MEI-CHI SHAW

Semi-global existence theorems of $\bar{\partial}_b$ for $(0, n - 2)$ forms on pseudo-convex boundaries in $\mathbb{C}^n$


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SEMI-GLOBAL EXISTENCE THEOREMS
OF $\partial_b$ FOR $(0, n-2)$ FORMS
ON PSEUDO-CONVEX BOUNDARIES IN $\mathbb{C}^n$

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INTRODUCTION

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Let $M$ be the boundary of a pseudo-convex domain $D$ in $\mathbb{C}^n$, $n \geq 2$. We consider the tangential Cauchy-Riemann equations

\begin{equation}
\partial_b u = \alpha
\end{equation}

on an open subset $\omega \subset M$, where $\alpha$ is a $(p, q)$ form in $\omega$, $0 \leq p \leq n$, $1 \leq q < n - 1$. Since $\partial_b^2 = 0$, in order for Eq.(0.1) to be solvable, $\alpha$ must satisfy the compatibility condition

\begin{equation}
\partial_b \alpha = 0 \quad \text{in} \quad \omega.
\end{equation}

Recently, the semi-global existence results have been obtained by the author for any $(p, q)$ form $\alpha$, where $1 \leq q \leq n - 3$, such that $\omega$ is a pseudo-convex boundary of finite type as defined in D'Angelo [8]. It is proved in [22] that when $\partial \omega$ lies in a flat or a Levi-flat hypersurface which has a Stein neighborhood basis, then Eq.(0.1) is solvable for all $(p, q)$ forms $\alpha$ satisfying condition (0.2), where $1 \leq q \leq n - 3$. When $q = n - 1$ Eq.(0.1) corresponds to the Lewy equation and it is well-known that for most $\alpha$ it is not solvable locally unless $\omega$ is Levi-flat (see Hörmander [11]).

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In this paper we shall discuss the local and semi-global existence theorems for the remaining case, i.e., when \( q = n - 2 \). It was observed by Rosay [19] that when \( q = n - 2 \), condition (0.2) is not sufficient for Eq.(0.1) to be solvable in \( \omega \). In fact, there is an additional compatibility condition that \( \alpha \) must satisfy in order for Eq.(0.1) to be solvable. This additional condition, called condition (A), is a condition on the boundary of \( \partial \omega \) and will be derived in Section I. Our main purpose in this paper is to show that condition (0.2) and condition (A) are the necessary and sufficient conditions for Eq.(0.1) to be solvable when \( \alpha \) is a smooth \((p, n - 2)\) form. We also characterize those domains on which condition (0.2) always implies condition (A) (see Proposition 1.2). This condition (A), though easy to derive, does not seem to have been observed before.

The local solvability of Eq.(0.1) has been studied by many people when \( M \) is strongly pseudo-convex (see [1,4,10,19,20,21,23,24]). In this case it was proved in Henkin [10] that one can construct explicit solution kernels for \( 1 \leq q \leq n - 2 \) when \( \partial \omega \) lies in a hyperplane. When \( 1 \leq q < n - 2 \), he actually derived a homotopy formula for \( \delta \). When \( q = n - 2 \), such a homotopy formula will not hold (see Nagel-Rosay [17]) and polynomial approximation arguments were used to construct the solution kernels. In fact, Henkin [10] showed that if \( \partial \omega \) is Runge, then condition (0.2) is sufficient for Eq.(0.1) to be solvable when \( q = n - 2 \). In this paper we shall characterize those domains such that condition (0.2) is sufficient for Eq.(0.1) to be solvable. These domains are more general than Runge ones. In the strongly pseudo-convex case, it is especially important to study the case when \( n = 3 \) and \( \alpha \) is a \((0,1)\) form, since this will give us some insight into the problem of embeddability of abstract \( CR \) structures of real dimension 5 (see Webster [25]).

The plan of the paper is as follows. In Section I we define the notation and state our main results in Theorems 1 and 2. In Section II we use the Cauchy problem for \( \delta \) for the top degree forms to prove Theorems 1 and 2. The Cauchy problem is different in this case from the lower degree cases and this is when the second compatibility condition was used. The rest of the proof is similar to the case when \( 1 \leq q \leq n - 3 \). In the end of this paper we give an example by Rosay [19] which shows that condition (0.2) is not sufficient for Eq.(0.1) to be solvable. The author would like to thank Professor Catlin for helpful discussions and to thank professor So-Chin Chen for pointing out the reference [3].

1. Notation and the main results

Let \( M \) be the boundary of a pseudo-convex domain \( D \) and \( \rho \) be its defining function, i.e., \( M = \{ z \in \mathbb{C}^n | \rho(z) = 0 \} \) and \( |d\rho| = 1 \) on \( M \). We assume that \( M \) is of finite type in the sense of D'Angelo [8]. Let \( \omega \subset M \) such that \( \omega = M \cap \{ z \in \mathbb{C}^n | r(z) < 0 \} \) and \( d\rho \wedge dr \neq 0 \) on the boundary of \( \partial \omega \). The space
$C^\infty_{(p,q)}(\omega)$ denotes all the $(p,q)$ forms in $\omega$ with coefficients in $C^\infty(\omega)$. For any $\alpha \in C^\infty_{(p,q)}(\omega)$, there exists a smooth $(p,q)$ form $\tilde{\alpha}$ in $C^n$ such that $\tau \tilde{\alpha} = \alpha$ where $\tau$ is the pointwise restriction operator to the boundary and projection to the parts which are orthogonal to the ideal generated by $\bar{\partial}$. Similarly we define the space $C^\infty_{(p,q)}(\omega)$ for $(p,q)$ forms in $\omega$ with $C^\infty(\omega)$ coefficients. The $\bar{\partial}$ operator is defined to be as follows: for any $\alpha \in C^\infty_{(p,q)}(\omega)$ and $\tilde{\alpha}$ that is any extension of $\alpha$ such that $\tau \tilde{\alpha} = \alpha$, then we define $\bar{\partial} \alpha = \tau(\bar{\partial} \tilde{\alpha})$. It is easy to see that the definition of $\bar{\partial}$ is independent of the choice of $\tilde{\alpha}$. For other definitions and the basic properties of the $\bar{\partial}$ complex, we refer the readers to Kohn-Rossi [16] or Folland-Kohn [9]. Since $p$ plays no role in the discussion of $\bar{\partial}$, we shall assume that $p = n$ for simplicity.

Let $K$ be a compact set in $\mathbb{C}^n$. We shall use the notation $\mathcal{O}(K)$ to denote the set of functions which are defined and holomorphic in some open neighborhood of $K$. Let $\omega_\epsilon \subset \omega$ such that $\omega_\epsilon$ increases to $\omega$ as $\epsilon \downarrow 0$ and each $\partial \omega_\epsilon$ is smooth. For any $\alpha \in C^\infty_{(n,n-2)}(\omega)$ such that Eq. (0.1) is solvable for some $u \in C^\infty_{(n,n-3)}(\omega)$, then for any $g \in \mathcal{O}(\partial \omega)$ we have, for small $\epsilon > 0$,

$$\int_{\partial \omega} \alpha \wedge g = \lim_{\epsilon \to 0} \int_{\partial \omega_\epsilon} \alpha \wedge g$$

$$= \lim_{\epsilon \to 0} \int_{\partial \omega_\epsilon} \bar{\partial} u \wedge g$$

$$= \lim_{\epsilon \to 0} \int_{\partial \omega_\epsilon} \bar{\partial} u \wedge g$$

$$= \lim_{\epsilon \to 0} \int_{\partial \omega_\epsilon} \bar{\partial}(u \wedge g)$$

$$= \lim_{\epsilon \to 0} \int_{\partial \omega_\epsilon} d(u \wedge g)$$

$$= 0$$

(1.1)

The third equality in (1.1) holds since the difference of $\bar{\partial} u$ and $\bar{\partial} u$ is a multiple of $\bar{\partial}(p)$ and $\bar{\partial}(p) = dp - dp$. Thus another necessary condition for Eq.(0.1) to be solvable for $\alpha \in C^\infty_{(n,n-2)}(\omega)$ is that

$$\int_{\partial \omega} \alpha \wedge g = 0$$ for all $g \in \mathcal{O}(\partial \omega)$. 

(A)

The following proposition characterizes all the domains $\omega$ such that condition (0.2) will imply condition (A). At the end of this paper we shall give an example of a $\bar{\partial}_b$-closed form which does not satisfy condition (A).
Proposition 1.2. If \( O(\omega) \) is dense in \( O(\partial \omega) \) (in the \( C(\partial \omega) \) norm), for any \( \alpha \) satisfying condition (0.2), \( \alpha \) satisfies condition (A). In particular, if polynomials are dense in \( O(\partial \omega) \), then condition (0.2) implies condition (A).

Proof. From our assumption, for any \( g \in O(\partial \omega) \), there exists a sequence of holomorphic functions \( g_n \in O(\omega) \) such that \( g_n \) converges to \( g \) in \( C(\partial \omega) \). We have, for any \( \alpha \) satisfying condition (2),

\[
\int_{\partial \omega} \alpha \wedge g = \lim_{n \to \infty} \int_{\partial \omega} \alpha \wedge g_n
\]

\[
= \lim_{n \to \infty} \int_{\omega} \delta(\alpha \wedge g_n)
\]

\[
= \lim_{n \to \infty} \int_{\omega} \delta_{\bar{\partial}} \alpha \wedge g_n
\]

\[
= 0.
\]

Thus condition (0.2) implies condition (A). If one can approximate any function \( g \in O(\partial \omega) \) by holomorphic polynomials, it is obvious that (2) implies (A) and the proposition is proved.

Our main results in this paper are the following theorems.

Theorem 1. Let \( M \) be the boundary of a smooth pseudo-convex domain in \( \mathbb{C}^n \), \( n \geq 3 \) and \( M \) is of finite type. Let \( \omega \subset M \) be a connected subset such that the boundary \( \partial \omega \) is the transversal intersection of \( M \) with a simply connected Levi-flat hypersurface \( M_0 \) which has a Stein neighborhood basis. Let \( \omega' \) be any relatively compact subset of \( \omega \). For any \( \alpha \in C_{(n,n-2)}^{(\infty)}(\omega) \) such that \( \alpha \) satisfies the compatibility conditions (0.2) and (A), there exists a \( u \in C_{(n,n-3)}^{(\infty)}(\omega') \) such that \( \bar{\partial}_u u = \alpha \) in \( \omega' \).

If one assumes that \( \omega \) can be exhausted by subsets whose boundaries lie in Levi-flat hypersurfaces, then we have the following semi-global existence result.

Theorem 2. Let \( M \) and \( \omega \) be the same as in Theorem 1. Furthermore we assume \( \omega = \cup \omega_i \) such that \( \omega_i \subset \omega_{i+1} \subset \omega \) and \( \partial \omega_i \) lies in a Levi-flat hypersurface for each \( i \). For any \( \alpha \in C_{(n,n-2)}^{(\infty)}(\omega) \) such that \( \alpha \) satisfies the conditions (0.2) and (A), there exists a \( u \in C_{(n,n-3)}^{(\infty)}(\omega) \) such that \( \bar{\partial}_u u = \alpha \) in \( \omega \).

Corollary 2.1. If \( M_0 \) is simply connected and defined by a pluriharmonic function, then the assertions in Theorem 2 holds. In particular, if \( M_0 \) is a hyperplane, then the assertions in Theorem 2 hold.

We also have the following local solvability result near a point of finite type.
Theorem 3. Let $M$ be a smooth pseudo-convex hypersurface in $\mathbb{C}^n$, $n \geq 3$ and $z_0 \in M$. Suppose $z_0$ is a point of finite type, then there exists a local neighborhood basis $\{\omega_\epsilon\}_{\epsilon > 0}$ of $z_0$ for $M$ such that the following holds: for any $\epsilon > 0$, if $\alpha \in C^\infty_{(n,n-2)}(\omega_\epsilon)$ such that $\alpha$ satisfies the conditions (0.2) and (A), there exists $u \in C^\infty_{(n,n-3)}(\omega_\epsilon)$ such that $\bar{\partial}_u = \alpha$ in $\omega_\epsilon$.

We note that Bedford-Fornaess (see the example on P. 21 in [3]) has given an example of an levi-flat hypersurface which does not have a Stein neighborhood basis. We mention that Bedford-de Bartolomeis [2] showed that Levi-flat hypersurfaces can not always be flattened locally even from one side. Thus our theorems generalize the results of Henkin [9] even in the strongly pseudo-convex case.

2. PROOF OF THE THEOREMS

To prove Theorems 1 and 2, we need to solve the Cauchy problem for $\bar{\partial}$ on the top degree forms. Let $L^2_{(p,q)}(G)$ denote forms on a domain $G$ with $L^2(G)$ coefficients. We denote the space of square integrable holomorphic functions by $H^2(G)$ and the space of holomorphic functions in $C^\infty(G)$ by $A^\infty(G)$. We have the following lemma.

Lemma 2.1. Let $G$ be a bounded pseudo-convex domain in $\mathbb{C}^n$, $n \geq 2$. For any $f \in L^2_{(n,n)}(\mathbb{C}^n)$, such that $f$ is supported in $\bar{G}$ and

$$\int_G f \wedge g = 0 \quad \text{for any} \quad g \in H^2(\bar{G}), \tag{2.2}$$

we can find a $u \in L^2_{(n,n-1)}(\mathbb{C}^n)$ such that $u$ is supported in $\bar{G}$ and $\bar{\partial}u = f$ in the distribution sense in $\mathbb{C}^n$. Furthermore, we have the following estimates:

$$\|u\|_G^2 \leq C \|f\|_G^2 \tag{2.3}$$

where the constant $C$ depends only on the diameter of the domain $G$.

If we assume that $G$ is a bounded pseudo-convex domain with smooth boundary, then we can substitute (2.2) by the condition

$$\int_G f \wedge g = 0 \quad \text{for any} \quad g \in A^\infty(\bar{G}), \tag{2.2'}$$

and the same conclusion holds.

If we assume that $G$ is a bounded pseudo-convex domain with a Stein neighborhood basis, then we can substitute (2.2) by the condition
(2.2’)
\[ \int_G f \wedge g = 0 \quad \text{for any } g \in \mathcal{O}(\overline{G}), \]

and the same conclusion holds.

**Proof.** We shall first prove the lemma assuming that \( G \) is a bounded pseudo-convex domain. Following Hörmander’s theory for \( \bar{\partial} \), the \( \bar{\partial} \)-Neumann operators for \((0,1)\) forms, denoted by \( N_1 \), exist on \( G \). One can also define the \( \bar{\partial} \)-Neumann operator on functions (denoted by \( N_0 \)) using \( N_1 \). In fact, let \( \theta \) be the formal adjoint of \( \bar{\partial} \), then it follows from Theorem 3.1.19 in Folland-Kohn [9] that

(2.4) \[ N_0 = \theta N_1 \bar{\partial} \]

whenever the formula is defined. We also have that

(2.5) \[ \theta \bar{\partial} N_0 = I - H \]

where \( \theta \) is the adjoint operator of \( \bar{\partial} \) and \( H \) is the Bergman projection operator from square-integrable functions into square integrable holomorphic functions \( H^2(G) \).

In fact, the formula (2.4) and (2.5) hold on all of \( L^2(G) \). To see this, we use the fact that \( N_1 \) is a bounded operator on \( L^2_{(0,1)}(G) \) and the bounds only depend on the diameter of \( G \). In fact, using the precise estimates obtained by Hörmander [12], we can have the following estimates:

(2.6) \[ \| N_1 \alpha \|_G^2 \leq e^{\delta^2} \| \alpha \|_G^2 \]

where \( \delta \) is the diameter of the domain \( G \) (for details of estimate (2.6), see Hörmander[12] and the proposition 2.3 in [21]). For any \( v \in C^\infty(\overline{G}) \), we have from (2.4)

(2.7) \[ \| N_0 v \|_G^2 = (\bar{\partial} \theta N_1^2 \bar{\partial} v, N_1^2 \bar{\partial} v) \\
= (N_1 \bar{\partial} v, N_1^2 \bar{\partial} v) \\
\leq \| N_1 \bar{\partial} v \| \| N_1^2 \bar{\partial} v \| \\
\leq e^{\frac{1}{4} \delta} \| N_1 \bar{\partial} v \|^2 \]

On the other hand, we have

(2.8) \[ (N_1 \bar{\partial} v, N_1 \bar{\partial} v) = (\bar{\partial} N_1^2 \bar{\partial} v, \bar{\partial} v) \\
= (\theta N_1^2 \bar{\partial} v, v) \\
\leq \| N_0 v \| \| v \| \]
Combining (2.10) and (2.11), we have

\begin{equation}
\|N_0 v\|^2 \leq e^{4} \|v\|^2
\end{equation}

Thus $N_0$ defined by (2.4) is a bounded on all smooth functions and it extends to $L^2(G)$. We also have that (2.5) holds for all functions in $L^2(G)$. Also, it follows from (2.5) that

\begin{equation}
\|\db N_0 v\|^2 = (\theta \db N_0 v, N_0 v) = ((I - H)v, N_0 v) \leq \|v\|\|N_0 v\| \leq e^{4} \|v\|^2
\end{equation}

Using $N_0$, we define

\begin{equation}
u = - \ast \db N_0 f
\end{equation}

where $\ast$ is the Hodge star operator extended naturally to the $L^2(G)$ forms. Using the relations that $\vartheta = - \ast \db \ast$ and $\ast \ast = I$, we have from (2.5),

\begin{align*}
\db u &= - \db (\ast \db N_0 f) \\
&= \ast \db (\db N_0 f) \\
&= \ast (\ast f - H(\ast f)) \\
&= f - \ast H(\ast f)
\end{align*}

For any $g \in H^2(G)$, from (2.2), we have

\begin{equation}
(\ast f, \bar{g}) = \int_G f \wedge g = 0
\end{equation}

Thus from (2.8), $H(\ast f) = 0$ and $\db u = f$ in $G$.

Extending $u$ to be zero outside $G$, we have for any $\phi \in C_0^\infty(n,n)(\mathbb{C}^n)$, that

\begin{align*}
(u, \vartheta \phi)_{\mathbb{C}^n} &= (\ast \db \phi, \ast \db u)_{G} \\
&= (\db \ast \phi, \db u)_{G} \\
&= (\ast \phi, \db \ast \db u)_{G} \\
&= (\ast \phi, \db \ast u)_{G} \\
&= (f, \phi)_{\mathbb{C}^n}
\end{align*}
where the third equality holds since $\star \tilde{u} \in \text{Dom}(\tilde{\delta}^*)$. This implies that $\tilde{\partial} u = f$ in $\mathbb{C}^n$ in the distribution sense. The estimate (2.3) holds from (2.11) with the constant $C = e^{\frac{1}{4} \delta}$.

When $G$ is a bounded pseudo-convex domain with smooth boundary, following Kohn's theorem in [9] on the global regularity of the solutions of $\tilde{\partial}$ on strongly pseudo-convex domains, we have that $A^\infty(G)$ is dense in $H^2(G)$ (in the $L^2(G)$ norm). Thus if $f$ satisfies condition (2.2'), it also satisfies condition (2.2) and the same results holds.

When $G$ is a pseudo-convex domain with a Stein neighborhood basis, we can assume that $\overline{G} = \cap G_i$ such that each $G_i$ is strongly pseudo-convex with smooth boundary. We note that on each $G_i$, the space $\mathcal{O}(G_i)$ is dense in $H^2(G_i)$ since $G_i$ is strongly pseudo-convex (this essentially follows from the Kohn's $\tilde{\partial}$-Neumann theory on strongly pseudo-convex domains). Since

$$\int_{G_i} f \wedge g = \int_G f \wedge g = 0$$

for every $g \in \mathcal{O}(G_i)$, thus condition (2.2) is satisfied for any $g \in H^2(G_i)$ and there exists a solution $u_i$ which is compactly supported in $\overline{G_i}$ and $\tilde{\partial} u_i = f$ in $\mathbb{C}^n$. Furthermore, it follows from (2.3) one has that

$$\| u_i \|_{\sigma_i} \leq C \| f \|_{\sigma}$$

where the constant $C$ can be chosen independent of $i$. Thus there exist a weak convergent subsequence of $u_i$ which converges to an element $u$ in $\mathbb{C}^n$ and the support of $u$ is contained in $\overline{G}$. One easily sees that $u$ satisfies (2.3) and $\tilde{\partial} u = f$ in the distribution sense in $\mathbb{C}^n$ and the lemma is proved.

**Proof of Theorem 1.** Let $D_0 = \{ z \in \mathbb{C}^n | r(z) < 0 \}$ and $\Omega = D \cap D_0$. From our assumption that $M_0$ has a Stein neighborhood basis, we have that $\tilde{\omega}_0 = M_0 \cap D$ also has a Stein neighborhood basis, since any domain of finite type has a Stein neighborhood basis. Let $G_i$ be a sequence of smooth decreasing pseudo-convex domains $G_i$ such that $\tilde{\omega}_0 = \cap G_i$. We define $\Omega_i = \Omega \setminus \tilde{G_i}$, $\Omega \cap G_i = D_i$ and $\Omega_i \cap \omega = \tilde{\omega}_i$. Then each $D_i$ is pseudo-convex since it is the intersection of two pseudo-convex domains. Also each $D_i$ has a Stein neighborhood basis since $D$ and $D_0$ both have Stein neighborhood basis. We also denote the boundary of $G_i$ by $\partial G_i$ and $\partial G_i \cap \Omega$ by $\omega_i$. Thus $\Omega_i \nearrow \Omega$ and $\omega_i \nearrow \omega$.

For any $\alpha$ that satisfies the conditions (0.2) and (A), we have, for any $g \in$
\[ \Omega(\overline{D_i}), \]
\[ \int_{\partial \omega_i^0} \alpha \wedge g = \int_{\partial \omega_i^0} \alpha \wedge g - \int_{\omega \setminus \omega_i} d(\alpha \wedge g) \]
\[ = \int_{\partial \omega_i^0} \alpha \wedge g - \int_{\omega \setminus \omega_i} \bar{\partial} \alpha \wedge g \]
\[ = \int_{\partial \omega} \alpha \wedge g - \int_{\omega \setminus \omega_i} \bar{\partial} \alpha \wedge g \]
\[ = \int_{\partial \omega} \alpha \wedge g \]
\[ = 0. \]

For a fixed \( i \), we extend \( \alpha \) to \( \bar{\alpha} \) on \( \Omega \) such that \( \bar{\partial} \bar{\alpha} \) vanishes to high order on \( \omega \). Let \( \zeta \in C^\infty_0(\mathbb{C}^n) \) be a cut-off function such that \( \zeta = 1 \) on \( \overline{\Omega_i} \) and \( \zeta = 0 \) on \( \Omega_i \setminus \Omega_{i+1} \). We define \( \alpha_1 = \zeta \bar{\partial} \bar{\alpha} \) and extend it to be zero outside \( \Omega \), then \( \alpha_1 \in C^\infty_{(n,n-1)}(\mathbb{C}^n) \) and \( \bar{\partial} \alpha_1 = \bar{\partial} \zeta \wedge \bar{\partial} \bar{\alpha} \). Setting \( \alpha_2 = \bar{\partial} \alpha_1 \). Then \( \alpha_2 \in C^\infty_{(n,n)}(\mathbb{C}^n) \) and \( \alpha_2 \) is supported in \( \overline{D_i} \).

For any \( g \in \Omega(\overline{D_i}) \), it follows from (2.14) that
\[ \int_{D_i} \alpha_2 \wedge g = \int_{D_i} \bar{\partial} \zeta \wedge \bar{\partial} \bar{\alpha} \wedge g \]
\[ = \int_{D_i} \bar{\partial}(\zeta \wedge \bar{\partial} \bar{\alpha} \wedge g) \]
\[ = \int_{\partial D_i} \zeta \bar{\partial} \bar{\alpha} \wedge g \]
\[ = \int_{\omega_i^0} \bar{\partial} \bar{\alpha} \wedge g \]
\[ = \int_{\omega_i^0} d(\bar{\alpha} \wedge g) \]
\[ = \int_{\partial \omega_i^0} \alpha \wedge g \]
\[ = 0. \]

Thus \( \alpha_2 \) satisfies (2.2). Using lemma 2.1 on the domain \( D_i \), there exists a \( u_1 \in L^2_{(n,n-1)}(\mathbb{C}^n) \) such that \( \bar{\partial} u_1 = \alpha_2 \) in \( \mathbb{C}^n \) and the support of \( u_1 \) is contained in \( \overline{D_i} \). We set
\[ \beta_1 = \alpha_1 - u_1 \]
then we have $\beta_1 = \bar{\partial}\alpha$ on $\Omega_i$ and $\bar{\partial}\beta_1 = \bar{\partial}\alpha_1 - \bar{\partial}u_1 = 0$ in $\mathbb{C}^n$. Using the $\bar{\partial}$-Neumann operator on $\Omega$ to solve the Cauchy problem for $\bar{\partial}$ for $(n, n-1)$ forms with support in $\Omega$ (see Proposition 2.7 in [22] for details), we have that there exists a $\beta_0$ such that

$$\bar{\partial}\beta_0 = \beta_1 \quad \text{in} \quad \mathbb{C}^n$$

and the support of $\beta_0$ is in $\Omega$. Furthermore, $\beta_0$ is smooth up to the boundary $\omega_i$. We define $\alpha_0 = \bar{\alpha} - \beta_0$, then $\bar{\partial}\alpha_0 = \bar{\partial}\alpha - \bar{\partial}\beta_0 = 0$ in $\Omega_i$ and $\alpha_0 = \alpha$ in $\omega_i$. This shows that we have extended $\alpha$ to be $\bar{\partial}$-closed on $\Omega_i$.

Since $\Omega_i \neq \Omega$, any compact subset $\omega'$ can be contained in the boundary of a pseudo-convex set $\Omega'$ and $\Omega' \subset \Omega_i$ for some $i$ sufficiently large. Thus one can use the $\bar{\partial}$-Neumann operator on $(n, n-2)$ forms to solve $\bar{\partial}u = \alpha_0$ on $\Omega'$ and the solution $u$ will be smooth to the part of boundary $\omega_i$ following the regularity of the $\bar{\partial}$-Neumann problem up to the boundary points of finite type proved by Kohn [15] and Catlin [7]. Since the method is similar to the case in [22] we omit the details. Restricting $u$ to $\omega'$, Theorem 1 is proved.

**Proof of Theorem 2.** Applying the argument of Theorem 1, we can construct a solution $u_i$ on every $\omega_i$. To extract the convergent subsequence $u_i$, we assume first that $n - 2 > 1$. Since $\bar{\partial}_b(u_i - u_{i+1}) = 0$ in $\omega_i$, from the results of [22], there exists a $v_i$ in $\omega_i$ such that $\bar{\partial}_b v_i = u_i - u_{i+1}$ in $\omega_i$. Extending $v_i$ smoothly to $\tilde{v}_i$ outside $\omega_{i-1}$, we have that $\bar{\partial}_b \tilde{v}_i = 0$ in $\omega_{i-1}$. Letting $\tilde{u}_{i+1} = u_{i+1} + \bar{\partial}_b \tilde{v}_i$, we have that $\bar{\partial}_b \tilde{u}_{i+1} = \alpha$ in $\omega_{i+1}$ and $\tilde{u}_{i+1} = u_i$ in $\omega_{i-1}$. Continuing this way one can construct a solution $u$ for Eq.(0,1) in $\omega$ and Theorem 2 is proved for $n > 3$. The case when $q = 1, n = 3$ is more involved and the argument involved holomorphic approximation. We refer the readers to the proof of lemma 3.1 in [22] and omit the details.

**Proof of Corollary 2.1.** If $M_0$ is defined by a pluriharmonic function $r(z) = 0$ and $M_0$ is simply connected, then there exists a holomorphic function $h$ such that $r(z) = \text{Im} h(z)$. After a holomorphic change of coordinates it is easy to see that the conditions of Theorem 2 are satisfied and the corollary is proved.

**Proof of Theorem 3.**

We shall construct a neighborhood basis $\{\omega_\varepsilon\}$ of $z_0$ such that $\partial\omega_\varepsilon$ lies in a holomorphically flat hypersurface. The following arguments were kindly provided by Catlin. Since $z_0$ is a point of finite type, it follows from Catlin [6] that $z_0$ is weakly regular. Thus there exist a family of strictly plurisubharmonic functions $\{\lambda_k\}_{k \in \mathbb{N}}$ defined in a neighborhood $U$ of $z_0$ such that $0 \leq \lambda_k \leq 1$ and
(2.17) \[ \sum_{i,j} \frac{\partial^2 \lambda_k}{\partial z_i \partial \overline{z}_j} a_i \overline{a}_j \geq k|a|^2 \quad \text{on} \quad M \cap U \]

for all \( a \in \mathbb{C}^n \).

Let \( \Omega \) be the pseudo-convex domain with boundary \( M \). Let \( B_\epsilon(z_0) \) denotes a ball of radius \( \epsilon \) with center \( z_0 \) and \( K_\epsilon = \overline{B}_\epsilon(z_0) \setminus B_\frac{3}{4}(z_0) \). We claim that there exists a strictly plurisubharmonic function \( \phi_\epsilon \) defined on \( \overline{\Omega} \cap \overline{B}_\epsilon(z_0) \) such that

\[
(2.18) \quad \phi_\epsilon(z_0) > \sup_{K_\epsilon} \phi_\epsilon(z).
\]

Let \( \eta \in C_0^\infty(B_\epsilon(z_0)) \) such that \( \eta = 3 \) on \( B_\frac{3}{4}(z_0) \) and \( \eta = 0 \) on \( K_\epsilon \). We define \( g_k = \eta + \lambda_k \). We have \( g_k(z_0) \geq 2 \) and

\[
g_k(z_0) > \sup_{K_\epsilon} g_k.
\]

If we choose \( k \) sufficiently large, from (2.17), we have that \( g_k \) is strictly plurisubharmonic near \( M \cap \overline{B}_\epsilon(z_0) \). Using Proposition 3.16 in Catlin [5], there exists a function \( \phi_\epsilon \) on \( \overline{\Omega} \cap \overline{B}_\epsilon(z_0) \) such that

\[
\phi_\epsilon = g_k \quad \text{on} \quad M \cap B_\epsilon(z_0)
\]

and

\[
\phi_\epsilon \leq g_k \quad \text{on} \quad \overline{\Omega} \cap \overline{B}_\epsilon(z_0).
\]

One easily sees that \( \phi_\epsilon \) satisfies (2.18). It follows from Theorem 3.15 in [5] that the holomorphic convex hull of \( K_\epsilon \) is the same as the hull of \( K_\epsilon \) with respect to plurisubharmonic functions. Thus there exists a holomorphic function \( f_\epsilon \in A^\infty(\overline{\Omega} \cap \overline{B}_\epsilon) \) such that

\[
f_\epsilon(z_0) > \sup_{K_\epsilon} |f(z)|
\]

Multiplying \( f_\epsilon \) by a nonzero constant and raise to high order if necessary, we can assume that \( f_\epsilon(z_0) = 1 \) and \( \sup_{K_\epsilon} |f(z)| \leq \frac{1}{2} \). Applying Sard's theorem, one can find a regular value \( \mu \) of the level set \( \{ \text{Re} f_\epsilon = \mu \} \), where \( \frac{1}{2} < \mu < 1 \). Let \( H_\epsilon \) is a smooth hypersurface and \( M \) intersects \( H_\epsilon \) transversally. Letting \( \omega_\epsilon \equiv M \cap \{ \text{Re} f_\epsilon > \mu \} \), it is easy to see that \( \omega_\epsilon \subset B_\epsilon(z_0) \cap M \). Thus we have constructed a neighborhood basis \( \{ \omega_\epsilon \} \) which satisfies all the hypothesis of Theorem 2 and Corollary 2.1 and Theorem 3 is proved.
3. An Example

In this section we shall examine some examples which were due to Rosay [19] and construct an explicit example of a \( \delta_b \)-closed form which does not satisfy condition (A). Let \( S_n \) be the unit sphere in \( \mathbb{C}^n \), \( n \geq 3 \) and \( \Sigma_1 = S_n \cap \{ |z_1|^2 < \frac{1}{2} \} \), \( \Sigma_2 = S_n \cap \{ |z_1|^2 > \frac{1}{2} \} \). It is proved in [19] that one can solve Eq.(0.1) for any \((p,q)\) form \( \alpha \) satisfying condition (0.2) in \( \Sigma_2 \) for all \( 1 \leq q \leq n - 2 \). While on \( \Sigma_1 \), this is only true when \( 1 \leq q < n - 2 \). We note that the boundary of \( \Sigma_1 \) and \( \Sigma_2 \) lie in the Levi flat hypersurface \( M_0 = \{ |z|^2 = \frac{1}{2} \} \) which has a Levi-flat Stein neighborhood basis. Let \( n = 3 \) and \( \zeta(z_3) \) be a cut-off function such that \( \zeta(z_3) = 0 \) when \( |z_3|^2 \geq \frac{1}{4} \) and \( \zeta(z_3) > 0 \) when \( |z_3|^2 < \frac{1}{4} \). We define

\[
    f = \frac{\zeta(z_3)}{z_2} \, dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3
\]

and

\[
    \alpha = \tau f
\]

where \( \tau \) is the projection operator from (3,1) form in \( \mathbb{C}^3 \) to \( \Sigma_1 \) defined in Section I. It is easy to see that \( \alpha \) is a smooth (3,1) form on \( \Sigma_1 \), since for \( z \in S_3 \), \( |z_2|^2 = 1 - |z_1|^2 - |z_3|^2 \geq \frac{1}{2} - |z_3|^2 > 0 \) on the support of \( \zeta \). Also \( \delta_b \alpha = 0 \) on \( \Sigma_1 \) which implies that \( \delta_b \alpha = 0 \) on \( \Sigma_1 \). If we set \( h(z) = \frac{1}{z_1} \), then \( h \in \mathcal{O}(\partial \Sigma_1) \) and

\[
    \int_{\partial \Sigma_1} \alpha \wedge h = \int_{\partial \Sigma_1} \frac{\zeta(z_3)}{z_1 \, z_2} \, dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3
\]

\[
    = \int_{|z_3| \leq \frac{1}{2}} \int_{|z_2| = \frac{1}{2} - |z_3|^2} \int_{|z_1|^2 = \frac{1}{2} - |z_3|^2} \frac{1}{z_1 z_2} \, dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3
\]

\[
    = (2\pi i)^2 \int_{|z_3| \leq \frac{1}{2}} \zeta(z_3) \, dz_3 \wedge d\bar{z}_3
\]

\[
    \neq 0
\]

Thus \( \alpha \) does not satisfy the necessary condition (A) and thus can not be solved on \( \Sigma_1 \) (or on arbitrarily large subset of \( \Sigma_1 \)). We note that \( \alpha \) is not smooth on \( \Sigma_2 \). On the other hand, since \( \mathcal{O}(\Sigma_2) \) is dense in \( \mathcal{O}(\partial \Sigma_2) \), Any \( \delta_b \)-closed (3,1) form can be solved on \( \Sigma_2 \) from Theorem 2. We also note that the boundary of \( \Sigma_2 \) is not Runge.
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Mei-Chi Shaw

Department of Mathematics

University of Notre Dame

Notre Dame, IN 46556 USA

e-mail address: mei.chi.shaw@nd.edu