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The rank of $J_0(N)$

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1. INTRODUCTION

We apply the machinery of Explicit Formulae to investigate the rank of some traditionally interesting abelian varieties. For any integer $N$, consider the Jacobian variety $J_0(N)$ of the modular curve $X_0(N)$ classifying isogenies of degree $N$, whose $C$-rational points are the Riemann surface $X_0(N)(C) = \mathbb{H}^*/\Gamma_0(N)$. Here $\mathbb{H} = \{x + iy | y > 0\}$, $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q}$ and $\Gamma_0(N)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices whose lower right corner is divisible by $N$, acting on the upper half-plane as linear fractional transformations.

These varieties owe much of their appeal to the generalized Shimura-Taniyama conjecture which asserts that any abelian variety $A$ with real multiplications, both defined over $\mathbb{Q}$, is isogenous to a factor of $J_0(N)$ for a suitable $N$.

Very little seems to be known about the rank of these abelian varieties, except for elliptic modular factors whose ranks have been studied numerically ([4], [12], [22]). For $N$ a prime, Barry Mazur [20] performed descents on the Eisenstein quotient $A$, the part of $J_0(N)$ with non-trivial torsion over $\mathbb{Q}$, and

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deduced that $A(\mathbb{Q})$ has rank 0. Despite their importance, one knows only that the Eisenstein factor has dimension at least $O(\log N)$ for infinitely many primes $N$ (cf. [20]). We have started a numerical investigation and we report some of our findings in the last section.

Since the abelian varieties $A$ we consider here admit endomorphisms by a ring of rank equal to $\text{dim} \ A$, we shall occasionally use the normalized rank of $A$, denoted $\text{nrk} \ A = \text{rank} \ A / \text{dim} \ A$.

Since the results of Gross-Zagier [14] and of Kolyvagin [18] provide good information about those factors of $J_0(N)$ whose rank over the Hecke algebra is at most one, it seemed of some interest to determine their prevalence. Our main results are the following conditional bounds.

**Theorem 3.15.** Assume the Riemann hypothesis for the $L$-series of newforms. For any positive integer $N$, let $\chi$ be a character of the Atkin-Lehner group $\mathcal{W}(N)$. For each $\epsilon > 0$, the analytic rank satisfies

$$\text{rank} \ A \leq \left( \frac{3}{2} + \epsilon \right) \text{dim} \ A,$$

for all sufficiently large $N$.

This improves the conditional bound obtained implicitly in [21] for a general abelian variety, namely $\text{rank} \ A = O(\text{dim} A \log N / \log \log N)$.

**Theorem 4.9.** Assume the Riemann Hypothesis for $L$-functions of newforms and Dirichlet $L$-series of quadratic characters. Then the average analytic normalized rank for Jacobians $J_0(N)$ with $N$ prime is bounded by $\frac{7}{6}$. More precisely, for any $c$ and any $\epsilon$, we have

$$\sum_{Y < N \leq cY} \text{rank} \ J_0(N) \leq \left( \frac{7}{6} + \epsilon \right) \sum_{Y < N \leq cY} \text{dim} \ J_0(N),$$

where the sum runs over primes $N$ and $Y$ is sufficiently large.

At the end of Section 4, we conclude that under suitable Riemann hypotheses, the Birch-Swinnerton-Dyer conjecture is true for a substantial proportion of the varieties we are considering.

Our methods are fairly traditional: we use explicit formulae. The improvements over [21] are due to the Eichler-Selberg trace formula, which gives an estimate better than the Hasse-Weil bound for small $p$. Since our estimates are fairly brutal, better techniques may provide substantial improvements.

The approach used here could be used in the function field case to get analogous unconditional bounds for the rank of the Jacobian of Drinfeld modular
curves (cf. [6], [8]). The missing ingredient is an explicit Selberg trace formula in this context.

In section 5, we deduce from conductor bounds, obtained in [9], that "large enough" wild ramification imposes conditions on the endomorphism ring of the variety. For instance, we have the following special case of Theorem 5.5.

**Theorem.** Let \( f(z) = \sum a_n e^{2\pi i n z} \) be a new form of weight 2 and level \( N \). Let \( E = \mathbb{Q} \langle \{a_n\} \rangle \) be the field of \( \mathbb{Q} \)-endomorphisms of the associated abelian variety \( A_f \).

Assume that \( p^{s_p} \mid N \) and define \( s_p \) by the following:

\[
s_p = \left\lfloor \frac{r_p}{2} - 1 - \frac{1}{p - 1} \right\rfloor
\]

where \( \lfloor x \rfloor \) is the least integer \( \geq x \). Then \( E \) contains the real subfield of the cyclotomic field of \( p^{s_p} \)-th roots of unity.

In contrast to what seems to happen in the tame case, this shows that small factors of \( J_0^{new}(N) \) cannot occur when \( N \) is divisible by high powers and we expect that non-trivial factors of the new part of dimension \( O(\sqrt{N}) \) may occasionally occur.

I wish to thank Ken Ribet for advising me that Saito [27] had obtained by modular methods a weaker result of the same type and for making some of their correspondence available.

We had earlier established upper bounds for the average rank of elliptic curves, first 2.3 in [6], later improved to 2 in [8], conditionally for curves over \( \mathbb{Q} \) and unconditionally for curves over \( \mathbb{F}_q(t) \). We make some comments on the lower ranks obtained here at the end of this paper.

In Section 6, we describe some experimental work on the rank of \( J_0(N) \) for primes \( N \) at most 10000, based on work of Birch and Gross. There is a tendency for \( J_0(N) \) to have a very large simple abelian factor, with non-trivial torsion and hence of rank 0 by [20]. It should be noted that a surprisingly large proportion of the small factors have normalized analytic rank at least 2.

We use the standard notation \( f \ll g \) to mean \( |f| = O(g) \). As usual, we write \( \omega(n) \) for the number of prime factors of \( n \) and \( \sigma_k(n) \) for the sum of the \( k \)-th powers of the divisors of \( n \). For any divisor \( m \) of \( N \), \( m \mid N \) means that \( m \) is prime to \( N/m \). The symbol \((\frac{x}{y})\) is the usual Jacobi symbol and \( x = \Box \) abbreviates the assertion that \( x \) be a square. The largest square factor of \( x \) will be denoted \( Q(x)^2 \), that is \( Q(x) = a \) if \( x = a^2 b \) with \( b \) squarefree. Set \( \delta(\mathcal{P}) = 1 \) or 0 depending on whether the property \( \mathcal{P} \) be true or not.
I wish to thank Henri Cohen for mentioning the trace formula in [32]. The key estimates were suggested by calculations made with Pari, the system created by Cohen and his collaborators.

I am grateful to Oisín McGuinness and the Department of Mathematics at Columbia University for allowing some of the calculations to be done on their machines. I also wish to thank him for valuable comments.

When Maple became too slow to compute the characteristic polynomials of Hecke operators, Robert Lewis kindly translated some of my Maple programs into his much faster symbolic language Fermat. This permitted us to complete our table.

Added intellectual stimulation, during the initial stages, was provided by an invitation to a workshop organized by the Centre de Mathématiques de Montreal and Concordia University in February 1992. I owe to J.-P. Serre a number of useful suggestions improving the exposition.

2. THE EICHLER-SELBERG TRACE FORMULA

We consider the trace of Hecke operators on various pieces of \( \text{Jo}(N) \). The clearest and cleanest form of these formulae seems to be in [32], whose notation we shall adopt. An introduction to this topic, in case \( N \) is a prime, may be found in [13].

For any negative discriminant \( d \), let \( h(d) \) be the class number of primitive binary integral quadratic forms of discriminant \( d \) and let \( w(d) \) be half of the number of units in the corresponding order. The Hurwitz class numbers \( H_1(D) \), for negative \( D \equiv 0 \) or \( 1 \) mod 4 are defined by

\[
H_1(D) = \sum_{\text{d}^2=\text{D}} \frac{h(d)}{w(d)},
\]

where \( H_1(0) = -1/12 \) and \( H_1(D) = 0 \) otherwise. We need a modification of the Hurwitz class number. Let \( (n, D) = a^2b \) with \( b \) squarefree and define

\[
H_n(D) = \begin{cases} 
  a^2b \left( \frac{D/a^2b^2}{n/a^2b} \right) H_1(D/a^2b^2) & \text{if } D \equiv 0 \text{ mod } a^2b^2, \\
  0 & \text{otherwise.}
\end{cases}
\]

In particular, \( H_n(0) = \frac{-1}{12} n \).
It is easy to check that for a fundamental discriminant \( d_0 \), we have
\[
H_1(F^2d_0) = \left( \sum_{t \mid F} \mu(t) \left( \frac{d_0}{t} \right) \sigma_1(F/t) \right) \frac{h(d_0)}{w(d_0)}
\]
(2.1)

We deduce the estimate
\[
|H_1(D)| \ll h(d_0) F \log \log^2(\max\{3, F\}) \ll |D|^\frac{1}{2} \max(1, \log^2 |D|)
\]
for all \( D = F^2d_0 < 0 \) from the well-known upper bound for the class number of imaginary quadratic fields, quoted on page 147 of [20], and the argument determining the size of \( \sigma_1(n) \) in §22.9 of [16].

More generally, there is an absolute constant \( c_1 \) such that
\[
|H_n(D)| \leq c_1 Q((n, D)) |D|^\frac{1}{2} \max(1, \log^2 |D|)
\]
for all \( D < 0 \).

We denote by \( \text{tr}(L|V) \) the trace of the operator \( L \) on the vector space \( V \). Let \( \ell \) be prime to \( N \) and let \( T_\ell \) be the Hecke operator on the space \( S_2^{\text{new}}(N) \) of newforms of weight 2 for \( \Gamma_0(N) \). If \( m||N \), let \( W_m \) be the corresponding Atkin-Lehner involution [1].

We use, in the special case of weight 2, a class number function introduced in [32] for \( n|m \):
\[
sz_m(\ell, n) = -\frac{1}{2} \sum_{d|n} \sum_{d^2t^2 \leq 4\ell} \frac{H_m(d^2t^2 - 4dt)}{d^4t^2} = -\frac{1}{2} \sum_{\ell' | \ell} \min(\ell', \frac{\ell}{\ell'}) \cdot \left( Q(n), \ell' + \frac{\ell}{\ell'} \right) \cdot \left( Q\left( \frac{m}{n} \right), \ell' - \frac{\ell}{\ell'} \right)
\]
(2.3)

\[ + \delta\left( \frac{m}{n} = \Box \right) \sigma_0(n)\sigma_1(\ell), \]

where the sum over the integers \( t \) is limited to those for which \( (t, Q(n/d)) = 1 \). We need an estimate for this function:

**Lemma 2.4.** There is an absolute constant \( c_2 \) such that for all \( u = rs \) with \((r, s) = 1\), and all \( \ell \) prime to \( u \), we have
\[
|sz_u(\ell, s) - \delta(l = \Box) \frac{r}{12}| \leq c_2 (\ell + \ell^\frac{1}{4}u^\frac{1}{2}) \sigma_0(l) \sigma_0^2(u) \log^2(4\ell s)
\]
Proof: Since \( \ell \) and \( rs \) are relatively prime, terms with \( H_r(0) \) can occur in the definition of the Skoruppa-Zagier function (2.3) only if \( \ell \) is a perfect square and both \( d = 1, t = \pm 2\sqrt{\ell} \) when \( s \) is not divisible by 4 (resp. \( d = 4, t = \pm \sqrt{\ell} \) when 4 divides \( s \)). The bound (2.2) on the class number gives

\[
|sz_{rs}(\ell, s) - \delta(\ell = \Box) \frac{r}{12}| \leq c_1 \sum_{d | \ell} \sum_{t^2 < \frac{4r}{d}} Q \left( (r, dt^2 - 4\ell) \right) \ell^\frac{1}{2} d^\frac{1}{2} \log^2(4\ell d) + u^\frac{1}{2} \ell^\frac{1}{2} \sigma_0(\ell) + \sigma_0(s) \sigma_1(\ell).
\]

We used \((r, d^2t - 4d\ell) = (r, dt - 4\ell)\), true because \( d \) must be prime to \( r \), since it divides \( s \).

Recall that the number of solutions of \( dt^2 \equiv 4\ell \mod k^2 \) is \( O(2^\omega(k)) \) to rewrite the inner sum above by summing over divisors \( k \) of \( Q(r) \). Let \( t \) run through values fixing \( k = Q((r, dt^2 - 4\ell)) \) to obtain:

\[
|sz_{rs}(\ell, s) - \delta(\ell = \Box) \frac{r}{12}| \leq c_3 \sum_{d | \ell} \sum_{k | Q(r)} 2^{\omega(k)} \left( 1 + \frac{\ell^\frac{1}{2}}{k^2 \ell^\frac{1}{2}} \right) k d^\frac{1}{2} \ell^\frac{1}{2} \log^2(4\ell d) + u^\frac{1}{2} \ell^\frac{1}{2} \sigma_0(\ell) + \sigma_0(s) \sigma_1(\ell)
\]

\[
\leq (1 + c_3) 2^{\omega(r)} (\ell + u^\frac{1}{2} \ell^\frac{1}{2}) \sigma_0(\ell) \sigma_0(u) \log^2(4\ell s),
\]

after some elementary manipulations using the inequality \( Q(r) \leq \sqrt{r} \). Finally \( 2^{\omega(n)} \leq \sigma_0(n) \) gives the assertion of the lemma.

From [32], we quote a special case of the Eichler-Selberg trace formula:

**Proposition 2.5.** For \( N, \ell, m \) positive integers, \( \ell \) prime to \( N \) and \( m \mid |N \).

Then

\[
\text{tr}(T_{\ell} W_m | S_2^{\text{new}}(N)) = \sum_{d \mid N} \alpha \left( \frac{N}{d} \right) sz_d(\ell, (m, d)),
\]

where \( \alpha(n) \) is the multiplicative function with \( \alpha(p) = \alpha(p^2) = -1, \alpha(p^3) = 1 \) and \( \alpha(p^r) = 0 \) for \( r \geq 4 \).

Two multiplicative functions involving \( \alpha \) will be used below, namely

\[
\beta(m) = \sum_{d \mid m} \alpha(d) = \begin{cases} 
\mu(a) & \text{if } m = a^2, \\
0 & \text{otherwise}.
\end{cases}
\]

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Finally, one obtains the following important estimate of traces:

**Proposition 2.8.** For $N, \ell, m$ positive integers, $\ell$ prime to $N$ and $m | N$.

$$
\left| \text{tr}(T_{\ell}W_m|S_2^{\text{new}}(N)) - \frac{1}{12} \delta(\ell = \square) \beta(m) \gamma \left( \frac{N}{m} \right) \right| \\
\leq c_2 (1 + e^{N \log(N)}) \sigma_0(N) \gamma(\ell) \log^2(4\ell N).
$$

**Proof:** In the trace formula (2.5), substitute the estimate (2.4) after writing $N = mm'$, $(d, m) = r$ and $(d, m') = s$ to get

$$
\left| \text{tr}(T_{\ell}W_m|S_2^{\text{new}}(N)) - \frac{1}{12} \delta(\ell = \square) \sum_{r | m \atop s | m'} \alpha \left( \frac{m'}{s} \right) \alpha \left( \frac{m}{r} \right) \right| \\
\leq c_2 \sum_{r | m \atop s | m'} (1 + e^{\frac{1}{2} \log(rs)}) \sigma_0(rs) \gamma(\ell) \log^2(4\ell s).
$$

The result now follows readily from the definitions and properties of multiplicative functions. □

3. The explicit formula

Assume that $A$ is an abelian variety defined over the global field $K$. Hasse defined an $L$-series $L(s, A)$ as a Dirichlet series with an Euler product converging for $s > \dim A + \frac{1}{2}$. Shimura [31] proved that $L(s, J_0^{\text{new}}(N))$ has an analytic continuation to an entire function with a functional equation, as a result of the factorization

$$
L(s, J_0^{\text{new}}(N)) = \prod_f L(s, f)
$$

where $f$ runs through a basis of newforms of $S_2^{\text{new}}(N)$, with Fourier series

$$
f(z) = \sum_{n=1}^{\infty} a_n(f) \exp(2\pi inz)
$$
and \( L(s, f) \) is the Mellin transform

\[
(3.2) \quad L(s, f) = \sum_{n=1}^{\infty} a_n(f)n^{-s} = \prod_{p \mid N} (1 - a_p(f)p^{-s})^{-1} \prod_{p \not\mid N} (1 - a_p(f)p^{-s} + p^{1-2s})^{-1}.
\]

Note that for \( p \nmid N \), \( a_p = \pm 1 \) while \( a_p = 0 \) if \( p^2 \mid N \). Hecke had already proved that \( L(s, f) \) has a functional equation of the shape

\[
\Lambda(2 - s, f) = w_f \Lambda(s, f)
\]

with \( \Lambda(s) = (2\pi)^{-s}N^{s/2}\Gamma(s)L(s, f) \) and \( w_f = \pm 1 \) is the famous root number. Only when \( N \) is squarefree is the formula for \( w_f \) simple: \( w_f = -\prod_{p\mid N}(-a_p(f)) \).

More precisely, for each newform \( f \), let \( \mathcal{O}_f \) be the ring generated over \( \mathbb{Z} \) by the coefficients of \( f \). Then \( \mathcal{O}_f \) is an order in a totally real algebraic number field \( \mathbb{E}_f \) of finite degree \( d_f \), say. Shimura defines a simple abelian variety \( A_f \) of dimension \( d_f \), along with an action by \( \mathcal{O}_f \), both defined over \( \mathbb{Q} \). The cotangent space of \( A_f \) is naturally identified with the \( \mathbb{Q} \)-span of the conjugates \( f^\sigma \) of \( f \), where \( f^\sigma = \sum a_n(f)^\sigma \exp(2\pi inz) \). The Eichler-Shimura relation yields the associated decomposition

\[
(3.3) \quad L(s, A_f) = \prod_{\sigma} L(s, f^\sigma),
\]

where \( \sigma \) runs through the embeddings of \( \mathbb{E}_f \) into \( \mathbb{R} \). The Mordell-Weil group \( A_f(\mathbb{Q}) \) is an \( \mathcal{O}_f \)-module. Hence \( A_f(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \) is an \( \mathbb{E}_f \)-vector space of dimension \( \text{nrk } A = \text{rank } A_f/d_f \), the normalized rank of \( A_f \). Finally, \( J_0^{\text{new}}(N) \) is isogenous to the product of the \( \mathbb{Q} \)-simple factors \( A_f \), where \( f \) runs through non-conjugate newforms in \( S^*_{2, \text{new}}(N) \).

The nature of \( A_f \) is quite mysterious, so we consider only the coarser splitting induced by the action of the elementary 2-group of Atkin-Lehner involutions \( \mathcal{W}(N) = \{ W_d : d \mid N \} \), generated by \( W_{q^r(a)} \) with \( q \) running through the prime factors of \( N \) and \( r(q) = \text{ord}_q(N) \). Note that \( |\mathcal{W}(N)| = 2^{\omega(N)} \). To each character \( \chi : \mathcal{W}(N) \mapsto \{ \pm 1 \} \), we may attach the idempotent

\[
e_{\chi} = \frac{1}{|\mathcal{W}(N)|} \sum_{d \mid N} \chi(W_d) W_d
\]

and the corresponding abelian subvariety \( A_{\chi}(N) = e_{\chi} J_0^{\text{new}}(N) \).

From Mestre [21], we borrow the following so-called explicit formula, as modified in [6].
Lemma 3.4. Let $A$ be a modular abelian variety of dimension $g$ defined over $\mathbb{Q}$ and let $h(u) = \max(1 - |u|, 0)$ be the triangle function, then under the Riemann Hypothesis for $L(s, A)$, we have

$$
\text{rank } A \leq \frac{\log \text{cond } A}{\log X} - \frac{2}{\log X} \sum_{p^m \leq X} \frac{b_{p^m}(A)}{p^m} \log p \frac{\log p^m}{\log X} + \frac{\dim A}{\pi} \int_{-\infty}^{\infty} \left( \frac{\Gamma'(1+it) - \log 2\pi}{\Gamma(t+1+it)} \right) \hat{h}(t \log X), \, dt
$$

where $\hat{h}$ is the Fourier transform of $h$, $\alpha_{p,i}$ for $1 \leq i \leq 2g$ are the eigenvalues of Frobenius at $p$ acting on the Tate module $T_{\ell}(A) \otimes \mathbb{Q}_{\ell} = H^1_{\text{et}}(A, \mathbb{Q}_{\ell})$ and $b_{p^m}(A) = \sum_{1 \leq i \leq 2g} \alpha_{p,i}^m$.

As in [6], we deduce the following inequality from Weil’s bound $|\alpha_{p,i}| = \sqrt{p}$:

$$
\text{rank } A \leq \frac{\log \text{cond } A}{\log X} - \frac{2}{\log X} \sum_{p \leq X} \frac{b_{p}(A)}{p} \log p \frac{\log p}{\log X} - \frac{2}{\log X} \sum_{p \leq \sqrt{X}} \frac{b_{p^2}(A)}{p^2} \log p \frac{\log p^2}{\log X} + O\left( \frac{\dim A}{\log X} \right),
$$

(3.5)

with an absolute implied constant.

Remark 3.6. In the special case $A = \text{Jac}(C)$, the Jacobian of a curve $C$ defined over $\mathbb{Q}$, we have the well-known interpretation $b_q(A) = q+1 - |C(\mathbb{F}_q)|$ for any prime power $q$.

In the case $A = J_0^\text{new}(N)$, we know from Eichler-Shimura-Igusa that the real algebraic numbers $a_{p,i} = \alpha_{p,i} + \overline{\alpha}_{p,i}$ are the eigenvalues of the Hecke operator $T_p$ acting on the space of newforms $S^\text{new}_2(N)$. Recall the identity $T_{p^2} = T_p^2 - pT_1$ and that the Hecke operators commute with $W(N)$, to conclude that

$$
\begin{align*}
&b_{p^2}(A_\chi(N)) = \sum_{i=1}^{g} (a_{p,i}^2 + \overline{a}_{p,i}^2) = \sum_i a_{p,i}^2 - 2p \dim A_\chi(N) \\
&= \text{tr}(T_{p^2}|e_\chi S^\text{new}_2(N)) - p \dim A_\chi(N),
\end{align*}
$$

(3.7)

where the $a_{p,i}$ consist of the eigenvalues of $T_p$ acting on the invariant differential forms of $A_\chi(N)$, identified with $e_\chi S^\text{new}_2(N)$. With this identification, we
see that
\[
\begin{align*}
\dim A_\chi(N) &= \frac{1}{|\mathcal{W}(N)|} \sum_{d|d|N} \chi(W_d) \text{tr}(T_1 W_d S_2^{\text{new}}(N)), \\
b_p(A_\chi(N)) &= \frac{1}{|\mathcal{W}(N)|} \sum_{d|d|N} \chi(W_d) \text{tr}(T_p W_d S_2^{\text{new}}(N)) \\
b_{p^2}(A_\chi(N)) &= \frac{1}{|\mathcal{W}(N)|} \sum_{d|d|N} \chi(W_d) \text{tr}(T_{p^2} W_d S_2^{\text{new}}(N)) \\
- &\quad p \cdot \dim A_\chi(N).
\end{align*}
\] (3.8)

The following estimate of the dimension of the Atkin-Lehner pieces is a consequence of (2.8) with \( \ell = 1 \):

\textbf{Lemma 3.9.} For every positive integer \( N \) and every character \( \chi \) of the Atkin-Lehner group \( \mathcal{W}(N) \), we have

\[
\left| \dim A_\chi(N) - \frac{1}{12} \frac{\phi_\chi(N)}{|\mathcal{W}(N)|} \right| \leq c_2 \sigma_0^3(N) N^{\frac{3}{4}} \log^2 4N,
\]

where
\[
\phi_\chi(N) = \sum_{d|d|N} \chi(W_d) \beta(d) \gamma \left( \frac{N}{d} \right),
\]
and \( \beta \) and \( \gamma \) were defined in (2.6) and (2.7). When \( N \) is squarefree, \( \phi_\chi(N) = \phi(N) \), where \( \phi(N) \) is Euler’s totient function. In general, there are absolute constants \( c_4 \) and \( c_5 \) such that
\[
c_4 \phi(N) \leq \phi_\chi(N) \leq c_5 \phi(N).
\]

\textbf{Proof:} Only the assertions about \( \phi_\chi(N) \) need comment. Let \( N_1 = \prod p^2 \) where \( p \) runs through the primes such that \( p^2 \mid N \) and let \( N_2 = N/N_1 \). Since all the functions are multiplicative, a calculation gives the factorization

\[
\phi_\chi(N) = \gamma(N_2) \sum_{d|d|N_1} \chi(W_d) \beta(d) \gamma \left( \frac{N_1}{d} \right) = \prod_{p^r \mid N} \phi_\chi(p^r).
\]

We have the following \( p \)-components:

\[
\phi_\chi(p^r) = \begin{cases} 
1 & \text{if } r = 0, \\
p - 1 & \text{if } r = 1, \\
p^2 - p - 1 - \chi(W_{p^2}) & \text{if } r = 2, \\
p^r - p^{r-1} - p^{r-2} + p^{r-3} & \text{if } r \geq 3.
\end{cases}
\]

The remaining claim is now clear from the convergence of \( \sum p^{-2} \). \( \blacksquare \)

We control the contribution of \( \text{tr}(T_1 W_m | S_2^{\text{new}}(N)) \), to the explicit formula by the prime number theorem (cf. Cor. 2.4 in [6]). Explicitly, use (2.8) and
(3.8) for \( \ell = p \leq X \), where \( X \geq 2 \) is a parameter to be chosen later as a function of \( N \), to obtain

\[
\left| \sum_{p \leq X} b_p(A_x(N)) \frac{\log p}{p} h \left( \frac{\log p}{\log X} \right) \right| 
\leq c_2 \sigma_0^3(N) \log^2(4XN) \sum_{p \leq X} \left( p + p^{\frac{1}{3}} N^{\frac{1}{3}} \right) \frac{\log p}{p} h \left( \frac{\log p}{\log X} \right) 
\leq c_6 \sigma_0^3(N) (X + X^{\frac{1}{2}} N^{\frac{1}{3}}) \log^2(4XN),
\]  

with an absolute constant \( c_6 \).

Similarly, the next inequality for \( A = A_x(N) \) follows from (3.7):

\[
\left| \sum_{p \leq \sqrt{X}} (b_{p^2}(A) + p \dim A) \frac{\log p}{p^2} h \left( \frac{\log p^2}{\log X} \right) \right| 
\leq c_7 \sigma_0^3(N) (X^{\frac{1}{2}} + N^{\frac{1}{6}} \log X) \log^2(4XN).
\]

In Cor. 2.4 in [6], we saw that

\[
\sum_{p \leq \sqrt{X}} \frac{\log p}{p} h \left( \frac{\log p^2}{\log X} \right) = \frac{1}{4} \log X + O(1),
\]

so the last estimate and (3.11) give

\[
\left| \sum_{p \leq \sqrt{X}} b_{p^2}(A) \frac{\log p}{p^2} h \left( \frac{\log p^2}{\log X} \right) + \frac{1}{4} \log X \dim A \right| 
\leq c_8 (\dim A + \sigma_0^3(N) (X^{\frac{1}{2}} + N^{\frac{1}{6}} \log X) \log^2(4XN)),
\]

for \( A = A_x(N) \).

Putting all this information into the Explicit Formula (3.5) for \( A = A_x(N) \), we conclude that

\[
\text{rank } A \leq \left[ \frac{\log N}{\log X} + \frac{1}{2} \right] \dim A 
+ c_9 \left( \sigma_0^3(N) (X + X^{\frac{1}{2}} N^{\frac{1}{3}}) \frac{\log^2(4XN)}{\log X} + \frac{\dim A}{\log X} \right),
\]

for all \( X \geq 2 \) and positive integers \( N \).
The inequalities $1 \leq N/\phi(N) \ll \log \log N$ and $\sigma_0(N) = O(N^{\log 2/\log \log N})$ can be found in §22.9 of [16]. It follows from (3.9) that for any $\epsilon > 0$, we have $\dim A_N(N) \geq N^{1-\epsilon/4}$, for all sufficiently large integers $N$.

Now set $X = N^c$ with $c = (1 + \frac{\epsilon}{2})^{-1}$ to obtain:

**Theorem 3.15.** For any positive integer $N$, let $\chi$ be a character of the Atkin-Lehner group $W(N)$. Assume the Riemann Hypothesis for the series $L(s, f)$ belonging to $A = A_N(\chi)$. For each $\epsilon > 0$, the analytic rank satisfies

$$\text{rank } A \leq \left( \frac{3}{2} + \epsilon \right) \dim A,$$

for all sufficiently large $N$.

4. AVERAGING OVER $N$

In this section, we use the trace formula with more care and avail ourselves of the cancellations afforded by averaging over a range of $N$. To do this painlessly, we restrict our attention to prime conductors and consider the full Jacobians $J_0(N)$.

We note some special cases of (2.5). The case $N = 1$ gives a classical class number relation due to Glaisher [17], which says that the trace of $T_n$ is zero on the trivial space $S_2(1)$.

**Lemma 4.1.** Let $\psi(n) = \sum_{d>\sqrt{n}} d - \sum_{d<\sqrt{n}} d$ be the excess of divisors greater than $\sqrt{n}$ over those smaller. Then

$$\sum_{s^2 \leq 4n} H_1(s^2 - 4n) = \sigma_1(n) + \psi(n).$$

For $N$ a prime, the trace formula (2.5) simplifies to:

**Lemma 4.2.** Let $N$ be prime and let $n$ be prime to $N$, then

$$\text{tr}(T_n|S_2(N)) = -\frac{1}{2} \sum_{s^2 \leq 4n} \left( \frac{s^2 - 4n}{N} \right) H_1(s^2 - 4n) - \frac{1}{2} \sum_{d|n} \text{min} \left( d, \frac{n}{d} \right)$$

$$- \frac{1}{2} \sum_{s^2 \equiv 4n \text{mod } N^2} NH_1 \left( \frac{s^2 - 4n}{N^2} \right)$$

and

$$\text{tr}(T_nW_N|S_2(N)) = \sigma_1(n) - \frac{1}{2} \sum_{s^2 \leq 4nN \atop N|s} H_1(s^2 - 4nN).$$

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We also need a consequence of the explicit Cebotarev Density Theorem found, for instance, as Theorem 4 of [28]:

**Lemma 4.3.** Assume the Riemann Hypothesis for quadratic fields. Then, for any $x \geq 2$ and any discriminant $d$ of a quadratic number field, we have

$$\left| \sum_{N \leq x} \left( \frac{d}{N} \right) \right| \leq c_9 x^{1/2} \log(|d|x^2),$$

with an absolute constant $c_9$.

We consider the rank of the abelian varieties $J_0(N)$ over a range $Y < N < cY$ of primes $N$, with a fixed $c > 1$. The parameter $X$ bounding the primes used in the explicit formula, will be chosen later. We shall assume that $4X < Y^2$ in order to simplify the contribution of the third sum in the trace of $T_n$ in (4.2). The estimates below will be valid for such $X$ and sufficiently large $Y$ (depending on $c$) with absolute implied constants.

Let $\Delta(Y, Z) = \sum_N \dim J_0(N)$ and $\text{Rk}(Y, Z) = \sum_N \text{rank } J_0(N)$, where $N$ runs through the primes in the interval $(Y, Z]$. We shall estimate the “average normalized rank”, namely $\text{Rk}(Y, Z)/\Delta(Y, Z)$. Recall that the dimension of $J_0(N)$ is $\frac{1}{12}N$, up to an error of at most 1. By the prime number theorem, we have for $Z = cY$

$$\Delta(Y, cY) = \frac{1}{12} \sum_{N \leq cY, N \text{prime}} (N + O(1)) = \frac{c^2 - 1}{24} \frac{Y^2}{\log Y} + O\left(\frac{c^2 Y^2 \log c}{\log^2 Y}\right).$$

where the implied constant does not depend on $c$.

Since the logarithm of the conductor of $J_0(N)$ is $\dim J_0(N) \log N$, the contribution of the conductors is similarly

$$\sum_{Y < N \leq Z} \log \text{cond } J_0(N) = \frac{1}{12} \sum_{Y < N \leq Z} N \log N + O\left(\sum_{Y < N \leq Z} \log N\right)
\frac{(c^2 - 1)}{24} Y^2 + O\left(\frac{c^2 Y^2 \log c}{\log Y}\right).$$

We add up, for a fixed $p \leq X < \frac{1}{4}Y^2$, the contributions of $\text{tr}(T_p|S_2(N)) =$
\[ b_p(J_0(N)) \text{ over primes } N. \text{ We use (4.2) and the bounds (2.2) and (4.3):} \]
\[
\sum_{Y < N \leq Z} \text{tr}(T_p \mid S_2(N))) = -\frac{1}{2} \sum_{s^2 \leq 4p} \sum_{N} \left( \frac{s^2 - 4p}{N} \right) H_1(s^2 - 4p) + O \left( \frac{cY}{\log Y} \right) \\
(4.5) \leq c_{10}\sqrt{c}pY^{\frac{1}{2}} \log(Xc^2Y^2) \log^2 X + \frac{cY}{\log Y}
\]
and \( N \) runs through \( Y < N \leq cY \) with \( Y \) large enough. Similarly, for any fixed \( p \leq \sqrt{X} < \frac{1}{2}Y \), we get
\[
\sum_{Y < N \leq Z} \text{tr}(T_p^2 \mid S_2(N)) = -\frac{1}{2} \sum_{s^2 \leq 4p^2} \sum_{Y < N \leq Z} \left( \frac{s^2 - 4p^2}{N} \right) H_1(s^2 - 4p^2) \\
(4.6) + \sum_{Y < N \leq Z} \frac{N}{12} + O \left( \frac{pcY}{\log Y} \right) \\
\leq c_{11}\sqrt{cp^2}Y^{\frac{1}{2}} \log(2cXY) \log^2 X + \frac{cpY}{\log Y} + \Delta(Y, cY).
\]
Next, the weighted sum (4.5) over \( p \leq X \) gives:
\[
\sum_{Y < N \leq Z} \sum_{p \leq X} b_p(J_0(N)) \frac{\log p}{p} h \left( \frac{\log p}{\log X} \right) \\
(4.7) \leq c_{12}\sqrt{c}XY^{\frac{1}{4}} \log(c^2XY^2) \log^2 X + cY \frac{\log X}{\log Y}
\]
Since \( \text{tr}(T_p^2 \mid S_2(N)) = b_p^2(J_0(N)) + p \dim J_0(N) \), we get from (4.6), the estimate
\[
\sum_{Y < N \leq Z} \sum_{p \leq \sqrt{X}} [b_p^2(J_0(N)) + p \dim J_0(N)] \frac{\log p^2}{p^2} h \left( \frac{\log p^2}{\log X} \right) \\
(4.8) \leq c_{13}\sqrt{c}X^{\frac{1}{4}}Y^{\frac{1}{4}} \log(2cXY) \log^2 X + cY \frac{\log X}{\log Y} + \Delta(Y, cY).
\]
We recall the explicit formula (3.5) in our context:
\[
\text{rank } J_0(N) \log X \leq \log \text{cond } J_0(N) - 2 \sum_{p \leq X} \frac{b_p(J_0(N))}{p} \log p h \left( \frac{\log p}{\log X} \right) \\
- 2 \sum_{p \leq \sqrt{X}} \frac{b_p^2(J_0(N))}{p^2} \log p h \left( \frac{\log p^2}{\log X} \right) + O (\dim J_0(N)).
\]
Add up over primes $N$ in the range $Y < N \leq cY$ and substitute (4.4), (4.7), (4.8) and the estimate for $\Delta(Y, cY)$ to obtain

$$\text{Rk}(Y, cY) \log X \leq \frac{1}{24} (c^2 - 1) Y^2 + 2\Delta(Y, cY) \sum_{p \leq \sqrt{X}} \frac{\log p}{p} h \left( \frac{\log p^2}{\log X} \right)$$

$$+ O \left( \frac{\sqrt{c} X Y^\frac{3}{4} \log(XY) \log^2 X + cY \log X \log^2 Y}{\log X} \right)$$

$$+ O \left( \sqrt{c} X^\frac{3}{4} Y^\frac{3}{4} \log(XY) \log^2 X + cY \log X \log^2 Y + \Delta(Y, cY) \right)$$

We conclude upon dividing this last result by $\Delta(Y, cY) \log X$, using (3.12), our earlier estimate for $\Delta(Y, cY)$, and taking $X = Y^{\frac{3}{2}} - \varepsilon$ which is adequate to give the final claim.

**Theorem 4.9.** Assume the Riemann Hypothesis for the $L$-functions of newforms and Dirichlet $L$-series for quadratic characters. Then the average analytic normalized rank for Jacobians $J_0(N)$ with $N$ prime is bounded by $\frac{7}{6}$. More precisely, for any fixed $c$ and $\varepsilon > 0$, we have

$$\sum_{Y < N \leq cY} \text{rank } J_0(N) \leq \left( \frac{7}{6} + \varepsilon \right) \sum_{Y < N \leq cY} \dim J_0(N),$$

where the sum runs over primes $N$ and $Y$ is sufficiently large.

Even our weak results imply entertaining consequences of the deep results of Kolyvagin, Bump-Friedberg-Hoffstein, Gross-Zagier and Murty-Murty ([18], [11], [14], [23]). For instance, Theorem (3.15) gives

**Corollary 4.10.** Let $N$ be any positive integer and let $A = A_x(N)$ be an Atkin-Lehner component with $x(W_N) = -1$, so that $A$ is of even parity. Under the Riemann hypotheses, $A$ has a quotient $B$ of analytic rank 0 and dimension at least $\frac{1}{4} \dim A$. Moreover $B(Q)$ and $\text{III}(B)$ are finite.

In particular, if $N$ is large and $A$ is simple, it will be of rank 0.

Similarly (4.9) implies that, among Jacobians $J_0(N)$ with $N$ prime with $Y < N \leq cY$, the abelian subvarieties of normalized analytic rank at most one, make up at least two-thirds of the total dimension. These will satisfy a form of the conjecture of Birch-Swinnerton-Dyer. To see the former claim, let $r$ be the proportion of the pieces of rank zero and let $s$ be that of the pieces of rank one. Since the odd and even pieces contribute roughly equal amounts, we conclude that the average normalized rank is at least $2(\frac{5}{2} - r) + s + 3(\frac{5}{2} - s) \leq \frac{7}{6} + \varepsilon$, so $r + s$ is at least $\frac{2}{3} - \varepsilon$ if $Y$ is large enough.
5. ABELIAN VARIETIES OF GL₂-TYPE AND WILD RAMIFICATION

In this section, we show that the endomorphism rings of abelian varieties with real multiplications over \( \mathbb{Q} \), whose conductor is divisible by high powers of a prime, contain large cyclotomic rings. This is an application of results obtained in [9], to which the reader is referred for a more leisurely treatment of conductors. We restrict our attention to modular forms of weight 2, since we are concerned here with the associated modular abelian varieties, but our methods apply to higher weights upon replacing the Tate module by the Deligne representation.

Let \( f \) be a newform of weight 2 on \( \Gamma_0(N) \) of Nebentypus \( \epsilon \) and let \( A_f \) be the corresponding abelian variety, as in [31]. It is defined over \( \mathbb{Q} \) and has multiplications by \( \mathcal{O}_f = \mathbb{Z}[\{a_n\}] \), the order defined by adjoining the coefficients of \( f \). The quotient field \( E_f \) of \( \mathcal{O}_f \) is a totally real field when \( \epsilon \) is trivial and a CM-field when \( \epsilon \) is non-trivial. The characteristic property of \( A_f \) is that its invariant differentials \( \Omega(A_f) \) may be identified with the \( \mathbb{Q} \)-span of the conjugates \( f^\sigma(z) = \sum a_n^\sigma q^n \), where \( \sigma \) runs through the isomorphisms of \( E_f \) into \( \mathbb{C} \). Thus \( \dim A_f = [E_f:Q] \).

More generally, let \( A \) be a simple abelian variety of dimension \( d \) defined over \( \mathbb{Q} \) with \( E \subseteq \text{End}_\mathbb{Q}(A) = \text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} \) for some number field \( E \) of dimension \( d \). Ribet [26] calls such abelian varieties “of GL₂-type”. Then \( E \) is either totally real or a CM-field, and \( E \) is a maximal commutative subfield of the simple algebra \( \text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} \), whose center will be denoted by \( F \). It is expected that, up to isogeny, all such abelian varieties \( A \) are modular ([29], [26]).

There is an abelian variety \( B \) isogenous to \( A \) over \( \mathbb{Q} \), with an action of the ring of integers \( \mathcal{O} \) of \( E \). Since the questions considered here are insensitive to isogenies, it is convenient to assume \( A = B \). Let \( \lambda \) run through the primes above \( \ell \) in \( E \). We have a natural representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the two-dimensional \( \mathcal{O}/\ell \mathcal{O} \)-module \( A[\ell] = \prod A[\lambda] \) of points of order \( \ell \) on \( A \). We obtain representations \( \rho_\lambda : \text{Gal}(L/\mathbb{Q}) \rightarrow GL_2(F_\lambda) \) with \( F_\lambda = \mathcal{O}/\lambda \) and \( L = \mathbb{Q}(A[\lambda]) \), because the Galois action commutes with that of \( \mathcal{O} \).

The conductor of \( A \) is \( N^d \) for some integer \( N \) which may be calculated from the Artin conductor of the representation \( \rho_\lambda \) on \( V = A[\lambda] \), for any sufficiently large prime \( \ell \) (cf. [15], [29], [26]). Precisely, let \( p \) be any prime, let \( r_p \) be the exponent of \( p \) in \( N \), and let \( v \) be any prime above \( p \) in \( L \). Let \( G \) denote the decomposition group of \( v \) and \( \{G_i\} \) be its higher decomposition groups, then \( r_p \) can be computed from

\[
[F_\lambda : F_\ell] r_p = \epsilon_\lambda(A/Q_\ell) + sw(A[\lambda]),
\]

where the “tame exponent” \( \epsilon_\lambda(A/Q_\ell) = \dim_{Q_\ell}(W/WG_\circ) \), which depends on the Tate module \( W = E_\lambda \otimes T_\lambda \) of \( A \), is known to be at most 2, while the
“wild” or Swan exponent is given by

\[
\text{sw}(V) = \sum_{i>0} \frac{g_i}{g_0} \dim_{F_i}(V/V^{G_i})
\]

(5.1)

\[\left[F:K\right] \sum_{i>0} \frac{g_i}{g_0} \dim_{F}(V/V^{G_i})\]

with \(g_i = |G_i|\). In the modular case, \(N\) is known to be the analytic conductor according to deep work of Deligne, Langlands and Carayol, for all sufficiently large \(\ell\). For higher weight, still unpublished work of Faltings-Jordan-Livné and Ribet would be needed for this assertion, but our applications need only the Swan conductor, to which the work of Carayol already applies.

We quote the following result from [9].

**Lemma 5.2.** Let \(L/K\) be a Galois extension of local fields with residue characteristic \(p\) and Galois group \(G\). Suppose \(G\) is a \(p\)-group and that \(\chi\) is the character of an irreducible \(\mathbb{C}[G]\)-representation \(V\). Then

\[
\text{sw}(V) \leq \chi(1)[\nu_K(\chi(1)[Q(\chi):Q]) + pe_K/(p-1)],
\]

where \(Q(\chi)\) is the field of values of the character, \(\chi(1)\) is its degree and \(e_K\) is the absolute ramification index of \(K\).

As a consequence, we deduce a bound on the Swan conductors in our setting:

**Corollary 5.3.** Let \(G\) be a \(p\)-group, let \(F = \mathbb{F}_q\) be the Galois field with \(q\) elements and let \(V\) be a faithful 2-dimensional \(F[G]\)-module. Let \(L/K\) be an extension of local fields with Galois group \(G\), whose residue characteristic \(p\) is prime to \(q\) and let \(e_K\) be the absolute ramification index of \(K\). Then we have the bound

\[
\text{sw}(V, L/K) \leq 2e_K [s + 1/(p-1)],
\]

with \(s = \text{ord}_p(q^2 - 1)\).

When \(G \subset \text{SL}_2(F)\), this bound may be refined by taking \(p^s\) as the exponent of \(G\).

**Proof:** Clearly, \(G\) is a subgroup of a \(p\)-Sylow subgroup \(S_p\) of \(GL_2(F)\). We shall apply the lemma after lifting the representation on \(V\) to characteristic 0. First, note that if \(G\) is abelian of exponent \(p^s\), then every faithful absolutely irreducible character \(\chi\) of \(G\) is one-dimensional with \(Q(\chi) = Q(\mu_{p^s})\). Our claim follows from the lemma since Swan conductors are additive. We proceed with a case by case analysis after recalling that \(|GL_2(F)| = q(q-1)^2(q+1)|\.
When \( p \) is odd, \( S_p = 1 \) unless \( q \equiv \pm 1 \mod p \) when we have:

\[
S_p \cong \begin{cases} 
\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t} & \text{if } p^t\|(q-1) \\
\mathbb{Z}_{p^t} & \text{if } p^t\|(q+1), 
\end{cases}
\]

so we are done, since these groups are abelian.

When \( p = 2 \), the story is slightly more complicated. Write \( \eta = (-1)^{\frac{q-1}{2}} \) and define \( t \geq 2 \) by \( 2^t\|(q-\eta) \). If \( q \equiv 3 \mod 4 \), then the two-dimensional \( \mathbb{F}_q \)-space \( \mathbb{F}_q^2 \) contains a primitive \( 2t+1 \)-st root of unity \( \zeta_{2t+1} \) and \( S_2 \cong \langle x, y \rangle \) with \( x \) induced by multiplication by \( \zeta_{2t+1} \) and \( y \) induced by \( \zeta \mapsto \zeta^q \). We have the relations \( x^{2t+1} = 1, y^2 = 1, yxy^{-1} = x^{-1+2^t} \). This is a semi-dihedral group whose intersection with \( SL_2 \) is a generalized quaternion group \( Q_2 \) of order \( 2^{t+1} \).

On the other hand, if \( q \equiv 1 \mod 4 \), denote by \( \omega \) a primitive \( 2^t \)-th root of unity in \( \mathbb{F}_q \). Then \( S_2 \) is the extension of \( \mu_{2^t} \oplus \mu_{2^t} \) along the diagonal by \( y = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right) \). The element \( z = \left( \begin{smallmatrix} 1 & \omega^t \\ 0 & 1 \end{smallmatrix} \right) \) has the largest order, namely \( 2^{t+1} \). The intersection of \( S_2 \) with \( SL_2 \) is the dihedral group of order \( 2^{t+1} \).

We verify readily that all the possible non-abelian 2-subgroups of \( GL_2 \) with cyclic maximal normal subgroup, and in particular all subgroups of \( SL_2 \), have irreducible 2-dimensional \( \mathbb{C} \)-representations whose field of character values is a subfield of index 2 of \( \mathbb{Q}(\mu_{2^t}) \), where \( 2^t \) is the exponent. The only remaining case occurs when \( q \equiv 1 \mod 4 \), and the field of character values is \( \mathbb{Q}(\mu_{2^t}) \). But in this case \( \text{ord}_2(q^2 - 1) = 1 + \text{ord}_2(q - 1) \) and the claim still obtains.

Recall the so-called Eichler-Shimura relation for any \( q \) prime to \( N \ell \):

\[
\begin{align*}
\text{tr}(\rho_\lambda(Frob_q)) & \equiv a_\lambda \pmod{\lambda} \\
\det(\rho_\lambda(Frob_q)) & \equiv \varepsilon(q)q \pmod{\lambda},
\end{align*}
\]

where \( Frob_q \) is the Frobenius class of any prime extending \( q \). This is of course classical in the modular case, and was extended by Ribet in [26].

**Theorem 5.5.** Let \( A \) be an abelian variety of \( GL_2 \)-type of dimension \( d \) and let \( \mathbb{E} \) be its field of \( \mathbb{Q} \)-endomorphisms. Then the conductor of \( A \) is \( N^d \) for some integer \( N \). Suppose that \( p^r || N \) and define \( s_p \) by the following:

\[
s_p = \left\lfloor \frac{r_p}{2} - 1 - \frac{1}{p-1} \right\rfloor
\]

where \( \lfloor x \rfloor \) is the least integer \( \geq x \). Let \( \zeta \) be a primitive \( p^s \)-th root of unity.

(i) \( \mathbb{E} \) contains \( \mathbb{Q}(\varepsilon, \zeta + \zeta^{-1}) \) if \( p \) is odd (resp. \( \mathbb{Q}(\varepsilon, \zeta^2) \) if \( p = 2 \)).
(ii) Suppose that $p^2$ does not divide the conductor of $\epsilon$ and that $A$ is not of CM-type. Then the center $F$ of $\text{End}_Q^2(A)$ contains $\mathbb{Q}(\zeta + \zeta^{-1})$ if $p > 2$ (resp. $\mathbb{Q}(\zeta^2 + \zeta^{-2})$ if $p = 2$).

**Proof:** We consider primes $\lambda$ of $E$ above rational primes $\ell \gg 0$ different from $p$. Let $G$ be the image $\rho_\lambda(G_1)$ in $GL_2(F_\lambda)$ of the first decomposition subgroup $G_1$ of $\text{Gal}(Q(A[\lambda])/Q)$ for a valuation extending $p$. Then $G$ is a $p$-group and Corollary 5.3 gives a lower bound for the exponent in terms of the exponent of $p$ in the conductor. We see that the norm of $\lambda$ satisfies $N(\lambda)^2 \equiv 1$ (mod $p^{*r}$) for almost all primes $\lambda$, since there must be an element of order $p^{*r}$ in the image $G$ of the higher ramification group in $GL_2(F_\lambda)$, according to (5.3). Apply the Cebotarev density theorem to conclude that the exponent of $\text{Gal}(Q(\zeta)/E \cap Q(\zeta))$ is at most 2, this proves (i).

To prove (ii), observe that our hypothesis implies that the abelian extension corresponding to the determinant is tamely ramified and so the group $G$ above is a subgroup of $SL_2(F_\lambda)$. From (5.3), we must have an element $g$ in $G$ of order $p^{*r}$ and determinant 1. We work in the field $E(\zeta)$. Replacing $g$ by a power if necessary, we may suppose that $\text{tr}(g) \equiv \zeta + \zeta^{-1}$. By Cebotarev applied to the extension $Q(A[\lambda])/Q$, there is a prime $r$ not dividing $\ell N$ such that $\text{Frob}_r = g$. It follows that $\alpha_r \equiv \text{tr}(\text{Frob}_r) \equiv \zeta + \zeta^{-1}$ mod $\lambda$.

Ribet ([25], [26]) showed that under our assumptions, $F$ is generated over $Q$ by $a_2^2/\epsilon(r)$ with $r$ running through almost all primes. Note that that (5.4) implies that for almost all primes $\lambda$ in $E$ there is a prime $r$ such that $r \equiv \epsilon(r) \equiv 1$ mod $\lambda$ and so $a_2^2 - 2 \equiv \zeta^2 + \zeta^{-2}$ mod $\lambda$. Ribet’s result and the congruence above shows that $\beta = \zeta^2 + \zeta^{-2}$ must belong to $F$, since the irreducible polynomial of $\beta$ over $F$ has a root modulo almost every prime. ■

**Remark 5.6.** Our result gives an entertaining explanation of the upper bound on the conductor of elliptic curves over $Q$ in [29].

**Remark 5.7.** Our argument shows that (i) holds in general, for weight $k$ at least 2 and arbitrary Nebentypus $\epsilon$.

**Remark 5.8.** As a consequence of special twisting operators $U_\chi$ he discovered, Saito [27] had given part (i) of the previous Theorem with the weaker bound $s_p = [r_p/3]$ in the modular case. Amusingly, this gives our result for the lowest non-trivial exponents and all $p$, making a computational discovery unlikely. Ribet, in a letter to Saito in 1980, suggested that Saito’s bound might be true of the smaller field $F$, and verified it in a special case. A modular explanation of our sharper result should be of interest: it is likely that Saito’s computations involving $U_\chi$ may be refined, but we have not attempted to do this.
6. DISCUSSION OF NUMERICAL EXPERIMENTS

In order to compare our bounds for the ranks of Jacobians with "reality", we decided to investigate the latter empirically, an elusive goal until we remembered some work of Gross, to be recalled later. Our numerical findings are in the Table at the end of this paper with a summary at the end of this section. In the next section, we mention some problems they suggested. We only hint at the ingredients of the lengthy and time-consuming calculations here.

In this section, \( N \) will always denote a prime number and denote by \( J_0(N) \) the maximal quotient of \( J_0(N) \) on which \( W_N \) acts as -1, so that the corresponding \( L \)-series have even order at \( s = 1 \). Ogg conjectured and Mazur [20] proved that the torsion subgroup of the abelian variety \( J_0(N) \) is cyclic of order \( n \) equal to the numerator of \( \frac{1}{12}(N - 1) \). For each \( p \) dividing \( n \), Mazur defined the Eisenstein quotient \( J'_0(N) \), which has nontrivial \( p \)-torsion. By a descent, he proved that these factors have rank 0.

For most primes \( N \), \( J_0(N) \) seems to be simple, in which case it is the Eisenstein quotient, and so has Mordell-Weil rank 0 over \( \mathbb{Q} \). Our table gives the splittings of \( J_0(N) \) for those \( N < 10000 \) for which it is not a simple variety. We estimate the rank of the comparatively few non-Eisenstein factors by using the following beautiful result due to Gross [13] to establish a lower bound for the analytic rank (presumably the actual analytic rank and, under the Birch-Swinnerton-Dyer conjecture, even the algebraic rank).

**Theorem 6.1.** Let \( N \) be a prime. Then the number of linear dependence relations between the theta functions of positive definite ternary quadratic forms of discriminant \( 32N^2 \) and level \( 4N \) is equal to the number of Hecke eigenforms \( f \) invariant under the Atkin-Lehner involution \( W_N \) for which the associated \( L \)-series has a zero of order at least 2 at \( s = 1 \). In particular, this counts the number of associated abelian varieties \( A_f \) with normalized rank at least 2.

The reader is invited to look at the paper of Birch [3] for an exposition of the rather complex interaction of this theorem with results of Ponomarev and Kitaoka on ternary and quaternary quadratic forms. The reader should also consult [19] and [5] which contains a generalization of Gross's theorem to the case of square-free \( N \).

Some parts of our table were computed independently by several people and our results are compatible with those I have had access to. We find as many linear factors as isogeny classes of elliptic of even rank, and get one relation

---

1We found this reference after most of this work was done. To our dismay, we found a table of the number of relations for \( p < 5000 \), computed by Hashimoto. His table agrees with ours except that he missed the 2 relations for \( N = 4483 \).
for each elliptic curve of rank 2, as given in the unpublished table of Mestre and Oesterlé [22]. There is agreement with partial tables of the splittings for conductors at most 1000 that H. Cohen had given me on the occasion of a visit to Bordeaux in 1990.

I used the algorithm of Birch [3] to find the Hecke graph of classes of positive definite ternary quadratic forms of discriminant $2N$. The characteristic polynomial of the adjacency matrix was obtained at first brutally. Then we used an algorithm based on continued fractions due to Mills and Wiedemann, which I learned from early fragments of the 1990 DEA of Dewisme, a student of Mestre. The relations between the theta functions were obtained by a calculation independent of the splitting, for additional confirmation, using about $N/3$ terms for $N$ at least 1000. The “small vectors” were calculated by an algorithm in the book of Pohst and Zassenhaus. The investigation was carried out in Maple at Fordham and at Columbia.

The first abelian surface of analytic rank 4 appears as one of the two 2-dimensional factors of $J_0(1061)$. Both have endomorphisms by the maximal order of $\mathbb{Q}(\sqrt{5})$. It may be of some interest to exhibit this example more explicitly.

Consider the three parameter family of hyperelliptic curves $C_{b,c,d}$

$$y^2 + (x^3 + x + 1 + c(x^2 + x))y = b + (1 + 3b)x + (1 - bd + 3b)x^2 + (b - 2bd - d)x^3 - bdx^4$$

(6.2)

It follows from some old work of ours [7] that their Jacobian admits endomorphisms by $\mathbb{Z}[\alpha]$, where $\alpha^2 + \alpha - 1 = 0$. If we specialize to the curve $X = C_{0,1,1}$, we find the hyperelliptic curve

$$y^2 = x^6 + 2x^5 + 5x^4 - 2x^3 + 10x^2 + 8x + 1$$

(6.3)

with at least 18 rational points with $x = 0, 1, 3, -1, -2, -1/2, 1/3, -10/3, \infty$. Its Jacobian has real multiplication by the maximal order in $\mathbb{Q}(\sqrt{5})$ and conductor $1061^2$. In fact, $J_0(1061)$ has 2 factors with that property but only 2 of the $L$-series vanish at $s = 1$, the corresponding pair of conjugate normalized Hecke eigenforms $f, f^\sigma$ have coefficients given by the list $[p, a_p]$ for $p$ from 2 to 149:

$[2, -1 + \alpha], [3, -1 + \alpha], [5, -3 - \alpha], [7, -3], [11, -3 - \alpha], [13, -4 - \alpha], [17, -3 + \alpha], [19, 1], [23, -3], [29, -3 + 3\alpha], [31, 0], [37, -1 + 2\alpha], [41, -6 + 2\alpha], [43, -5], [47, -10 - \alpha], [53, 3 + 6\alpha], [59, 6\alpha], [61, -3 + 4\alpha], [67, -5 - 9\alpha], [71, -3 + 2\alpha], [73, 1 - 6\alpha], [79, -1 - 6\alpha], [83, 8 - 2\alpha], [89, 4\alpha], [97, 6 - 2\alpha], [101, -4 - 8\alpha], [103, 5\alpha + 2], [107, -11 - 10\alpha], [109, -8 + \alpha], [113, -9 - 6\alpha], [127, -1], [131, -9 - 14\alpha], [137, -11 - 4\alpha], [139, 17 - 3\alpha], [149, -6 + 3\alpha].$
Since we have agreement to many terms with the \( L \)-series of the Jacobian of \( X \), we expect that the method of Faltings and Serre should prove that
\[ \text{Jac}(X) = A_f. \]

7. OPEN QUESTIONS

This investigation began in the hope of better understanding the ranks of modular elliptic curves. While the bounds on ranks we have obtained here are better than those we had obtained earlier for elliptic curves ([6], [8]), the table at the end of this paper shows that we are still far from the truth.

Our experiments with \( J_0^-(N) \) for \( N \) prime suggest that the Atkin-Lehner subvarieties \( A_X(N) \) might consist mainly of simple abelian subvarieties whose dimension grows with \( N \). When \( N \) is highly divisible by 2 or 3, the additional special Atkin-Lehner operators should also be used, since they provide additional splitting. Such abelian varieties are likely to have normalized rank 0 or 1 according to their parity, since they have “small conductors” compared to their dimension.

We mention a few interesting questions.

**Problem 1:** Is the average normalized rank of \( J_0(N) = \frac{1}{2} \) ? More precisely, does one have \( \text{nrk}(A_X(N)) = \eta + o(\dim A_X(N)) \), with \( \eta = \frac{1}{2}(1 + \chi(N)) \)?

**Problem 2:** Consider abelian varieties of fixed dimension, perhaps even with fixed endomorphism structure. Is there a definite positive proportion of each (normalized) rank? What can be said about the average normalized rank?

Some condition on the endomorphism structure must be imposed to ensure that the moduli space have infinitely many rational points, since families of “twists” are likely to have average normalized rank \( \frac{1}{2} \). There are probably very few totally real number fields \( \mathbb{E} \) which can be the field of endomorphisms of infinitely many isomorphism \( \mathbb{C} \)-classes of abelian varieties of \( \text{GL}_2 \)-type defined over \( \mathbb{Q} \). It may be worthwhile to pursue calculations similar to ours for composite conductors to see whether the behaviour suggested by the splittings of \( J_0^-(N) \) in our table prevails. In the case of elliptic curves, our numerical investigations [10] suggested Problem 2 (cf. the table on the next page) while ([6], [8]) addressed the second question.

**Problem 4:** Assume that \( N \) is prime to 6 and that \( \chi \) is a character of the Atkin-Lehner group. Is \( A_X(N) \) a simple abelian variety up to a piece of dimension at most \( O(\sqrt{N}) \)?

When \( N \) is squarefree, small factors of dimension at most 3 or 4 occasionally appear in \( A_X(N) \), but note \( N = 4751 \) in our main Table. In contrast, when all
exponents in $N$ are large, we deduce from (5.5) that the dimensions of simple factors have size at least of the order of $\sqrt{N}$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Field} & \text{Eis. pieces} & \text{Non-Eis. rank 0} & \text{nrk} \geq 2 & \text{Total} \\
\hline
\mathbb{Q} & 23 & 107 & 59 & 189 \\
\mathbb{Q}(\sqrt{5}) & 7 & 27 & 33 & 67 \\
\mathbb{Q}(\sqrt{2}) & 3 & 12 & 2 & 17 \\
\mathbb{Q}(\sqrt{13}) & 2 & 6 & 1 & 9 \\
\text{Other quad.} & 1 & 3 & 0 & 4 \\
\mathbb{Q}(\cos(\frac{2\pi}{7})) & 0 & 2 & 13 & 15 \\
\text{Other cubics} & 6 & 11 & 2 & 19 \\
\text{All quartics} & 4 & 5 & 3 & 12 \\
\hline
\end{array}
\]

The table above gives a summary of our more detailed table in the next section. It highlights the relative frequency of factors of normalized analytic rank at least two when there is any splitting at all among the varieties $J_0^-(N)$ with $N$ prime at most 10 000. The first column gives the field of endomorphisms, the second lists the number of Eisenstein factors, the third the non-Eisenstein pieces of rank 0, while the fourth contains the number whose analytic rank is at least twice their dimension. We hope the reader will agree that the proportions are quite remarkable given the small size of the conductors involved.

8. TABLE OF SPLITTINGS OF $J_0(N)$

This section contains the table for those primes $N$ for which $J_0^-(N)$ splits non-trivially, up to isogeny. The column after the conductor $N$ contains the dimensions of all the simple factors of $J_0^-(N)$, the "Comments" column contains the discriminant $D$ of the corresponding totally real field of multiplications, for those factors of dimension between 2 and 5 and the number of relations between the theta functions of the ternary quadratic forms of determinant $2N$ (cf. [19]). This provides a lower bound for the analytic rank of $J_0^-(N)$ and presumably an upper bound for the algebraic rank, as a tedious descent might verify.

The factor of largest dimension is usually the only Eisenstein piece and when that is the case, no special mention is made. Otherwise, we use the convention introduced by Mazur in [20] of indicating in bold face with subscript $p$ quotients of the Eisenstein factor $\tilde{J}^p$. 

63
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### The Rank of \( J_0(N) \)

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### REFERENCES


(A. Brumer) MATHEMATICS DEPT, FORDHAM UNIVERSITY, BRONX NY 10458

E-mail address, A. Brumer: brumer@murray.fordham.edu