\alpha$-Hölder on $M$, $R$ is a rectangle and $V$ is defined as in Definition 8, then $V$ is $\beta$-Hölder on $R$ with exponent

$$
\beta = \alpha \frac{|\ln \lambda_n|}{\ln \Lambda_n + |\ln \lambda_n|} < \alpha.
$$

**Proof.** — We divide the proof into four steps:

**Step one.** — If $d(x, x') < \varepsilon^*$ and $x, x'$ are on the same stable leaf, then

$$
\Delta^s(x, x') \leq \sum_{n \geq 0} |A \circ f^n(x) - A \circ f^n(x')| \leq C(M) \frac{\text{Höld}_\alpha(A)}{1 - \lambda_n^\alpha} d(x, x')^{\alpha},
$$

for some constant $C(M)$ depending only on $M$ and the metric.

Indeed, it follows from the contraction $d(f^k(x), f^k(x')) \leq C(M) \lambda_n^k d(x, x')$ for $k \geq 0$ and the fact that $A$ is $\alpha$-Hölder.

**Step two.** — For every $n \geq 1$, $x, x' \in M$ such that $d(f^k(x), f^k(x')) < \varepsilon^*/K^*$ for all $0 \leq k \leq n$, then

$$
\sum_{k=0}^{n-1} |A \circ f^k(x) - A \circ f^k(x')| \leq K(M, f) \max(d(x, x')^\alpha, d(f^n(x), f^n(x'))^\alpha),
$$

where $K(M, f) = C(M) \frac{\text{Höld}_\alpha(A)}{\min(1 - \lambda_n^{-\alpha}, 1 - \lambda_n^\alpha)^2}$.

Indeed, one can build $w = [x, x']$; then on the one hand, $d(x, w) \leq \varepsilon^*$ and $x, w$ are on the same stable leaf; on the other hand, $d(f^n(w), f^n(x')) \leq \varepsilon^*$ and $f^n(w)$ and $f^n(x)$ are on the same unstable leaf. We conclude by applying step one and the estimates:

$$
d(x, w) \leq K^* d(x, x'), \quad d(f^n(w), f^n(x')) \leq K^* d(f^n(x), f^n(x')).
$$

**Step three.** — We show that $V(x)$ is finite for every $x \in R$. It is precisely here that the choice of the normalizing constant $m(A, f)$ is important.

Indeed, since a transitive Anosov diffeomorphism is mixing (see [3]), there exists an integer $\tau^* \geq 1$ such that, for every finite orbit $\{f^{-n}(y), \ldots, f^{-1}(y), y\}$, $n$ arbitrary, $f^{\tau^*}(B(y, \varepsilon^*/K^*))$ contains $f^{-n}(y)$. Thanks to the shadowing lemma, there exists a periodic orbit $z$, of period $n + \tau^*$, satisfying

$$
d(f^{-k}(z), f^{-k}(y)) \leq \varepsilon^* \quad (\forall k = 0, 1, \ldots, n).
$$

Using step two, $\sum_{k=1}^{n} (A \circ f^{-k}(y) - A \circ f^{-k}(z))$ is uniformly bounded in $n$ by some constant $C(M, f)$. As any periodic orbit is associated to an invariant probability, then, $\sum_{k=1}^{n+\tau^*} (A \circ f^{-k}(z) - m(A, f)) \leq 0$. 

SOCIÉTÉ MATHÉMATIQUE DE FRANCE 2003
Without lost of generality we can assume $m(A, f) = 0$. Therefore, we get

$$
\sum_{k=1}^{n} A \circ f^{-k}(y) \leq C(M, f) + \sum_{k=1}^{n+\tau^*} A \circ f^{-k}(z) + \tau^* \|A\|_{\infty}
$$

$$
\leq C(M, f) + \tau^* \|A\|_{\infty}.
$$

Step four. — We finally prove that $V$ is Hölder on $R$. Let $n \geq 0$, $x, x' \in R$, $y \in W^s(x, R)$ and define $y' = [x', y]$ belonging to $R$ since $R$ is a rectangle and to the same local unstable manifold as $y$. Then for some $N$ we are going to choose soon: let $B = A - m(A, f)$,

$$
S_n B \circ f^{-n}(y) + \Delta^s(y, x) \leq S_n B \circ f^{-n}(y') + \Delta^s(y', x')
$$

$$
+ \sum_{k=-n}^{N-1} |A \circ f^k(y) - A \circ f^k(y')| \quad (= \Sigma_1)
$$

$$
+ \sum_{k=0}^{N-1} |A \circ f^k(x) - A \circ f^k(x')| \quad (= \Sigma_2)
$$

$$
+ |\Delta^s(f^N(y), f^N(x))| \quad (= \Sigma_3)
$$

$$
+ |\Delta^s(f^N(y'), f^N(x'))| \quad (= \Sigma_4)
$$

We now bound from above each $\Sigma_i$ with respect to $d(x, x')$:

$$
\Sigma_1 \leq C(M) \frac{\text{Höld}_s(A)}{1 - \lambda_s^\alpha} d(f^N(y), f^N(y'))^\alpha,
$$

$$
\Sigma_2 \leq C(M) \frac{\text{Höld}_s(A)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^\alpha)^2} \max(d(x, x')^\alpha, d(f^N(x), f^N(x'))^\alpha),
$$

$$
\Sigma_3 \leq C(M) \frac{\text{Höld}_s(A)}{1 - \lambda_s^\alpha} d(f^N(y), f^N(x)),
$$

$$
\Sigma_4 \leq C(M) \frac{\text{Höld}_s(A)}{1 - \lambda_s^\alpha} d(f^N(y'), f^N(x'))^\alpha.
$$

We now choose $N = N(x, x')$ by $\lambda_s^\alpha \epsilon^* = \Lambda_u^{\tau} d(x, x')$, $N = [\ell] + 1$ and then choose $\tilde{\epsilon} \geq \epsilon^*$ so that $\lambda_s^N \tilde{\epsilon} = \Lambda_u^{N} d(x, x')$. Then

$$
d(f^N(x), f^N(x')) \leq C(M) \Lambda_u^{N} d(x, x') \leq C(M) \lambda_s^N \tilde{\epsilon},
$$

$$
d(f^N(y), f^N(x)) \text{ or } (f^N(y'), f^N(x')) \leq C(M) \lambda_s^N \epsilon^* \leq C(M) \lambda_s^N \tilde{\epsilon}.
$$

In particular, we get first $d(f^N(y), f^N(y')) \leq 3C(M) \lambda_s^N \tilde{\epsilon}$ and next:

$$
\Sigma_1 + \cdots + \Sigma_4 \leq 6C(M) \frac{\text{Höld}_s(A)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^\alpha)^2} (\lambda_s^N \tilde{\epsilon})^\alpha = K(M, f)(\lambda_s^N \tilde{\epsilon})^\alpha,
$$

$$
S_n B \circ f^{−n}(y) + \Delta^s(y, x) \leq S_n B \circ f^{−n}(y') + \Delta^s(y', x') + K(M, f)(\lambda_s^N \tilde{\epsilon})^\alpha,
$$

$$
V(x) \leq V(x') + K(M, f)(\lambda_s^N \tilde{\epsilon})^\alpha.
$$
But

\[ \lambda_s^N \tilde{\varepsilon} = d(x, x')^{\ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)}. \]

**Remark 10.** — We have not used explicitly the fact that the stable foliation $W^s$ is Hölder but our proof (step four) is close to showing $W^s$ is Hölder of exponent $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

**Proof.** — We show that if $\varepsilon < \varepsilon^*/K^*$, $d(x, x') \leq \varepsilon$, $y \in W^s_\varepsilon(x)$, $y' \in W^s_\varepsilon(x')$ and $y \in W^s_{\varepsilon^N}(y')$ then

\[ d(y, y') \leq 3C(M)^2 d(x, x')^{\gamma} \]

where $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Indeed we choose $t > 0$ real such that $\lambda^N_s \tilde{\varepsilon} = \Lambda^t_u d(x, x')$, $N = [t] + 1$, and $\tilde{\varepsilon}$ close to $\varepsilon$ so that $\lambda^N_s \tilde{\varepsilon} = \Lambda^N_u d(x, x')$ where $\tilde{\varepsilon}/\varepsilon$ varies between 1 and $\Lambda_u/\lambda_s$. Then

\[ d(f^N(x), f^N(y)) \text{ or } d(f^N(x'), f^N(y')) \text{ or } d(f^N(x), f^N(x')) \leq C(M) \lambda^N_s \tilde{\varepsilon}, \]

\[ d(f^N(y), f^N(y')) \leq 3C(M) \lambda^N_s \tilde{\varepsilon}, \]

\[ d(y, y') \leq 3C(M)^2 (\lambda_u/\lambda_s)^N \tilde{\varepsilon} = 3C(M)^2 d(x, x')^{\gamma}. \]

\[ \square \]

**3. Maximizing periodic measures**

The proof of Theorem 4 requires two ingredients: the first one is the notion of sub-actions we have already studied, the second is the notion of strongly non-wandering points we are going to explain.

**Definition 11.** — Given $A \in C^2(M)$ and $m = m(A, f)$, a point $x \in M$ is said to be strongly non-wandering with respect to $A$, if for any $\varepsilon > 0$, there exist $n \geq 1$ and $y \in M$ such that

\[ y \in B(x, \varepsilon), \quad f^n(y) \in B(x, \varepsilon) \quad \text{and} \quad \left| \sum_{k=0}^{n-1} (A - m) \circ f^k(y) \right| < \varepsilon \]

where $B(x, \varepsilon)$ denotes the ball centered at $x$ and radius $\varepsilon$. We call $\Omega(A, f)$ the set of strongly non-wandering points.

The first non-trivial but easy observation is that $\Omega(A, f)$ is non-empty; more precisely:

**Lemma 12.** — The set $\Omega(A, f)$ is compact forward and backward $f$-invariant and contains the support of any maximizing measure.

**Proof.** — If $\mu$ is maximizing, by Atkinson’s theorem [1], for almost $\mu$-point $x$, the Birkhoff’s sums $\sum_{k=0}^{n-1} (A - m) \circ f^k$ are recurrent (in the sense of random walk theory)
to $\int (A - m) \, d\mu = 0$: that is, for any Borel set $B$ of positive $\mu$-measure and for any $\varepsilon > 0$, the set

$$\left\{ x \in B \mid \exists n \geq 1 \ f^n(x) \in B \text{ and } |\sum_{k=0}^{n-1} (A - m) \circ f^k(x)| < \varepsilon \right\}$$

has positive $\mu$-measure. Since by definition of the support of a measure, any ball $B(x, \varepsilon)$ has positive $\mu$-measure, we have proved that $\text{supp}(\mu)$ is included in $\Omega(A, f)$. \hfill $\Box$

The second observation is that any Hölder function $A$ is cohomologous to $m(A, f)$ on $\Omega(A, f)$, more precisely:

**Lemma 13.** — Let $A$ be a $C^0$-function and assume $A$ admits a $C^0$ sub-action $V$, then

$$\Omega(A, f) \subseteq \Sigma_V(A, f) = \{ x \in M \mid A - m = V \circ f - V \}$$

and any $f$-invariant measure $\mu$ whose support is contained in $\Omega(A, f)$ is maximizing.

The set $\Sigma_V(A, f)$ will play an important role later and it is convenient to give it a name:

**Definition 14.** — Let $A$ be a $C^0$-function and $V$ be a sub-action of $A$.

(i) We call the set $\Sigma_V(A, f) = \{ x \in M \mid A - m = V \circ f - V \}$, the $V$-action-set of $A$.

(ii) Two points $x$, $y$ of the $V$-action-set are said to be $V$-connected and we shall write $x \xrightarrow{V} y$, if for every $\varepsilon > 0$, there exist $n \geq 1$ and $z \in M$ (not necessarily in $\Sigma_V(A, f)$) such that

$$x \in B(z, \varepsilon), \quad y \in B(f^n(z), \varepsilon), \quad |S_n(A - m)(z) - (V(y) - V(x))| < \varepsilon.$$

Notice that, if $V$ is $\beta$-Hölder for some $\beta > 0$, using the shadowing lemma, one can prove that $x \xrightarrow{V} y$ and $y \xrightarrow{V} u$ imply $x \xrightarrow{V} u$. This is so, because if $z_x$ and $n_x$ are the ones for $x \xrightarrow{V} y$ in (ii) above, and if $z_y$ and $n_y$ are the ones for $y \xrightarrow{V} u$ in (ii) above, then considering the pseudo-orbit $z_x, \ldots, f^{n_x}(z_x), z_y, \ldots, f^{n_y}(z_y)$, we can find by shadowing the $z$ for $x \xrightarrow{V} u$ in (ii) above.

**Proof of Lemma 13.** — Define $R = V \circ f - V - A + m$ and choose $x \in \Omega(A, f)$. Then

$$\sum_{k=0}^{n_i-1} (A - m) \circ f^k(y_i)$$

converges to $0$ for a sequence of points $y_i$ and a sequence of integers $n_i$ such that $y_i$ converges to $x$, $n_i$ converges to $+\infty$ and $f^{n_i}(y_i)$ converges to $x$. Since $R$ is non-negative,

$$0 \leq R(y_i) \leq \sum_{k=0}^{n_i-1} R \circ f^k(y_i) = V \circ f^{n_i}(y_i) - V(y_i) - \sum_{k=0}^{n_i-1} (A - m) \circ f^k(y_i)$$

converges to $0$ and by continuity of $R$: $R(x) = 0$. \hfill $\Box$

**Definition 15.** — For any $\beta > 0$, define

$$G_\beta = \{ A \in C^\beta(M) \mid \Omega(A, f) \text{ is a periodic orbit} \}.$$
Our next goal is to show that \( G_\beta \) is open in the \( C^3 \) topology. We could have chosen a bigger set: the set of \( A \) in \( C^3(M) \) such that \( \Omega(A, f) \) is minimal and is dynamically isolated (i.e. there exists \( U \), open, containing \( \Omega(A, f) \) as the only \( f \)-invariant compact set inside \( U \)) and the proof below would again be the same.

**Lemma 16.** — For any \( \beta > 0 \), \( G_\beta \) is open in the \( C^3 \) topology and \( \Omega(A, f) \) is locally constant as a function of \( A \) in \( G_\beta \).

**Proof.** — Let \( A \in G_\beta \). We want to show that \( \Omega(A, f) = \Omega(B, f) \) whenever \( B \) is sufficiently close to \( A \) in the \( C^3 \) topology. By contradiction: let \( U \) be an isolating open set of the periodic orbit \( \Omega(A, f) = \text{orb}(p) \) and \( \{A_n\} \) be a sequence of \( \beta \)-Hölder observables converging to \( A \) in the \( C^3 \) topology such that \( \Omega(A, f) \) is not included in \( U \) for each \( n \).

Each \( A_n \) admits (Theorem 1) a \( \gamma \)-Hölder subaction \( V_n \) with \( \gamma \)-Hölder norm uniformly bounded and \( \gamma = \beta \ln(1/\lambda_s)/\ln(A_n/\lambda_s) \). By Ascoli, \( \{V_n\} \) admits a subsequence converging in the \( C^0 \) topology to some \( \gamma \)-Hölder function \( V \). Since the set of non-empty compact sets is compact with respect to the Hausdorff topology, we may assume that \( \{\Omega(A_n, f)\} \) has a sub-sequence converging to some compact invariant set \( K \). Each \( A_n \) satisfies:

\[
A_n - m(A_n, f) \leq V_n \circ f - V_n \quad (\forall x \in M),
\]

\[
A_n - m(A_n, f) = V_n \circ f - V_n \quad (\forall x \in \Omega(A_n, f)).
\]

By continuity of \( m(A, f) \) with respect to \( A \) (for the \( C^0 \) topology),

\[
A - m(A, f) \leq V \circ f - V \quad (\forall x \in M)
\]

\[
A - m(A, f) = V \circ f - V \quad (\forall x \in K).
\]

We have assumed that each \( \Omega(A_n, f) \setminus U \) is not empty, then \( K \setminus U \) is not empty too. Let \( x_0 \in K \setminus U \), the \( \omega \)-limit set \( \omega(x_0) \) and the \( \alpha \)-limit set \( \alpha(x_0) \) of \( x_0 \) are compact invariant sets included in \( \Omega(A, f) \), necessarily:

\[
\omega(x_0) = \alpha(x_0) = \text{orb}(p) \subset \overline{\text{orb}(x_0)} \subset \Sigma_V(A, f).
\]

Since \( p \) is \( V \)-connected to \( x_0 \) and \( x_0 \) is \( V \)-connected to \( p \), \( x_0 \) is \( V \)-connected to itself which is equivalent to \( x_0 \in \Omega(A, f) \). We just have obtained a contradiction. \( \square \)

**Proof of Theorem 4.** — Let \( \beta \) given and \( A, \alpha \)-Hölder with:

\[
\beta < \tilde{\beta} = \alpha \frac{\ln(1/\lambda_s)}{\ln(A_n/\lambda_s)}.
\]

According to Theorem 1, there exists \( V, \tilde{\beta} \)-Hölder, satisfying:

\[
A - m \leq V \circ f - V \quad (\forall x \in M).
\]
Define \( R = V \circ f - V - A + m, \phi_n = \min(R, 1/n) \) and \( B_n = A + \phi_n \). Then \( \phi_n \) is \( \tilde{\beta} \)-Hölder with \( \text{Höld}_{\tilde{\beta}}(\phi_n) \leq \text{Höld}_{\tilde{\beta}}(R) \) and

\[
A - m \leq B_n - m \leq V \circ f - V \quad (\forall x \in M)
\]

\[
B_n - m = V \circ f - V \quad (\forall x \in \{R < 1/n\}).
\]

In particular \( m(B_n, f) = m(A, f) \) and the \( V \)-action set of \( B_n \) contains a neighborhood \( \{R < 1/n\} \) of \( \Omega(A, f) \). Using the shadowing lemma, we construct a periodic orbit \( \text{orb}(p) \) inside \( \{R < 1/n\} \) and we just have proved a perturbation \( B_n \) of \( A \) satisfies

\[
\text{orb}(p) \cup \Omega(A, f) \subset \Omega(B_n, f).
\]

Let \( \psi_n \) be any \( \tilde{\beta} \)-Hölder function with small \( \tilde{\beta} \)-Hölder norm satisfying:

\[
\begin{align*}
\psi_n(x) &= 0 \quad (\forall x \in \text{orb}(p)) \\
\psi_n(x) &> 0 \quad (\forall x \in M \setminus \text{orb}(p)).
\end{align*}
\]

Then \( A_n = B_n - \psi_n = A + \phi_n - \psi_n \) is such that the minimizing measure has support on \( \text{orb}(p) \), and \( A_n \) has small \( C^0 \) norm and (possibly large) uniform \( \tilde{\beta} \)-Hölder norm. Therefore \( (A_n) \) converges to \( A \) in the \( C^\beta \)-topology and each \( A_n \) has a unique maximizing measure which is supported on a periodic orbit. \( \Box \)

References


