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AN ANALOG OF FREIMAN'S THEOREM IN GROUPS

by

Imre Z. Ruzsa

Abstract. — It is proved that in a commutative group G, where the order of elements is bounded by an integer r, any set A having n elements and at most αn sums is contained in a subgroup of size Cn with $C = f(r, \alpha)$ depending on r and α but not on n. This is an analog of a theorem of G. Freiman which describes the structure of such sets in the group of integers.

Let A be a set of integers, $|A| = n$, and suppose that $|A + A| \leq cn$. A famous theorem of Freiman [1, 2] provides a certain structural description of these sets; in one of the possible formulations, it says that A can be covered by a generalized arithmetic progression

$$\{a + q_1x_1 + q_2x_2 + \cdots + q_dx_d : 0 \leq x_i \leq l_i - 1\},$$

where $d < c$ and $\prod l_i \leq Cn$ with C depending on c.

One can ask for a description of sets with few sums in every Abelian group. In this paper we consider groups which are in a sense very far from $\mathbb{N}$.

Theorem. — *Let $r \geq 2$ be an integer, and let G be a commutative group in which the order of every element is at most r. Let $A \subset G$ be a finite set, $|A| = n$. If there is another $B \subset G$ such that $|B| = n$ and $|A + B| \leq \alpha n$ (in particular, if $|A + A| \leq \alpha n$ or $|A - A| \leq \alpha n$), then A is contained in a subgroup H of G such that*

$$|H| \leq f(r, \alpha)n,$$

where

$$f(r, \alpha) = \alpha^2 r^{\alpha^4}.$$

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The proof goes along similar lines to my proof of Freiman's theorem [3, 4], but is considerably simpler.

For a nonnegative integer k and a set $A \subset G$ we introduce the notation

$$kA = A + \cdots + A, \quad k \text{ summands},$$

$$0A = \{0\}, \quad 1A = A.$$

Lemma. — *If $A, B \subset G$, $|B| = n$ and $|A + B| \le \alpha n$, then for arbitrary nonnegative integers k, l we have*

$$|kA - lA| \le \alpha^{k+l} n.$$

See [3], Lemma 3.3. Observe the asymmetric role of A and B. No a priori bound is assumed for $|A|$; an alternative formulation (like in the Theorem) would be "if A is such that the union of n suitable translations has at most αn elements, then A is so small that even the sets $kA - lA$ are small".

Proof of the Theorem. Let $b_1, b_2, \ldots, b_k$ be a maximal collection of elements such that $b_i \in 2A - A$ and the sets $b_i - A$ are all disjoint. We have

$$b_i - A \subset 2A - 2A,$$

hence

$$\Big| \bigcup (b_i - A) \Big| = kn \le |2A - 2A| \le \alpha^4 n$$

(the last inequality follows from the Lemma). This implies $k \le \alpha^4$.

Take an arbitrary $x \in 2A - A$. Since the collection $b_1, \ldots, b_k$ was maximal, there must be an i such that

$$(x - A) \cap (b_i - A) \ne \varnothing,$$

that is, $x - a_1 = b_i - a_2$ with some $a_1, a_2 \in A$, which means

$$x = b_i + a_1 - a_2 \in b_i + (A - A).$$

Hence

$$2A - A \subset \bigcup (b_i + (A - A)) = B + A - A, \tag{1}$$

where $B = \{b_1, \ldots, b_k\}$.

Now we prove

$$jA - A \subset (j-1)B + A - A \quad (j \ge 2) \tag{2}$$

by induction on j. By (1), this holds for $j = 2$. Now we have

$$\begin{aligned}(j+1)A - A &= (2A - A) + (j-1)A \\ &\subset B + A - A + (j-1)A \text{ by (1)} \\ &= B + (jA - A) \\ &\subset B + (j-1)B + A - A \\ &= jB + A - A,\end{aligned}$$

which provides the inductive step.

Let H and I be the subgroups generated by A and B, respectively. By (2) we have

$$jA - A \subset I + (A - A) \tag{3}$$

for every j. We have also

$$\bigcup (jA - A) = H, \tag{4}$$

which easily follows from the fact that the order of elements of G is bounded. Relations (3) and (4) imply that

$$H \subset I + (A - A).$$

Since I is generated by k elements of order $\leq r$ each, we have

$$|I| \leq r^k \leq r^{\alpha^4},$$

consequently

$$|H| \leq |I||A - A| \leq \alpha^2 r^{\alpha^4} n$$

(the estimate for $|A - A|$ follows from the Lemma). QED

Remarks. — Take a group of the form $G = Z_r^m$, where Z_r is a cyclic group of order r, and a set $A \subset G$ of the form

$$A = (a_1 + G') \cup \cdots \cup (a_k + G')$$

with a subgroup G'. Here $|A| = n = k|G'|$, and if all the sums $a_i + a_j$ lie in different cosets of G', then

$$|A + A| = \frac{k(k+1)}{2}|G'| = \alpha n, \quad \alpha = \frac{k+1}{2}.$$

The subgroup generated by A can have as many as $r^k|G'|$ elements, hence our function

$$f(r, \alpha) = \alpha^2 r^{\alpha^4}$$

cannot be replaced by anything smaller than

$$r^k = r^{2\alpha - 1}.$$

Conjecture. — *The Theorem holds with $f(r, \alpha) = r^{C\alpha}$ with a suitable constant C.*

The following conjecture of Katalin Marton would yield a more efficient covering in a slightly different form.

Conjecture. — *If $|A| = n$, $|A + A| \leq \alpha n$, then there is a subgroup H of G such that $|H| \leq n$ and A is contained in the union of α^c cosets of H, where the constant c may depend on r but not on n or α.*

This also suggests that perhaps in Freiman's original problem a better result can be formulated in terms of covering by a small number of generalized arithmetical progressions than just one.

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