

Astérisque

MARCO BRUNELLA

**Vanishing holonomy and monodromy of
certain centres and foci**

Astérisque, tome 222 (1994), p. 37-48

[<http://www.numdam.org/item?id=AST_1994__222__37_0>](http://www.numdam.org/item?id=AST_1994__222__37_0)

© Société mathématique de France, 1994, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

VANISHING HOLONOMY AND MONODROMY OF CERTAIN CENTRES AND FOCI

MARCO BRUNELLA

Introduction

Let $\omega(x, y) = A(x, y)dx + B(x, y)dy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at the origin: $A(0, 0) = B(0, 0) = 0$, $\dim_{\mathbf{R}} \frac{\mathbf{R}\{x, y\}}{(A, B)} < +\infty$.

The singularity $\omega = 0$ is called *monodromic* if there are not separatrices at 0. In this case, given a germ of an analytic embedding $(\mathbf{R}^+, 0) \xrightarrow{\tau} (\mathbf{R}^2, 0)$ transverse to ω outside 0, it is possible to define a *monodromy map* $P_{\omega, \tau}: (\mathbf{R}^+, 0) \rightarrow (\mathbf{R}^+, 0)$, following clockwise the solutions of $\omega = 0$; $P_{\omega, \tau}$ is a germ of homeomorphism of $(\mathbf{R}^+, 0)$ analytic outside 0. If $P_{\omega, \tau} = id$ then $\omega = 0$ is called *centre*. Otherwise $P_{\omega, \tau}$ is a contraction or an expansion (by the results of Écalle, Il'yashenko, Martinet, Moussu, Ramis... on "Dulac conjecture") and $\omega = 0$ is called *focus*.

The simplest monodromic singularities are those for which the linear part M_{ω} of the dual vector field $v(x, y) = B(x, y)\frac{\partial}{\partial x} - A(x, y)\frac{\partial}{\partial y}$ is nondegenerate, i.e. invertible. We distinguish two situations:

- i) the eigenvalues λ, μ of M_{ω} are complex conjugate, non real, with real part different from zero. Then $\omega = 0$ is a focus and it is analytically equivalent to $\omega_{lin} = 0$, where ω_{lin} denotes the linear part of ω (Poincaré's linearization theorem).
- ii) the eigenvalues λ, μ of M_{ω} are complex conjugate, non real, with zero

real part. Then if $\omega = 0$ is a centre there exists an analytic first integral (Lyapunov-Poincaré theorem, see [Mou1] and references therein) and $\omega = 0$ is analytically equivalent to $x dx + y dy = 0$. If $\omega = 0$ is a focus the analytic classification is a difficult problem, which requires the theory of Écalle-Martinet-Ramis-Voronin to pass from the formal classification to the analytic one ([M-R]). The monodromy is an analytic diffeomorphism tangent to the identity, and two such equations are analytically equivalent if and only if their monodromies are ([M-R]).

In this paper we shall study the simplest degenerate monodromic singularities, i.e. those with $\lambda = \mu = 0$, $\omega_{lin} \neq 0$, and with “generic” higher order terms. Modulo a change of coordinates ([Mou2]), we may work in the following class.

Definition. Let $\omega = A dx + B dy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at 0. This singularity is called *monodromic semidegenerate* if the first nonzero quasihomogeneous jet of type (1,2) of ω is

$$\omega_0(x, y) = x^3 dx + (y + ax^2) dy$$

with $a^2 < 2$. Notation: $\omega \in MSD(a)$.

We will denote by P_ω the monodromy map of $\omega \in MSD(a)$ corresponding to the embedding $(\mathbf{R}^+, 0) \hookrightarrow (\mathbf{R}^2, 0)$, $t \mapsto (t, 0)$. P_ω is a germ of analytic diffeomorphism tangent to the identity ([Mou2]), and we may consider P_ω as the restriction to $\mathbf{R}^+$ of a germ of biholomorphism of $(\mathbf{C}, 0)$, tangent to the identity, again denoted by P_ω .

Let $\omega \in MSD(a)$ and let Ω be the germ of holomorphic 1-form on $\mathbf{C}^2$ obtained by complexification of ω . Using a resolution of the singularity we may define as in [Mou3] and [C-M] the *vanishing holonomy* of Ω : it is a

subgroup $H(\Omega) \subset Bh(\mathbf{C}, 0) = \{ \text{group of germs of biholomorphisms of } (\mathbf{C}, 0) \}$, generated by $f, g \in Bh(\mathbf{C}, 0)$ satisfying the relation $(f \circ g)^2 = id$.

Our result is a computation of P_ω in terms of $H(\Omega)$. A similar result was remarked by Moussu in the (simpler) case of nondegenerate monodromic singularities ([Mou1]).

Theorem. *Let $\omega = 0$ be monodromic semidegenerate, then*

$$P_\omega = [f, g]$$

In particular, $H(\Omega)$ is abelian if and only if $\omega = 0$ is a centre. This means, by [C-M], that a nontrivial space of “formal-analytic moduli” can appear only if $\omega = 0$ is a centre (and $a = 0$, see below): for the foci, formal equivalence $\Rightarrow$ analytic equivalence. Hence our situation is very different from the situation of equations of the type $x dx + y dy + \dots = 0$, where the difficult case is the case of foci whereas all the centres are analytically equivalent (here the vanishing holonomy is always abelian, generated by a single $f \in Bh(\mathbf{C}, 0)$, and the monodromy is given by f^2 , see [Mou1]). On the other hand, it is no more true that the monodromy characterizes the equation: it may happen that $\omega_1, \omega_2 \in MSD(a)$ have the same monodromy without being analytically equivalent.

A consequence of the above relation between monodromy and vanishing holonomy is the following normal form theorem for centres, based again on the results of [C-M]. Let us before remark that $\omega_0(x, y) = x^3 dx + (y + ax^2) dy = 0$ is a centre for any $a \in \mathbf{R}$ (but a first integral exists if and only if $a = 0$).

Corollary 1. *Let $\omega \in MSD(a)$ be a centre and let $a \neq 0$, then the germ $\omega = 0$ is analytically equivalent to $\omega_0 = 0$.*

We don’t know a similar explicit and “simple” (polynomial?) normal form for foci, even in the case $a \neq 0$; but the triviality of the space of formal-

analytic moduli seems here a useful tool. The classification of centres with $a = 0$ requires arguments of the type Écalle - Martinet - Ramis - Voronin (cfr. [C-M]).

As another corollary of the above theorem we give a positive answer to a question posed by Moussu in [Mou2].

Corollary 2. *Let $\omega = 0$ be a monodromic semidegenerate centre, then there exists a nontrivial analytic involution $I : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ which preserves the solutions of $\omega = 0$: $I^*(\omega) \wedge \omega = 0$.*

The above computation may be generalized to the case of germs ω whose first nonzero quasihomogeneous jet of type $(1, n)$ is

$$\omega_0(x, y) = x^{2n-1}dx + (y + ax^n)dy$$

with $a^2 < \frac{1}{4}n$ ([Mou2]). The vanishing holonomy $H(\Omega)$ for these germs is generated by $f, g \in Bh(\mathbf{C}, 0)$ satisfying $(f \circ g)^n = id$ ([C-M]). But now, if $n \geq 3$, the relation between commutativity of $H(\Omega)$ and triviality of P_ω becomes more complicated; in particular, it is no more true that there is equivalence between “ $H(\Omega)$ abelian” and “ $P_\omega = id$ ”.

The computation of P_ω in terms of $H(\Omega)$ for $n \geq 3$ is straightforward, once one has understood the case $n = 2$. Hence, for sake of simplicity and clarity, we have choose to limit ourselves to the semidegenerate monodromic singularities.

Acknowledgements: I thank R. Moussu and A. Verjovsky who read the manuscript and suggested me some improvements of the exposition.

Resolution of singularities and vanishing holonomy

Let $\omega \in MSD(a)$ and let Ω be its complexification. We recall the desingularization of Ω and the construction of $H(\Omega)$ ([C-M], [Mou3]).

We denote by M the complex manifold of dimension two covered by 3 charts $U_j = \{(x_j, y_j)\} \simeq \mathbf{C}^2$, $j = 1, 2, 3$, glued together by the identifications

$$\begin{cases} x_1 = \frac{1}{y_2} \\ y_1 = x_2 y_2^2 \end{cases} \quad \begin{cases} x_3 = x_2 y_2 \\ y_3 = \frac{1}{x_2} \end{cases} \quad \begin{cases} x_3 = x_1 y_1 \\ y_3 = \frac{1}{x_1^2 y_1} \end{cases}$$

Let $h : M \rightarrow \mathbf{C}^2$ be the holomorphic map whose expressions h_j in the charts U_j are

$$h_1(x_1, y_1) = (x_1 y_1, y_1), \quad h_2(x_2, y_2) = (x_2 y_2, x_2 y_2^2), \quad h_3(x_3, y_3) = (x_3, x_3^2 y_3)$$

The *divisor* $Z \stackrel{\text{def}}{=} h^{-1}((0, 0))$ is a union of two copies of $\mathbf{CP}^1$, which intersect transversally at a point p . If $Z_j = Z \cap U_j$, then

$$Z_1 = \{y_1 = 0\}, \quad Z_2 = \{x_2 = 0\} \cup \{y_2 = 0\}, \quad Z_3 = \{x_3 = 0\}$$

The map $h|_{M \setminus Z} : M \setminus Z \rightarrow \mathbf{C}^2 \setminus \{(0, 0)\}$ is a biholomorphism.

On M there is naturally defined an involution $j : M \rightarrow M$ given, in every chart U_j , by $j(x_j, y_j) = (\bar{x}_j, \bar{y}_j)$. The set $M^{\mathbf{R}}$ of fixed points of j is a real analytic manifold, the map h restricts to a real analytic map $h^{\mathbf{R}} : M^{\mathbf{R}} \rightarrow \mathbf{R}^2$, $Z^{\mathbf{R}} \stackrel{\text{def}}{=} (h^{\mathbf{R}})^{-1}((0, 0)) = Z \cap M^{\mathbf{R}}$ is a union of two copies of $\mathbf{RP}^1$ intersecting transversally at p . The map $h^{\mathbf{R}}|_{M^{\mathbf{R}} \setminus Z^{\mathbf{R}}} : M^{\mathbf{R}} \setminus Z^{\mathbf{R}} \rightarrow \mathbf{R}^2 \setminus \{(0, 0)\}$ is a real analytic diffeomorphism. The manifold $M^{\mathbf{R}}$ is covered by the charts $U_j^{\mathbf{R}} \stackrel{\text{def}}{=} U_j \cap M^{\mathbf{R}} \simeq \mathbf{R}^2$, $j = 1, 2, 3$, and we will denote again with (x_j, y_j) the corresponding coordinates.

From now on we will consider only the germs of the previous objects (M , h , $M^{\mathbf{R}}$, etc.) along Z or $Z^{\mathbf{R}}$, denoted by the same symbols.

Define $\tilde{\Omega} = h^*(\Omega)$. Its local expressions are

$$\tilde{\Omega}_1 = y_1[(1 + \mathcal{O}(|y_1|))dy_1 + (\mathcal{O}(|y_1|))dx_1]$$

$$\tilde{\Omega}_2 = x_2 y_2^3[(2x_2 + 2ax_2^2 + x_2^3 + \mathcal{O}(|x_2 y_2|))dy_2 + (y_2 + \mathcal{O}(|x_2 y_2|))dx_2]$$

$$\tilde{\Omega}_3 = x_3^3[(1 + 2ay_3^3 + 2y_3^2 + \mathcal{O}(|x_3|))dx_3 + (\mathcal{O}(|x_3|))dy_3]$$

The only singularities of the foliation $\tilde{\mathcal{F}}$ defined by $\tilde{\Omega}$ are:

- the point p ($= (0,0)$ in U_2), where $\bar{\Omega}_2 \stackrel{def}{=} \frac{1}{x_2 y_2^3} \tilde{\Omega}_2$ has a singularity of the type “2:1 resonant saddle”:

$$\bar{\Omega}_2 = 2x_2 dy_2 + y_2 dx_2 + h.o.t.$$

- the points

$$q_1 = (x_2 = -a + i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a - i\sqrt{2-a^2}))$$

$$q_2 = (x_2 = -a - i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a + i\sqrt{2-a^2})) = j(q_1)$$

where $\bar{\Omega}_2$ has hyperbolic singularities (the ratio of the eigenvalues is not real) if $a \neq 0$, and saddles with 4 : 1 resonance if $a = 0$.

We denote by $W_0 \simeq \mathbf{CP}^1$ the component of Z containing q_1 and q_2 , and by $W_1 \simeq \mathbf{CP}^1$ the other component; $W_0 \setminus \{p, q_1, q_2\}$ and $W_1 \setminus \{p\}$ are regular leaves of $\tilde{\mathcal{F}}$.

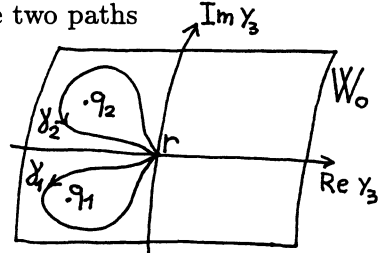
Let $L = cl(h^{-1}(\{y = 0\}) \setminus Z) = \{y_3 = 0\}$ and $r = L \cap W_0 = (x_3 = 0, y_3 = 0)$. Let $\gamma_j : [0, 1] \rightarrow W_0 \setminus \{p, q_1, q_2\}$, $j = 1, 2$, be two paths

such that $\gamma_j(0) = \gamma_j(1) = r$, $ind_{\gamma_j}(q_i) = \delta_{ij}$.

To these paths there correspond two germs

of biholomorphisms $f_j : (L, r) \rightarrow (L, r)$,

given by the holonomy of the foliation $\tilde{\mathcal{F}}$.



We set $f, g \in Bh(\mathbf{C}, 0)$ equal respectively to f_1, f_2 expressed using the coordinate x_3 on L .

Definition ([C-M]). The *vanishing holonomy* $H(\Omega)$ of Ω is the subgroup of $Bh(\mathbf{C}, 0)$ generated by f and g .

An elementary computation shows that

$$f'(0) = -i \cdot \exp\left(-\frac{a}{2\sqrt{2-a^2}}\right) \quad g'(0) = -i \cdot \exp\left(+\frac{a}{2\sqrt{2-a^2}}\right)$$

in particular, $H(\Omega)$ is *hyperbolic* (i.e. $|f'(0)| \neq 1$, $|g'(0)| \neq 1$) if and only if $a \neq 0$.

Near the singularity p there exists a first integral, of the form $x_2 y_2^2 + h.o.t.$ ([C-M]). This implies that the holonomy of $\tilde{\mathcal{F}}$ along $\gamma_1 * \gamma_2$ or $\gamma_2 * \gamma_1$ (which are freely homotopic in $W_0 \setminus \{p, q_1, q_2\}$ to small paths around p) is periodic, of period 2. Hence:

$$(f \circ g)^2 = (g \circ f)^2 = id$$

Remark that from the fact that Ω has real coefficients we deduce that

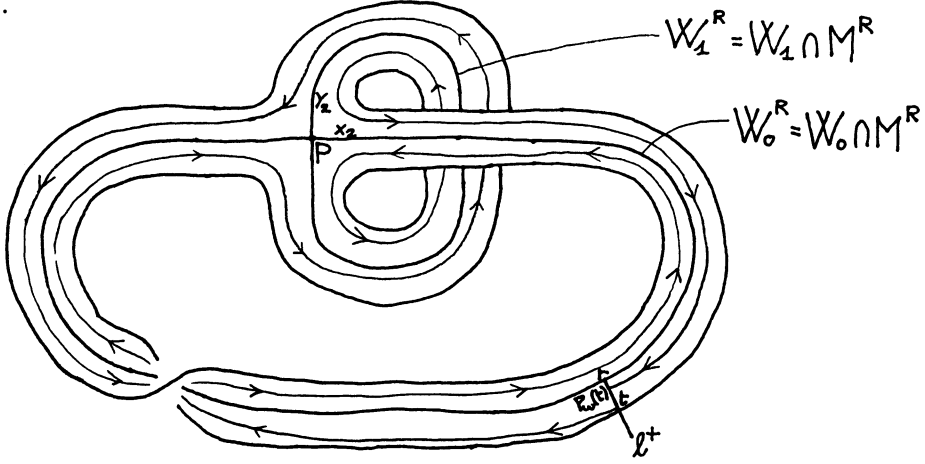
$$g(z) = \overline{f^{-1}(\bar{z})}$$

and, because p is a real point of M , $f \circ g$ and $g \circ f$ are real:

$$(f \circ g)(\bar{z}) = \overline{(f \circ g)(z)} \quad \text{and} \quad (g \circ f)(\bar{z}) = \overline{(g \circ f)(z)}$$

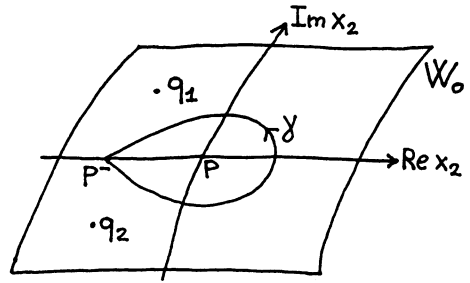
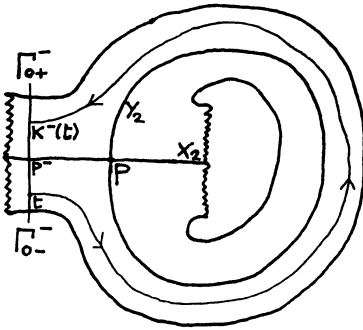
Now we turn to the real 1-form ω . Clearly, $h^{\mathbf{R}} : M^{\mathbf{R}} \rightarrow \mathbf{R}^2$ gives a resolution of the singularity. We set $l^+ = cl((h^{\mathbf{R}})^{-1}(\{y = 0, x \geq 0\}) \setminus Z^{\mathbf{R}}) = L \cap M^{\mathbf{R}} \cap \{x_3 \geq 0\}$, then the holonomy of the foliation $\tilde{\mathcal{G}}$ defined by $\tilde{\omega} = (h^{\mathbf{R}})^*(\omega)$ along the “polycycle” $Z^{\mathbf{R}}$ produces a germ of analytic diffeomorphism of (l^+, r) . Using the coordinate x_3 on l^+ and complexifying the result we obtain a germ of biholomorphism of $(\mathbf{C}, 0)$ which is nothing else than the (complex) monodromy P_ω of ω (for the appropriate choice of orientation of

$\tilde{g}$).



Proof of the theorem

Let $\Gamma^- \subset M$ be a germ of complex line, j -symmetric, transverse to $\tilde{\mathcal{F}}$, passing through a point p^- of W_0 near p with coordinates (in the chart U_2) $(-\epsilon, 0)$, $\epsilon \in \mathbb{R}^+$ small. Let $\Gamma_0^- = \Gamma^- \cap M^{\mathbb{R}}$ be its real part; it is a germ of real line transverse to $\tilde{g}$. The holonomy of $\tilde{g}$ along the polycycle composed by the segment from p^- to p and $W_1^{\mathbb{R}} \stackrel{\text{def}}{=} W_1 \cap M^{\mathbb{R}}$ gives a germ of homeomorphism $k^- : (\Gamma_{0-}^-, p^-) \rightarrow (\Gamma_{0+}^-, p^-)$, where $\Gamma_{0-}^- = \Gamma_0^- \cap \{y_2 \leq 0\}$, $\Gamma_{0+}^- = \Gamma_0^- \cap \{y_2 \geq 0\}$.



On the other hand, we may consider a path $\gamma : [0, 1] \rightarrow W_0 \setminus \{p, q_1, q_2\}$ with $\gamma(0) = \gamma(1) = p^-$ and $\text{ind}_\gamma(p) = 1$, $\text{ind}_\gamma(q_1) = \text{ind}_\gamma(q_2) = 0$. The holonomy of $\tilde{\mathcal{F}}$ along γ induces a germ of biholomorphism $K^- : (\Gamma^-, p^-) \rightarrow (\Gamma^-, p^-)$, which is an involution because of the first integral of $\tilde{\Omega}$ near p .

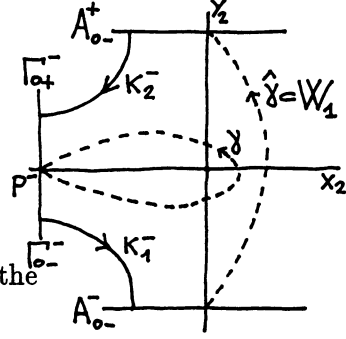
Lemma.

$$k^- = K^-|_{\Gamma_{0-}^-}$$

Proof.

let $A^+, A^- \subset M$ be two germs of complex lines, j -symmetric, transverse to $\tilde{\mathcal{F}}$ and passing through $(0, \epsilon)$, $(0, -\epsilon)$ (in the chart U_2). Let A_0^+, A_0^- be their real parts, $A_{0-}^+ = A_0^+ \cap \{x_2 \leq 0\}$, $A_{0-}^- = A_0^- \cap \{x_2 \leq 0\}$. The homeomorphism k^- is the composition of a homeomorphism k_1^- from Γ_{0-}^- to A_{0-}^- , an analytic diffeomorphism $k^* : A_0^- \rightarrow A_0^+$, and a homeomorphism k_2^- from A_{0-}^+ to Γ_{0+}^- .

Because $W_1 \setminus \{p\}$ is simply connected, the path which joins $(0, -\epsilon)$ to $(0, \epsilon)$ along $W_1^{\mathbf{R}} \setminus \{p\}$ is contractible in $W_1 \setminus \{p\}$ (endpoints fixed) to a path $\hat{\gamma}$ contained in $\{|y_2| \leq \epsilon\}$. Hence k^* is the restriction to A_0^- of a biholomorphism $K^* : A^- \rightarrow A^+$, obtained from the holonomy of $\tilde{\mathcal{F}}$ along this path $\hat{\gamma}$.



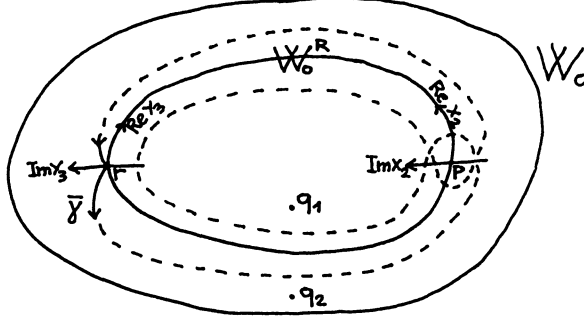
We choose ϵ so small that $\tilde{\Omega}$ has a first integral $x_2 y_2^2 + \dots$ defined on $U_\epsilon = \{|x_2| \leq \epsilon, |y_2| \leq \epsilon\}$. Hence every leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ different from a separatrix at p either does not intersect $U_\epsilon^{\mathbf{R}} = U_\epsilon \cap M^{\mathbf{R}}$, or it intersects $U_\epsilon^{\mathbf{R}}$ along two segments “symmetric” w.r. to the x -axis. We deduce that:

- i) if $t \in A_{0-}^-$, then $K^*(t) \in A_{0-}^+$ is the only intersection of A_{0-}^+ with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through t ;
- ii) if $s \in \Gamma_{0-}^-$, then $K^-(s) \in \Gamma_{0+}^-$ is the only intersection of Γ_{0+}^- with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through s .

From these two remarks, it is clear that $k_2^- \circ (K^*|_{A_{0-}^-}) \circ k_1^-$ is equal to $K^-|_{\Gamma_{0-}^-}$, i.e. $k^- = K^-|_{\Gamma_{0-}^-}$. Q.E.D.

Obviously, a similar result holds if we start from $\Gamma^+ =$ germ of complex line j -symmetric passing through $(\epsilon, 0) = p^+$, etc..

As a consequence, the complex monodromy $P_\omega : (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$ may be computed as the holonomy of $\tilde{\mathcal{F}}$ along a path $\bar{\gamma} : [0, 1] \rightarrow W_0 \setminus \{p, q_1, q_2\}$, $\bar{\gamma}(0) = \bar{\gamma}(1) = r$, as in the following picture:



This path is homotopic to $\gamma_2^{-1} * \gamma_1^{-1} * \gamma_1^{-1} * \gamma_2^{-1}$, hence

$$P_\omega = g^{-1} \circ f^{-1} \circ f^{-1} \circ g^{-1}$$

and from $(g \circ f)^2 = id$ we conclude

$$P_\omega = f \circ g \circ f^{-1} \circ g^{-1} = [f, g]$$

Q.E.D.

Proof of corollary 1

It is sufficient, using the path-lifting argument of [C-M], to show that the vanishing holonomies $H(\Omega)$ and $H(\Omega_0)$ are holomorphically conjugate. The “assertion 1” of [C-M], pag. 478, is here replaced by

Assertion 1_a: if $\omega \in MSD(a)$ then there are analytic coordinates (x, y) near $(0, 0)$ s.t. ω is, modulo multiplication by a nonvanishing germ:

$$\omega(x, y) = x^3 dx + (y + ax^2) dy + f(x, y)(2y dx - x dy), \quad f \in \mathbf{R}\{x, y\}$$

The proof of this normal form lemma is achieved as in [C-M]: $\Omega = 0$ has a separatrix $X^4 + 2aX^2Y + 2Y^2 = 0$ (in suitable coordinates, preserving the

class $MSD(a)$, which is a separatrix also for $2YdX - XdY = 0$. Hence:

$$\Omega \wedge (X^3dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_1(X, Y)dX \wedge dY$$

$$\Omega \wedge (2YdX - XdY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_2(X, Y)dX \wedge dY$$

$$(2YdX - XdY) \wedge (X^3dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot dX \wedge dY$$

and from these formulae the assertion 1_a follows.

Let us denote by f_0, g_0 the generators of $H(\Omega_0)$; $[f_0, g_0] = id$ because ω_0 is a centre, moreover

$$\begin{aligned} f'_0(0) &= f'(0) = \lambda \\ g'_0(0) &= g'(0) = \frac{1}{\lambda} = \frac{-1}{\lambda} \end{aligned}$$

and $|\lambda| \neq 1$ because $a \neq 0$.

From the commutativity and the hyperbolicity of $H(\Omega)$ we deduce that $H(\Omega)$ is holomorphically conjugate to the group generated by

$$z \mapsto \lambda z \quad \text{and} \quad z \mapsto \frac{1}{\lambda} z$$

For the same reasons, $H(\Omega_0)$ also is holomorphically conjugate to that linear group, hence to $H(\Omega)$. Q.E.D.

Proof of corollary 2

Consider the germ $f \circ g \in Bh(\mathbf{C}, 0)$: it is real ($((f \circ g)(\bar{z}) = \overline{(f \circ g)(z)})$), periodic with period 2, and conjugates $H(\Omega)$ with itself thanks to $[f, g] = id$:

$$(f \circ g) \circ g = (g \circ f) \circ g = g \circ (f \circ g)$$

$$(f \circ g) \circ f = f \circ (g \circ f) = f \circ (f \circ g)$$

As in corollary 1, we use the path-lifting technique of [C-M] to suspend $(f \circ g)$ and to obtain a germ of biholomorphism $\tilde{I} : M \rightarrow M$, which preserves $\tilde{\mathcal{F}}$:

$\tilde{I}^*(\tilde{\Omega}) \wedge \tilde{\Omega} = 0$. From the fact that $(f \circ g)$ is a real involution, we obtain that $\tilde{I}$ is also a real involution. Taking the projection on $\mathbf{C}^2$ and the restriction to $\mathbf{R}^2$ we obtain the required analytic involution. Q.E.D.

Remark: if $a \neq 0$ the result follows also from corollary 1: ω_0 is invariant by $(x, y) \mapsto (-x, y)$.

References

[C-M] D. Cerveau, R. Moussu: *Groupes d'automorphismes de $(\mathbf{C}, 0)$ et équations différentielles $ydy + \dots = 0$* , Bull. Soc. Math. France 116 (1988), 459-488.

[Mou1] R. Moussu: *Une démonstration géométrique d'un théorème de Poincaré - Lyapunov, in Bifurcations, théorie ergodique et applications (Dijon)*, Astérisque 98-99 (1982), 216-223.

[Mou2] R. Moussu: *Symétrie et forme normale des centres et foyers dégénérés*, Erg. Th. and Dyn. Sys. 2 (1982), 241-251.

[Mou3] R. Moussu: *Holonomie évanescence des équations différentielles dégénérées transverses*, in *Singularities and dynamical systems (ed. Pneumatikos)*, Elsevier (1985), 161- 173.

[M-R] J. Martinet, J.-P. Ramis: *Analytic classification of resonant saddles and foci*, in *Singularities and dynamical systems (ed. Pneumatikos)*, Elsevier (1985), 109-135.

Marco Brunella
S.I.S.S.A.
Strada Costiera 11
34014 Trieste, Italy