ERIC AMAR

A problem of ideals

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Recently U. Cegrell [2] proved the following result:

**Theorem.** Let $\mathcal{B}$ be the unit ball of $\mathbb{C}^n$ and $f_1 \in A(\mathcal{B})$, $f_2 \in H^\infty(\mathcal{B})$ such that $\forall z \in \mathcal{B}$, $|f_1(z)| + |f_2(z)| \geq \delta$ then there are two functions $g_1$, $g_2$ in $H^\infty(\mathcal{B})$ such that: $f_1g_1 + f_2g_2 = 1$ in $\mathcal{B}$.

where, as usual, if $\Omega$ is a domain in $\mathbb{C}^n$, $A(\Omega)$ is the algebra of all holomorphic functions in $\Omega$ continuous up to the boundary, $A^k(\Omega)$ is the algebra of all holomorphic functions in $\Omega$, $C^k$ up to $\partial \Omega$ and $H^\infty(\Omega)$ is the algebra of all holomorphic and bounded functions in $\Omega$.

This means, in this special case, that the Corona is true. He uses a nice analysis of pic functions and representing measures in the ball, of independant interest.

The aim of this note is to give a very simple proof of this theorem which is also more general.

In order to state the result, let me give the following definition:

**Definition.** We say that the bounded pseudo-convex domain in $\mathbb{C}^n$ has the $L^\infty_q$ property if: for any $(0, q)$ form $\omega$ in $C^\infty(\Omega) \cap L^\infty(\Omega)$, there is a $(0, q - 1)$ form $u$ in $C^\infty(\Omega) \cap L^\infty(\Omega)$ such that: $\bar{\partial}u = \omega$.

As usual, a $(0,0)$ form is just a function.
There are many examples of such domains: the strictly pseudo-convex ones [6], the polydiscs [8], the ellipsoids [4], [11], the domains of finite type in \( \mathbb{C}^2 \) [3], [5].

We shall prove the following:

**Theorem 1.** Let \( \Omega \) be a pseudo-convex bounded domain in \( \mathbb{C}^n \) verifying the \( L_1^\infty \) condition and let \( f_1 \in A(\Omega) \), \( f_2 \in H^\infty(\Omega) \) such that \( \forall z \in \Omega, |f_1(z)| + |f_2(z)| \geq \delta \) then there are two functions \( g_1, g_2 \) in \( H^\infty(\Omega) \) such that: \( f_1 g_1 + f_2 g_2 = 1 \) in \( \Omega \).

**Proof:**

because \( f_1 \) is continuous up to \( \partial \Omega \), it is easy to make a function \( \chi \in C^\infty(\overline{\Omega}) \) such that:

\[
\chi = \begin{cases} 
1 & \text{in } \{ |f_1| > \delta/2 \} \\
0 & \text{in } \{ |f_1| < \delta/4 \}
\end{cases}
\]

Now let \( \omega := \frac{\partial \chi}{f_1 f_2} \), then \( \omega \in C^\infty(\Omega) \cap L^\infty(\Omega) \) because on the set where \( \overline{\partial} \chi \neq 0 \), \( |f_1 f_2| > \delta^2/16 \). Moreover, \( \overline{\partial} \omega = 0 \) in \( \Omega \), hence, by the \( L_1^\infty \) condition, there is a \( u \in L^\infty(\Omega) \) such that: \( \overline{\partial} u = \omega \).

Let us define

\[
g_1 := \frac{\chi}{f_1} - u f_2 \quad \text{and} \quad g_2 := \frac{1 - \chi}{f_2} + u f_1;
\]

then we get:

\[
\overline{\partial} g_1 = 0, \quad \overline{\partial} g_2 = 0
\]

hence these functions are holomorphic in \( \Omega \) and:

\[
f_1 g_1 + f_2 g_2 = 1.
\]

Moreover the \( g_i \)'s are easily seen to be bounded in \( \Omega \), hence the theorem.  

Now using the Koszul’s Complex method as in [9], it is easy to prove, using exactly the same lines the:

**Theorem 2.** Let \( \Omega \) be a pseudo-convex bounded domain in \( \mathbb{C}^n \) verifying the \( L_q^\infty \) condition for \( q \leq p - 1 \) and let \( f_1, \ldots, f_{p-1} \in A(\Omega), f_p \in H^\infty(\Omega) \) such that \( \forall z \in \mathbb{B} \sum_{i=1}^{p} |f_i(z)| \geq \delta; \) then there are \( p \) functions \( g_1, \ldots, g_p \) in \( H^\infty(\Omega) \) such that: \( \sum_i f_i g_i = 1 \) in \( \Omega \).

Now let us define the \( C_p^k \) property for a pseudo-convex bounded domain in an analogous way:
**Definition.** We say that the bounded pseudo-convex domain in $\mathbb{C}^n$ has the $C^k_q$ property if: for any $(0, q)$ form $\omega$ in $C^k(\Omega)$, there is a $(0, q - 1)$ form $u$ in $C^k(\Omega)$ such that: $\bar{\partial}u = \omega$.

For $k$ finite, the domains listed above with the $L^\infty_q$ property have the $C^k_q$ property too. For $k = \infty$ a very famous theorem by J.J. Kohn [10] says that all pseudo-convex bounded domains with smooth boundary has the $C^\infty_q$ property.

The same way has above, we can show:

**Theorem 3.** Let $\Omega$ be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the $C^k_q$ condition for $q \leq p - 1$ and let $f_1, ..., f_p \in A^k(\Omega)$, such that $\forall z \in \Omega \sum_{i=1}^{p} |f_i(z)| \geq \delta$; then there are $p$ functions $g_1, ..., g_p$ in $A^k(\Omega)$ such that: $\sum_i f_ig_i = 1$ in $\Omega$.

As a classical corollary we get:

**Corollary.** Let $\Omega$ be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the $C^k_q$ condition for $q \leq n$, then the spectrum of the algebra $A^k(\Omega)$ is $\overline{\Omega}$.

In the case $k = \infty$, M. Catlin [1] and M. Hakim and N. Sibony [7] already proved this result, the method they used is also a division method but slightly different and their method cannot give theorem 1 and 2 here.

**References**

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Amar Eric
Université Bordeaux I
351 Cours de la Libération
33405 Talence, France