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Condenser Potentials

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Abstract. — Under appropriate hypotheses, the potential theory of a transient Markov process can be recovered from the condenser charges.

A central object in both the theory of Markov processes and in potential theory is the cone of excessive or superharmonic functions. This cone provides a critical link between the two subjects, and many important and useful theorems in Markov processes aim at a deeper understanding of the cone and its properties. Various subsets of this cone have been studied, also. For example, the collection of hitting probabilities proves to contain enough information to recover the potential theory of the process [5,6]. In this article, we suggest that another important link between Markov processes and potential theory is forged by the collection of condenser potentials and the associated collection of condenser charges. Condenser potentials receive little attention even in comprehensive tomes on potential theory. The condenser theorem for classical potential theory in $\mathbb{R}^d$ is the following and has a standard extension in the theory of Dirichlet spaces [9].

Theorem. *Let K and L be open sets with disjoint closures $\overline{K}$ and $\overline{L}$, and assume that $\overline{K}$ is compact. Then there exists a potential p of a signed measure μ such that:*
(i) $0 \leq p \leq 1$ *a.e. on* $\mathbb{R}^d$.
(ii) $p = 0$ *a.e. on* L *and* $p = 1$ *a.e. on* K.
(iii) The support of μ^+ *is contained in* $\overline{K}$ *and the support of* μ^- *is contained in* $\overline{L}$.

The potential p is in fact unique in $\mathbb{R}^d$ and uniqueness holds also in Dirichlet spaces.

We are aware of only one probabilistic study of condenser potentials, that being the 1977 note by K. L. Chung and R. K. Getoor [4]. They "guessed" that the condenser potential is simply the probability starting at x that Brownian motion hits K before it hits L. In their article, they deal with a Hunt process on a locally compact state space satisfying the duality assumptions in [1]. Throughout this article, we adopt the same assumptions: $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a Hunt process on a locally compact

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state space E satisfying the duality assumptions in Section VI–1 of [1]. In particular, $u(x,y)$ will denote the potential density and $U\gamma$ will denote the potential of a measure γ. For K and L Borel sets in E with disjoint closures, we define $p(x) = P^x[T_K < T_L]$ to be the condenser potential of the pair (K,L). Define $p_n = (P_K P_L)^n P_K 1$. Chung and Getoor's final result may be stated as follows.

Theorem. *If $\sum_n p_n$ converges, then $p = U\mu$ is the condenser potential of (K,L), where the measure μ is obtained as follows:*

(i) μ^+ is the capacitary measure μ_K of K relative to (X, T_L), the process X killed when it hits L.

(ii) $\mu^- = \widehat{P}_L \mu_K$ is the co–balayage onto L of μ_K.

Chung and Getoor also investigate several conditions guaranteeing $\sum_n p_n$ converges.

In this article, we investigate what rôle the condenser potential p and the condenser measure μ can play in potential theory, now that they have interesting probabilistic interpretations. We begin by characterizing the condenser potential in the symmetric case in a style akin to Hunt's balayage theorem. Our real interest in this article lies in studying the non–symmetric case, so we present this result in passing to help the reader with intuition about condenser potentials. Let $\mathcal{K}$ denote the collection of probability measures on $\overline{K}$, and let $\mathcal{L}$ denote the collection of positive measures on $\overline{L}$. Recall that the mutual energy of two measures λ and ρ is defined by

$$\ll \lambda, \rho \gg = \int_E \int_E u(x,y)\lambda(dx)\rho(dy)$$

It is by now a standard exercise to extend this to be an inner product on the space of signed measures π with $\ll |\pi|, |\pi| \gg < \infty$. This space is a pre–Hilbert space. We shall denote the norm of π by $||\pi||$.

Theorem. *Assume that $u(x,y) = u(y,x)$. Assume also that every potential $U\gamma$ of a measure γ is lower semicontinuous on E and continuous off the support of γ. Let $\overline{K}$ and $\overline{L}$ be compact and disjoint. The unique measure $\gamma - \nu$ which minimizes*

$$\inf_{\gamma \in \mathcal{K}, \nu \in \mathcal{L}} ||\gamma - \nu||$$

is a constant times the condenser measure μ.

Proof. We show first that the inf above is achieved by measures $\rho \in \mathcal{K}$ and $P_L\rho \in \mathcal{L}$. Choose $(\rho_n) \subset \mathcal{K}$ and $(\pi_n) \subset \mathcal{L}$ such that the sequence $||\rho_n - \pi_n||$ converges to $e = \inf_{\gamma \in \mathcal{K}, \nu \in \mathcal{L}} ||\gamma - \nu||$. Then, by reducing to a subsequence if necessary, ρ_n converges weakly to a measure $\rho \in \mathcal{K}$ and π_n converges weakly to a measure $\pi \in \mathcal{L}$. A standard Dirichlet space computation yields $||\rho - \pi|| \le \liminf_{n\to\infty} ||\rho_n - \pi_n|| = e$. But $||\rho - \pi|| \ge ||\rho - P_L\rho||$ since $P_L\rho$ is the unique measure in $\mathcal{L}$ minimizing the distance between ρ and $\mathcal{L}$. Thus $e = ||\rho - P_L\rho||$ with $\rho \in \mathcal{K}$ and $P_L\rho \in \mathcal{L}$.

Now let $\xi \in \mathcal{K}$ and $\beta \in \mathcal{L}$ be any pair of measures such that $||\xi - \beta|| = e$. Note that $\beta = P_L\xi$ since $P_L\xi$ is the unique measure in $\mathcal{L}$ minimizing the distance between ξ and $\mathcal{L}$. Take another measure $\lambda \in \mathcal{K}$ of finite energy, and let $t > 0$. Then

$$||t(\lambda - \xi) + \xi - \beta|| = ||(1-t)\xi + t\lambda - \beta|| \ge ||\xi - \beta||$$

Thus

$$t^2||\lambda - \xi||^2 + 2t \ll \lambda - \xi, \xi - \beta \gg \geq 0$$

for every $t > 0$, and we conclude that

$$\ll \lambda - \xi, \xi - \beta \gg \geq 0$$

That is,

$$\int U(\xi - \beta) d(\lambda - \xi) \geq 0$$

so

$$\int U(\xi - \beta) d\lambda \geq \int U(\xi - \beta) d\xi = \int U(\xi - \beta) d(\xi - \beta) = e^2$$

(the first equality holding since $U(\xi - \beta) = 0$ on L). Since $\lambda \in \mathcal{K}$ is arbitrary, $U(\xi - \beta) \geq e^2$ on K, except perhaps on a set of capacity zero. Taking $\lambda = \xi$, we get

$$\int U(\xi - \beta) d\xi = \int U(\xi - \beta) d(\xi - \beta) = e^2$$

so $U(\xi - \beta) = e^2$ a.e. ξ on K. Since $U\xi \leq U\beta + e^2$ a.e. ξ, we have $U\xi \leq U\beta + e^2$ on E by the maximum principle. To summarize, $U(\xi - \beta) \leq e^2$, $U(\xi - \beta) = e^2$ a.e. ξ on K, $\beta = P_L\xi$ and $U(\xi - \beta) = 0$ on L. By uniqueness of condenser potentials, $\xi = \rho$ and $\beta = P_L\rho$. $\square$

In [7], Glover, Hansen and Rao observed that the potential theory of a symmetric process can be reconstructed from the capacities. This result can also be found in Choquet [3], at least in the case where points are polar. In the case where the process hits points, Glover, Hansen and Rao proved the following formula, which will be useful for the purposes of comparison later.

Theorem. *Assume that $u(z,w) = u(w,z)$ and $P^z(T_{\{z\}} < \infty) > 0$ for all z and w in E. Fix x and y in E, let a be the capacity of $\{x\}$, let b be the capacity of $\{y\}$, and let c be the capacity of the set $\{x,y\}$. Then*

$$u(x,y) = \frac{1 - \sqrt{1 - c\left(\frac{1}{a} + \frac{1}{b} - \frac{c}{ab}\right)}}{c}$$

If $a = b$, then

$$u(x,y) = \frac{2}{c} - \frac{1}{a}$$

Symmetry is needed in the theorem above: one cannot recover the potential theory of a nonsymmetric process from the capacities, in general. However, one can recover neatly the potential theory of a nonsymmetric process from the condenser charges, as follows.

Definition. *Let $U\mu$ be the condenser potential of the sets (K, L). The condenser charge associated with the sets (K, L) is defined to be $\mu(E)$, and will be denoted by $c(K, L)$.*

Under the hypotheses of Chung and Getoor's theorem, $c(K, L) > 0$, and, for fixed L, the map $K \to c(K, L)$ is a capacity which is alternating of order infinity, since it is simply the capacity of the process X killed the first time it hits L. We will need some notation to state the next result. For $x \neq y$, let $c_{xy} = c(\{x\}, \{y\})$ and $c_{yx} = c(\{y\}, \{x\})$, let $a = c(\{x\}, \emptyset)$ and $b = c(\{y\}, \emptyset)$.

Theorem. *Assume that $P^z(T_{\{z\}} = 0) = 1$ for all $z \in E$. Then $u(x, x) = a^{-1}$.*

- *If $c_{xy} = 0$, then $u(y, x) = b^{-1}$ and $u(x, y) = 0$.*
- *If $c_{yx} = 0$, then $u(x, y) = a^{-1}$ and $u(y, x) = 0$.*
- *If $c_{xy} \neq 0$ and $c_{yx} \neq 0$, then*

$$u(y, x) = \frac{1}{c_{yx}} - \frac{c_{xy}}{ac_{yx}}$$
$$u(x, y) = \frac{1}{c_{xy}} - \frac{c_{yx}}{bc_{xy}}$$

Proof. $u_{xx} = a^{-1}$, so u_{xx} can be determined from condenser charges. If $x \neq y$, let $\mu = \mu_{\{x\},\{y\}}$, $\mu_x = \mu(\{x\})$ and $\mu_y = \mu(\{y\})$. Then

$$\begin{aligned} u_{xx}\mu_x + u_{xy}\mu_y &= 1 \\ u_{yx}\mu_x + u_{yy}\mu_y &= 0 \end{aligned} \tag{1}$$

If the determinant $D = u_{xx}u_{yy} - u_{xy}u_{yx} = 0$, then $u_{xx} = u_{yy} = u_{xy} = u_{yx}$ since the maximum principle guarantees $u_{xx} \geq u_{xy}$, $u_{yy} \geq u_{yx}$, $u_{xx} \geq u_{yx}$, and $u_{yy} \geq u_{xy}$. But this would imply that the restriction of u to $\{x, y\} \times \{x, y\}$ is not the potential density of a transient two-state Markov chain, which would be a contradiction. So $D > 0$. Then $\mu_x = u_{yy}/D$ and $\mu_y = -u_{yx}/D$, so $c_{xy} = (u_{yy} - u_{yx})/D$. Similarly, $c_{yx} = (u_{xx} - u_{xy})/D$. If $c_{xy} = 0$, then $u_{yx} = u_{yy} = b^{-1}$ and $c_{yx} = b$. By Chung and Getoor's result [4], $c_{yx} = b - b\widehat{P}_{\{x\}}(y, \{x\})$, and we conclude that $\widehat{P}_{\{x\}}(y, \cdot) = 0$. By time reversal, it follows that $P^x(T_y < \infty) = 0$ which implies $u(x, y) = 0$.

If $c_{xy} \neq 0$ and $c_{yx} \neq 0$, then $(u_{yy} - u_{yx})/c_{xy} = (u_{xx} - u_{xy})/c_{yx}$. Solving for u_{xy} and substituting into equation (1), we obtain a quadratic in u_{yx} with solution

$$u_{yx} = \frac{\left(1 + \frac{c_{yx}}{b} - \frac{c_{xy}}{a} \pm \sqrt{\left(\frac{c_{xy}}{a} - 1 - \frac{c_{yx}}{b}\right)^2 - 4c_{yx}\left(\frac{1}{b} - \frac{c_{xy}}{ab}\right)}\right)}{2c_{yx}}$$
$$= \frac{1 + \frac{c_{yx}}{b} - \frac{c_{xy}}{a} \pm \left|1 - \left(\frac{c_{xy}}{a} + \frac{c_{yx}}{b}\right)\right|}{2c_{yx}}$$

Similarly,

$$u_{xy} = \frac{1 + \frac{c_{xy}}{a} - \frac{c_{yx}}{b} \pm \left|1 - \left(\frac{c_{xy}}{a} + \frac{c_{yx}}{b}\right)\right|}{2c_{xy}}$$

So we have two possibilities at the moment for u_{yx}: either $u_{yx} = 1/c_{yx} - c_{xy}/ac_{yx}$ or $u_{yx} = 1/b = u_{yy}$. However, in this second case, $U\mu(y) = 0$ would force c_{xy} to be zero, contradicting our assumption that $c_{xy} > 0$. Thus we conclude that

$$u_{yx} = \frac{1}{c_{yx}} - \frac{c_{xy}}{ac_{yx}}$$
$$u_{xy} = \frac{1}{c_{xy}} - \frac{c_{yx}}{bc_{xy}}$$

□

So the previous theorem shows that, if points are regular for X, then the potential density can be explicitly recovered from the condenser charges. We now turn our attention to the case where points are polar. The condenser charges still determine the potential theory of the process, at least if the process is a diffusion. Following the theorem, we discuss an extension to the case of discontinuous processes.

Recall that we are always dealing with a transient Hunt process X in duality with another transient Hunt process $\widehat{X}$ with respect to a duality measure ξ as described in VI–1 of [1]. Let $E_\Delta = E \cup \{\Delta\}$ be the one point compactification of the LCCB state space E, and let C_0 denote the collection of bounded functions on E_Δ vanishing at Δ. If L is any Borel set, let $H_L f(x) = E^x[f(X(T_L)); T_L < \infty]$. If K and L are two Borel sets, let $p_n(K, L) = (P_K P_L)^n P_K 1$. If $x \in E$, let $B(r, x)$ denote the closed ball of radius r around x. Let $\mathcal{Z}$ be the set consisting of all pairs (x, L) with $x \in L^c$ and L compact, having the property: for some $r > 0$, $P^y[T_{B(s,x)} < T_L] > 0$ for every $y \in L^c$ and for every $s < r$.

Theorem. *Assume*

(i) $X_{\zeta-} = \Delta$ *and* $\widehat{X}_{\zeta-} = \Delta$ *a.s.*

(ii) $\sum p_n(K, L) < \infty$ *whenever* K *and* L *are closed disjoint balls.*

(iii) $H_L f \in C_0$ *for every* $f \in C_0$ *and every closed ball* L.

Then the hitting distributions $H_L(x, \cdot)$ *can be recovered from the condenser charges for every pair* $(x, L) \in \mathcal{Z}$.

Proof. Fix two closed balls L and M with $L \subset M^\circ$. Since $K \to c(K, L)$ is a capacity, by Choquet's theorem [10], we can construct a unique measure Q_M on $\mathcal{K}(M)$, the closed subsets of M, which is characterized by

$$Q_M[\{A \in \mathcal{K}(M) : A \cap K \neq \emptyset\}] = c(K, L)$$

for every closed subset K of M. To identify this measure Q_M in terms of X, we define $\Gamma : \Omega \to \mathcal{K}(M)$ by letting $\Gamma(\omega)$ be the closure of $\{X_t(\omega) : 0 < t < T_L(\omega)\} \cap M$. Let

η be the coequilibrium measure of M, so $\eta\widehat{U} = 1$ on M°: the coequilibrium measure exists since M is compact and $\widehat{X}$ is transient. Then

$$c(K,L) = P^\eta[T_K < T_L] = P^\eta[\Gamma^{-1}\{A \in \mathcal{K}(M) : A \cap K \neq \emptyset\}]$$

By the uniqueness portion of Choquet's theorem, $Q_M = \Gamma(P^\eta)$. Let $C \subset \partial L$. Thus from the condenser charges, we can calculate

$$Q_M[A \cap K \neq \emptyset, A \cap C \neq \emptyset] = P^\eta[X(T_L) \in C; T_K < T_L < \infty]$$

and $Q_M[A \cap K \neq \emptyset] = P^\eta[T_K < T_L]$. It follows that we can calculate

$$\begin{aligned} P^\eta[f(X(T_L)); T_K < T_L < \infty] &= P^\eta[P^{X(T_K)}[f(X(T_L)); T_L < \infty]; T_K < T_L] \\ &= P^\eta[H_L f(X(T_K)); T_K < T_L] \end{aligned}$$

for any bounded continuous function f. If $(x, L) \in \mathcal{Z}$, we can take $K = B(s,x)$, so $P^\eta[T_{B(s,x)} < T_L] > 0$, and we can compute

$$\frac{P^\eta[H_L f(X(T_K)); T_K < T_L]}{P^\eta[T_K < T_L]}$$

As s decreases to zero, $B(s,x)$ decreases to $\{x\}$, and the quotient converges to $H_L f(x)$. □

The set $\mathcal{Z}$ is used in the proof above simply to insure that we do not divide by zero. The same proof applies for any pair (x, L) provided $P^\eta[T_K < T_L] \neq 0$.

There is a large literature on the construction of Markov processes from hitting distributions which covers the present situation. See [2, 8, 11], and for the case of discontinuous processes, [13].

The previous theorem works for continuous processes since $X(T_L) \in C \subset \partial L$ if and only if $\overline{\{X_t : 0 < t < T_L\}} \cap C \neq \emptyset$. In the case where X_t is only right continuous, $X(T_L)$ will not be in ∂L, in general, and the point $\{X(T_L)\}$ will not be in $\overline{\{X_t : 0 < t < T_L\}}$. However, if we use a modified condenser charge, we can obtain a similar theorem. Define $\kappa(K,L) = P^\eta(T_K \leq T_L)$. If one knows κ, then one can recover $H_L f$ even for discontinuous processes, since, in this case, $Q_M(\{A \in \mathcal{K}(M) : A \cap K \neq \emptyset\}) = P^\eta(\Lambda^{-1}\{A \in \mathcal{K}(M) : A \cap K \neq \emptyset\})$, where $\Lambda(\omega)$ is the closure of $\{X_t(\omega) : 0 < t \leq T_L\} \cap M$. One could then calculate $P^\eta(T_K \leq T_L, X(T_L) \in C) = Q_M(A \cap K \neq \emptyset, A \cap L \neq \emptyset)$.

While we have concentrated most of our attention on condenser charges, a tremendous amount of well-organized information is contained in the modified condenser potentials discussed in the last paragraph. Here, for example, is a simple necessary and sufficient condition for the hitting distributions of X to dominate those of another Hunt process $Y = (\Omega, \mathcal{G}, \mathcal{G}_t, Y_t, \theta_t, Q^x)$.

Theorem. *Suppose*

$$P^x(T_A \le T_C) \ge Q^x(T_A \le T_C) \tag{2}$$

for all disjoint Borel sets A and C and for every $x \in E$. Then

$$P^x(X(T_B) \in dy) \ge Q^x(Y(T_B) \in dy) \tag{3}$$

for every Borel set B and for every $x \in E$. Conversely, if (3) holds, then (2) holds.

Proof. Fix $x \notin B$, and suppose there exists a compact Borel set $A \subset B$ with $P^x(X(T_B) \in A) < Q^x(Y(T_B) \in A)$. Since $A \subset B$ is compact, $\{X(T_B) \in A\} = \{T_A \le T_{B-A}\}$ a.s., so $P^x(T_A \le T_{B-A}) = P^x(X(T_B) \in A) < Q^x(Y(T_B) \in A) = Q^x(T_A \le T_B)$, a contradiction. The converse follows by applying Shih's theorem [12] and Choquet's theorem. If (3) holds, then Y can be obtained by killing X, so the range of Y is contained in the range of X. □

REFERENCES

1. R.M. Blumenthal and R.K. Getoor, *Markov Processes and Potential Theory*, Academic Press, 1968.
2. N. Boboc, C. Constantinescu, and A. Cornea, *Semigroups of transitions on harmonic spaces*, Revue Roumaine Math. pur. appl. **12** (1967), 763–805.
3. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier **5** (1955), 131–295.
4. K.L. Chung and R.K. Getoor, *The condenser problem*, Ann. Probab. **5** (1977), 82–86.
5. P. Fitzsimmons, *Markov processes with identical hitting probabilities*, Math. Zeitschrift **194**, 547–554.
6. J. Glover, *Markov processes with identical hitting probabilities*, Trans. AMS **275** (1983), 131–141.
7. J. Glover, W. Hansen and M. Rao, *Capacities of symmetric Markov processes*, in Seminar on Stochastic Processes, 1987 (Cinlar, Chung and Getoor, eds.), Birkhauser, 1988, pp. 159–170.
8. W. Hansen, *Konstruktion von helbgruppen und Markoffschen prozessen*, Inventiones math. **5** (1968), 335–348.
9. N.S. Landkof, *Foundations of Modern Potential Theory*, Springer–Verlag, 1972.
10. G. Matheron, *Random Sets and Integral Geometry*, John Wiley and Sons, 1975.
11. P.–A. Meyer, *Brelot's axiomatic theory of the Dirichlet problem and Hunt's theory*, Ann. Inst. Fourier **13/2** (1963), 357–372.
12. C.–T. Shih, *Markov processes whose hitting distributions are dominated by those of a given process*, Trans. AMS **129** (1967), 157–179..
13. C.–T. Shih, *Construction of Markov processes from hitting distributions*, Z. Wahr. verw. Geb. **18** (1971), 47–72.

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