KARL OELJEKLAUS

On the automorphism group of certain hyperbolic domains in $\mathbb{C}^2$


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1 Introduction and Results

Let $Q = Q(z, \bar{z})$ be a subharmonic and non-harmonic polynomial on the complex plane $\mathbb{C}$ with real values. Then the degree the non-harmonic part $Q^N$ of $Q$ is an even positive number $2k \in \mathbb{N}^*$. In their paper [1], F. Berteloot and G. Cœuré proved that the domain $\Omega_Q = \{(w, z) \in \mathbb{C}^2 \mid \text{Re } w + Q(z, \bar{z}) < 0\}$ is hyperbolic for every $Q$ like above. In this note, we consider the positive cone $M$ of all such polynomials and the associated domains $\Omega_Q \subset \mathbb{C}^2$.

Let $Q_1, Q_2 \in M$ and $\Omega_{Q_1}, \Omega_{Q_2}$ be the associated domains. In what follows, we use also $\Omega, \Omega_1, \Omega_2$ instead of $\Omega_Q, \Omega_{Q_1}, \Omega_{Q_2}$ if there is no confusion possible. First, we introduce an equivalence relation on the cone $M$.

Definition 1.1 Let $Q_1, Q_2 \in M$. We say that $Q_1$ and $Q_2$ are equivalent $Q_1 \sim Q_2$, if there is a real number $\rho > 0$, a holomorphic polynomial $p(z)$ and an automorphism $g(z)$ of $\mathbb{C}$ such that

$$Q_1(z, \bar{z}) = \rho \text{Re}(p(z)) + \rho Q_2(g(z), \overline{g(z)}).$$

On the other hand, there is another equivalence relation on $M$ given by the biholomorphy of the domains $\Omega_{Q_1}$ and $\Omega_{Q_2}$. The first results states that these two equivalence relations are the same.

Theorem 1.2 Let $Q_1, Q_2 \in M$. Then $\Omega_1$ and $\Omega_2$ are biholomorphic, if and only if the two polynomials $Q_1$ and $Q_2$ are equivalent in the sense of definition 1.1. In particular the degrees of the non-harmonic parts $Q_1^N$ and $Q_2^N$ are equal, if the domains $\Omega_1$ and $\Omega_2$ are biholomorphic.

The fact that $\Omega$ is hyperbolic implies that the holomorphic automorphism group $\text{Aut}_\mathbb{C}(\Omega)$ is a real Lie group and that all isotropy groups of the action
of $\text{Aut}_\mathcal{O}(\Omega)$ on $\Omega$ are compact [3]. We denote by $G, G_1, G_2$ the connected identity components of $\text{Aut}_\mathcal{O}(\Omega), \text{Aut}_\mathcal{O}(\Omega_1), \text{Aut}_\mathcal{O}(\Omega_2)$. Clearly, if $\Omega_1$ and $\Omega_2$ are biholomorphic, then $G_1$ and $G_2$ are isomorphic.

Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ denote the Lie algebras of $G, G_1, G_2$.

Let $J, J_1, J_2$ denote the subgroups of $G, G_1, G_2$ generated by the translation $\{(w, z) \mapsto (w + it, z) \mid t \in \mathbb{R}\}$ and $j, j_1, j_2$ their Lie algebras. Hence the dimension of $G, G_1, G_2$ is at least one.

The second result gives a "canonical" defining polynomial for the domain $\Omega$ if $\dim_{\mathbb{R}} \mathcal{G} \geq 2$.

**Theorem 1.3** Let $\Omega = \{\text{Re} w + Q(z) < 0\}$ as above. Assume that $\dim_{\mathbb{R}} G \geq 2$. Then there are the following cases:

a) $\Omega$ is homogeneous. Then $\Omega \simeq \mathbb{B}_2 = \{|w|^2 + |z|^2 < 1\}$ and $Q \sim P_1 \sim P_2$, where $P_1(z, \bar{z}) = (\text{Re} z)^2$ and $P_2(z, \bar{z}) = |z|^2$.

b) $\Omega$ is not homogeneous.

1) $\dim_{\mathbb{R}} G = 2$. Then $\deg Q^N \geq 4$ and either i) $Q \sim P_1$ or ii) $Q \sim P_2$, or iii) $Q \sim P_3$, where

i) $P_1(z, \bar{z}) = P_1(\text{Re} z)$ is an element of $M$ depending only on $\text{Re} z$ and $G \simeq \mathbb{R}^2, +$,

ii) $P_2(z, \bar{z}) = P_2(|z|^2)$ is an element of $M$ depending only on $|z|^2$, and $G \simeq \mathbb{R} \times S^1$,

iii) $P_3(z, \bar{z})$ is a homogeneous polynomial of degree $2k$, $k \geq 2$, i.e. $P_3(\lambda z, \lambda \bar{z}) = \lambda^{2k} P_3(z, \bar{z})$ for all $\lambda \in \mathbb{R}$ and $G$ is the non-abelian two dimensional real Lie group.

2) $\dim_{\mathbb{R}} G \geq 3$. Then $\deg Q^N \geq 4$ and either i) $Q \sim P_1$ or ii) $Q \sim P_2$ where

i) $P_1(z, \bar{z}) = (\text{Re} z)^{2k}$ and $G$ is 3-dimensional and solvable,

ii) $P_2(z, \bar{z}) = |z|^{2k}$ and $G$ is 4-dimensional and contains a finite covering of $SL_2(\mathbb{R})$.

We are going to prove the two theorems simultaneously by distinguishing the dimension of $G$. First we handle the one and two-dimensional cases, then the homogeneous case and we finish with the three and higher dimensional cases.

Before doing so, we prove the easy direction of theorem 1.1.

**Lemma 1.4** If $Q_1 \sim Q_2$, then $\Omega_1$ and $\Omega_2$ are biholomorphic.
Proof: Assume (1.1). Let \( \Psi = (\Psi_1, \Psi_2) \) be the biholomorphic map of \( \mathbb{C}^2 \) defined by

\[
(*) \quad \begin{cases} 
\Psi_1(w, z) = \frac{1}{\rho} w + p(z) \\
\Psi_2(w, z) = g(z)
\end{cases}
\]

Then \( \Psi(\Omega_1) = \Omega_2. \)

Remark 1.5 In what follows we will often make a global coordinate change in \( \mathbb{C}^2 \) like \((*)\), which is coherent with the equivalence of the defining polynomials. In the following, we take the notation from above.

## 2 The one-dimensional case

Let \( \Psi : \Omega_1 \to \Omega_2 \) be a biholomorphic map. For a subgroup \( N \subset G_2 \) let \( \Psi^*(N) \) be the group \( \Psi^{-1} \circ N \circ \Psi \subset G_1. \)

**Lemma 2.1** Assume that \( \Psi^*(J_2) = J_1. \) Then \( Q_1 \sim Q_2. \)

**Proof**: From our hypothesis it follows that there is a non-zero real number \( \rho \) such that

\[
\Psi^{-1} \circ T_t \circ \Psi = T_{\rho t}, \; (T_t(w, z) = (w + it, z)),
\]

since \( \Psi^* \) is a continuous group isomorphism of two copies of \( \mathbb{R} \).

So we get with \( \Psi = (\Psi_1, \Psi_2) \)

\[
\Psi_1(w, z) + it = \Psi_1(w + i\rho t, z) \\
\Psi_2(w, z) = \Psi_2(w + i\rho t, z)
\]

which implies:

\[
\Psi_1(w, z) = \frac{1}{\rho} w + p(z) \\
\Psi_2(w, z) = g(z)
\]

with \( p \in \mathcal{O}(\mathbb{C}) \) and \( g \in \text{Aut}_{\mathcal{O}}(\mathbb{C}) \), since the projection \( \pi : \mathbb{C}^2 \to \mathbb{C}, \; (w, z) \mapsto z \) is surjective on \( \Omega_1 \) and \( \Omega_2. \)

Therefore \( \Psi \) is a biholomorphic map of \( \mathbb{C}^2 \) which maps \( \Omega_1 \) to \( \Omega_2 \) and so we have

\[
\begin{align*}
\Omega_1 &= \{ \text{Re} w + Q_1(z, \bar{z}) < 0 \} = \Psi^{-1}(\Omega_2) \\
&= \{ \text{Re} \left( \frac{1}{\rho} w + p(z) \right) + Q_2(g(z), \bar{g}(z)) < 0 \} \\
&= \{ \text{Re} \; w + \text{Re} p(z) + \rho Q_2(g(z), \bar{g}(z)) < 0 \}. 
\end{align*}
\]
It follows that
\[ Q_1(z, \bar{z}) = \rho \, \text{Re} \, p(z) + \rho Q_2(g(z), g(\bar{z})). \]

This equality implies the positivity of \( \rho \) and the fact that the holomorphic function \( p(z) \) is already a polynomial. Hence \( Q_1 \sim Q_2 \).

We mention the following direct consequence, which is the statement of theorem 1.2 in the case \( \dim_{\mathbb{R}} G_1 = 1 \).

**Corollary 2.2** If \( \dim_{\mathbb{R}} G_1 = 1 \), then \( Q_1 \) and \( Q_2 \) are equivalent.

**Proof**: Here we have \( G_1 = J_1 \) and \( G_2 = J_2 \), hence \( \Psi^*(J_2) = J_1 \).

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### 3 The two-dimensional case

We are going to handle this case in a sequence of lemmas. We always assume that there is a two-dimensional subgroup \( H \subset G \) such that \( J \subset H \). Since \( J \subset G \) is a closed subgroup isomorphic to \( \mathbb{R} \) there are two possibilities for \( H \):

i) \( H \) is abelian and non-compact.

ii) \( H \) is the solvable two dimensional non-abelian Lie group.

**Lemma 3.1** Suppose that \( H \) is abelian. Then \( Q \sim P_1 \) or \( Q \sim P_2 \), where \( P_1(z, \bar{z}) = P_1(\text{Re} \, z) \) is an element of \( M \) which depends only on \( \text{Re} \, z \), or \( P_2(z, \bar{z}) = P_2(|z|^2) \) is an element of \( M \) which depends only on \( |z|^2 \).

In the first case, the domain \( \{\text{Re} \, w + P_1(\text{Re} \, z) < 0\} \) realizes the domain \( \Omega \) as a tube domain.

**Proof**: Let \( L = \{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbb{R}\} \) be a one parameter group of \( H \) such that \( L \) and \( J \) generate \( H \). The group \( H \) being abelian implies that \( L \) and \( J \) commute and so we get for all \( s, t \in \mathbb{R} \):

\[
\sigma_1^t(w + is, z) = \sigma_1^t(w, z) + is \\
\sigma_1^t(w + is, z) = \sigma_1^t(w, z).
\]

The restriction of the projection \( \pi : (w, z) \to z \) from \( \mathbb{C}^2 \) to \( \Omega \) being surjective and the second equality imply that
\[
\sigma_2^t(w, z) = \sigma_2^t(z)
\]
is a non-trivial one-parameter subgroup of \( \text{Aut}_\mathcal{O}(\mathbb{C}) \simeq \mathbb{C}^* \ltimes \mathbb{C} \). Furthermore \( \sigma_t^1(w, z) = w + f(t, z) \), where \( f(t, \cdot) \in \mathcal{O}(\mathbb{C}) \). Since \( \sigma_t^1 \in \text{Aut}_\mathcal{O}(\mathbb{C}^2) \) and stabilises \( \Omega \), it follows that \( f(t, \cdot) \) is a holomorphic polynomial for all \( t \in \mathbb{R} \).

After a holomorphic change of coordinates in \( \{ z \in \mathbb{C} \} \), which is in fact polynomial and therefore coherent with the equivalence of defining polynomials, we have that

- a) \( \sigma_2^1(z) = z + it \) or
- b) \( \sigma_2^1(z) = e^{\alpha \cdot t} \cdot z \) for \( \alpha \in \mathbb{C}^* \) fixed.

ad (a) : Here we have

\[
\begin{align*}
\sigma_1^1(w, z) &= w + f(t, w) \\
\sigma_2^1(w, z) &= z + it \quad \text{for all} \quad t \in \mathbb{R}.
\end{align*}
\]

It follows that

\[
(3.1) \quad f(t_1 + t_2, z) = f(t_1, z + it_2) + f(t_2, z) \quad \text{for all} \quad t_1, t_2 \in \mathbb{R}.
\]

and therefore there is a holomorphic polynomial \( \tilde{f} \) such that

\[
(3.2) \quad f(t, z) = \tilde{f}(z + it) - \tilde{f}(z).
\]

After the change of coordinates in \( \mathbb{C}^2 \)

\[
\begin{pmatrix}
\bar{w} \\
\bar{z}
\end{pmatrix} = \begin{pmatrix}
w - \tilde{f}(z) \\
z
\end{pmatrix},
\]

we have that \( \Omega \) is given by \( \{ \text{Re} \bar{w} + \tilde{Q}(\bar{z}, \bar{z}) < 0 \} \), with a polynomial \( \tilde{Q} \) equivalent to \( Q \). The action of \( L \) is then given by

\[
\begin{align*}
\sigma_1^1(\bar{w}, \bar{z}) &= \bar{w} \\
\sigma_2^1(\bar{w}, \bar{z}) &= \bar{z} + it.
\end{align*}
\]

This means that \( \tilde{Q}(\bar{z}, \bar{z}) \) is invariant under translations of the form \( \{ \bar{z} \mapsto \bar{z} + it \mid t \in \mathbb{R} \} \), which implies that \( \tilde{Q}(\bar{z}, \bar{z}) = \tilde{Q}(\text{Re} \bar{z}) \) and that \( \Omega \) is realized as a tube domain. The group \( H \) is isomorphic to \( (\mathbb{R}^2, +) \).

ad (b) : In this case, we have

\[
\begin{align*}
\sigma_1^1(w, z) &= w + f(t, z) \\
\sigma_2^1(w, z) &= e^{\alpha t} \cdot z
\end{align*}
\]
for all $t \in \mathbb{R}$ with fixed $\alpha = a + ib \in \mathbb{C}^*$. By the same argument as in case (a), we see that $f(t, \cdot)$ is a holomorphic polynomial and that $\sigma^t \in \text{Aut}_0(C^2)$ for all $t \in \mathbb{R}$. So we have:

$$\Omega = \{(w, z) \in C^2 | \text{Re } w + Q(z, z) < 0\}$$

$$= \{(w, z) \in C^2 | \text{Re } f(t, z) + Q(e^{\alpha t} \cdot z, e^{\alpha t} \cdot z) < 0\} \text{ for all } t \in \mathbb{R},$$

i.e. $Q(z, \bar{z}) = \text{Re } f(t, z) + Q(e^{\alpha t} \cdot z, e^{\alpha t} \cdot z)$. Without loss of generality, we may assume that the harmonic part of $Q$ is trivial, which implies that $\text{Re } f(t, z) \equiv 0$ for all $t \in \mathbb{R}$, i.e. $f(t, z) = f(t) \in i\mathbb{R}$ for all $t \in \mathbb{R}$. Hence $f(t) = i\beta t$ with $\beta \in \mathbb{R}$. Then we have that $Q(z, \bar{z}) = Q(e^{\alpha t} \cdot z, e^{\alpha t} \cdot \bar{z})$ for all $t \in \mathbb{R}$. This implies that $\alpha \in i\mathbb{R}^*$ and that $Q(z, z) = Q(|z|^2)$, i.e. the polynomial $Q$ depends only on $|z|^2$.

The action of $L$ then is given by

$$\sigma^t_1(w, z) = w + i\beta t$$

$$\sigma^t_2(w, z) = e^{\alpha t} \cdot z, \text{ for all } t \in \mathbb{R}.$$

The group $H$ is isomorphic to $\mathbb{R} \times S^1$.

**Lemma 3.2** Suppose that $H$ is the two dimensional solvable non-abelian Lie group. Then the polynomial $Q$ is equivalent to a polynomial $P_{2k}$, which is homogeneous of degree $2k$, i.e. $P_{2k}(\lambda z, \lambda \bar{z}) = \lambda^{2k} P_{2k}(z, \bar{z})$ for all $\lambda \in \mathbb{R}$ and $J$ is a normal subgroup of $H$.

**Proof:** Let $L = \{\sigma^t = (\sigma^t_1, \sigma^t_2) | t \in \mathbb{R}\}$ be a one parameter subgroup of $H$ such that $L$ and $J$ generate $H$. Then there are two cases:

a) $J$ is not the normal subgroup of dimension one in $H$.

b) $J$ is normal in $H$.

**ad(a):** We may assume that $L$ is normal in $H$.

Let $X = i \frac{\partial}{\partial w} - i \frac{\partial}{\partial \bar{w}}$ and $Y = f \frac{\partial}{\partial w} + g \frac{\partial}{\partial \bar{w}} + \bar{f} \frac{\partial}{\partial w} + \bar{g} \frac{\partial}{\partial \bar{w}}$ be the two holomorphic infinitesimal transformations induced by $J$ and $L$ on $\Omega$. By our assumption there is a $\lambda \in \mathbb{R}^*$ such that $[X, Y] = \lambda \cdot Y$. This equation yields $f(w, z) = e^{-i\lambda w} h_1(z)$ and $g(w, z) = e^{-i\lambda w} h_2(z)$, $h_1, h_2 \in O(C)$. It follows that $Y$ is a global infinitesimal holomorphic transformation of $C^2$, since $\pi : \Omega \to C$, $(w, z) \mapsto z$ is surjective.

Furthermore $h_2$ vanishes nowhere, since $h_2(z_0) = 0$ implies that the set $\{(w, z_0) | \text{Re } w + Q(z_0, \bar{z}_0) < 0\}$ is stabilized by $H$ with $J$ as a non-normal subgroup which is impossible. Now we have $Y((\text{Re } w + Q(z, \bar{z}))) |_{\{\text{Re } w + Q(z, \bar{z}) = 0\}} \equiv 0$. 198
This yields the equation

\[ h_1(z) + h_2(z) \frac{\partial Q}{\partial z}(z, \bar{z}) + e^{2i\lambda Q(z, \bar{z})} \left( \frac{\partial Q}{\partial z}(z, \bar{z}) + h_2(z) \frac{\partial Q}{\partial \bar{z}}(z, \bar{z}) \right) \equiv 0. \]

The expression \( h_1(z) + h_2(z) \frac{\partial Q}{\partial z}(z, \bar{z}) \) being a polynomial in \( \bar{z} \) implies that the expression \( e^{2i\lambda Q(z, \bar{z})} \left( \frac{\partial Q}{\partial z}(z, \bar{z}) + h_2(z) \frac{\partial Q}{\partial \bar{z}}(z, \bar{z}) \right) \) is also a polynomial in \( \bar{z} \). By differentiating \( n \) times, \( n \in \mathbb{N} \) with respect to \( \bar{z} \) this yields that \( h_2(z) = 0 \) for all \( z \in \mathbb{C} \), a contradiction to the fact mentioned above.

**ad (b):** Assume that \( J \) is normal in \( H \). We get

\[
\sigma_1^t(w + is, z) = \sigma_1^t(w, z) + ie^{\alpha t} \cdot s \\
\sigma_2^t(w + is, z) = \sigma_2^t(w, z), \alpha \in \mathbb{R}^* \text{ fixed.}
\]

So we have again \( \sigma_2^t(w, z) = \sigma_2^t(z) \) and \( \sigma_2^t \in \text{Aut}_\mathcal{O}(\mathbb{C}) \) for all \( t \in \mathbb{C} \).

Furthermore \( \sigma_1^t(w, z) = e^{\alpha t}w + f(t, z) \) with \( f(t, \cdot) \in \mathcal{O}(\mathbb{C}) \) for all \( t \in \mathbb{R} \). Hence \( \sigma^t \in \text{Aut}_\mathcal{O}(\mathbb{C}^2) \) and \( f(t, z) \) is a holomorphic polynomial for all \( t \in \mathbb{R} \).

Since \( \dim_{\mathbb{R}} H = 2 \), the one parameter group \( \{ \sigma_2^t(z) \mid t \in \mathbb{R} \} \subset \text{Aut}_\mathcal{O}(\mathbb{C}) \) cannot be trivial. So after a change of coordinates in the \( z \)-variable, we have

(i) \( \sigma_2^t(z) = z + it \) or

(ii) \( \sigma_2^t(z) = e^{\beta t} \cdot z, \beta \in \mathbb{C}^* \text{ fixed.} \)

If (i) \( \sigma_2^t = z + it \), we get

\[
\sigma_1^t(w, z) = e^{\alpha t}w + f(t, z) \\
\sigma_2^t(w, z) = z + it \text{ and } \sigma^t \in \text{Aut}_\mathcal{O}(\mathbb{C}^2)
\]

This yields

\[ Q(z, \bar{z}) = e^{-\alpha t} \Re f(t, z) + e^{-\alpha t}Q(z + it, \bar{z} - it). \]

It is easy to see that this is not possible by considering the highest degree homogeneous summand of the non-harmonic part \( Q^N \) of \( Q \).

So we may assume (ii), \( \sigma_2^t(z) = e^{\beta t} \cdot z, \beta \in \mathbb{C}^* \text{ fixed.} \)

Hence

\[
\sigma_1^t(w, z) = e^{\alpha t} \cdot w + f(t, z) \\
\sigma_2^t(w, z) = e^{\beta t} \cdot z, \alpha \in \mathbb{R}^*, \beta = a + ib \in \mathbb{C}^* \text{ fixed}
\]

and it follows that

\[ Q(z, \bar{z}) = e^{-\alpha t} \Re f(t, z) + e^{-\alpha t}Q(e^{\beta t}z, e^{\beta t}\bar{z}). \]
We may assume that \( Q \) has no harmonic summands and therefore

\[
Q(z, \bar{z}) = e^{-\alpha t} Q(e^{\beta t} \cdot z, e^{\beta t} \cdot \bar{z}), \text{ for all } t \in \mathbb{R}.
\]

The highest degree of \( Q \) is an even number \( 2k, k \in \mathbb{N}^* \). Let

\[
Q_{2k}(z, \bar{z}) = \sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j}, \quad (a_j = a_{2k-j})
\]

be the highest degree homogeneous summand of \( Q \). We get

\[
Q_{2k}(z, \bar{z}) = e^{-\alpha t} Q_{2k}(e^{\beta t} \cdot z, e^{\beta t} \cdot \bar{z}), \text{ i.e.}
\]

\[
a_j = a_j e^{-\alpha t} \cdot e^{(2k-j)\beta t}, 1 \leq j \leq 2k - 1.
\]

A necessary condition for this is

\[
\alpha = 2k \cdot \text{Re} \beta
\]

and that there are no summands in \( Q \) of degree smaller then \( 2k \).

Hence \( Q = Q_{2k} = P_{2k} \) and the lemma is proved. \( \blacksquare \)

**Remark 3.3** Lemma 3.1 and Lemma 3.2 give the proof of theorem 1.3 in the case \( \dim_{\mathbb{R}} G = 2 \).

**Lemma 3.4** Let \( \Omega_1 = \{ \text{Re} w + Q_1(z, \bar{z}) < 0 \} \) and \( \Omega_2 = \{ \text{Re} w + Q_2(z, \bar{z}) < 0 \} \) like above. Assume that \( \Psi : \Omega_1 \rightarrow \Omega_2 \) is biholomorphic and that \( J_1 \) and \( \Psi^*(J_2) \) are both contained in a two-dimensional subgroup \( H \subset G_1 \). Then \( J_1 = \Psi^*(J_2) \) and \( Q_1 \sim Q_2 \).

**Proof**: We have again to consider the following two cases:

a) \( H \) is abelian,

b) \( H \) is not abelian.

In both cases, we assume \( J_1 \neq \Psi^*(J_2) \) and produce a contradiction. Let \( \Psi^*(J_2) = \{ \sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbb{R} \} \).

\( \text{ad}(a) \): i) Assume that \( H = (\mathbb{R}^2, +) \). Then by Lemma 3.1, we may suppose that \( \Omega_1 = \{ \text{Re} w + Q_1(\text{Re} w) < 0 \} \) and \( \Omega_2 = \{ \text{Re} w + Q_2(\text{Re} z) < 0 \} \) are already realized as tube domains and that the biholomorphism \( \Psi \) is equivariant with respect to the action of \( H \simeq i\mathbb{R}^2 \) as imaginary translations on both domains. Hence \( \Psi \) is an affine linear automorphism of \( \mathbb{C}^2 \), i.e. \( \Psi_1(w, z) = aw + bz + e \), \( \Psi_2(w, z) = cw + dz + f \) with

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{R}) \quad \text{and} \quad \left( \begin{array}{c} e \\ f \end{array} \right) \in \mathbb{R}^2
\]
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We get

$$\Omega_1 = \{\text{Re } w + Q_1(\text{Re } z) < 0\} = \{a \text{Re } w + b \text{Re } z + e + Q_2(c \text{Re } w + d \text{Re } z + f) < 0\},$$

which implies $c = 0$, i.e. $J_1 = \Psi^*(J_2)$ and that the two polynomials are equivalent.

ii) Assume that $H = \mathbb{R} \times S^1$. Then by Lemma 3.1, we may assume that $\Omega_1 = \{\text{Re } w + Q_1(|z|^2) < 0\}$ and $\Omega_2 = \{\text{Re } w + Q_2(|z|^2) < 0\}$ where $Q_1$ and $Q_2$ depend only on $|z|^2$. Furthermore, the action of $S^1$ on both domains is given by rotations in the $z$-variable. Hence there is an $\alpha \in \mathbb{R}^*$ such that

$$\Psi_1(w, e^{i\alpha t} \cdot z) = \Psi_1(w, z),$$

$$\Psi_2(w, e^{i\alpha t} \cdot z) = e^{it} \cdot \Psi_2(w, z), \quad \text{for all } t \in \mathbb{R}.$$

We get $\alpha = 1$ and

$$(1) \quad \left\{ \begin{array}{l}
\Psi_1(w, z) = \Psi_1(w) \\
\Psi_2(w, z) = z \cdot g(w).
\end{array} \right.$$

Furthermore there exist $b \in \mathbb{R}$, $\beta \in \mathbb{R}^*$ such that $\Psi^*(J_2) = \{\sigma^t \mid t \in \mathbb{R}\}$ looks like:

$$\sigma_1^t(w, z) = w + i\beta t$$

$$\sigma_2^t(w, z) = e^{ibt} z$$

We get

$$\Psi_1(w + i\beta t, e^{ibt} \cdot z) = \Psi_1(w, z) + it$$

$$\Psi_2(w + i\beta t, e^{ibt} \cdot z) = \Psi_2(w, z).$$

Now the above expression (1) yields

$$\Psi_1(w, z) = \Psi_1(w) = \frac{1}{\beta} w$$

$$\Psi_2(w, z) = z \cdot g(w) = e^{ibt} \cdot z \cdot g(w + i\beta t), \quad \text{for all } t \in \mathbb{R},$$

i.e. $e^{-ibt} g(w) = g(w + i\beta t)$, for all $t \in \mathbb{R}$.

It follows:

$$-ibg(w) = g'(w)i\beta, \quad \text{i.e.}$$

$$g'(w) = -\frac{b}{\beta}g(w), \quad \text{hence}$$

$$g(w) = c \cdot e^{-\frac{1}{\beta}w}$$

and $\Psi$ is a global automorphism of $\mathbb{C}^2$. This yields easily that $b = 0$ and $c \neq 0$, i.e. $\Psi_2(w, z) = c \cdot z$. But then $\Psi^*(J_2) = J_1$ and $Q_1 \sim Q_2$. 

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ad (b): Assume that $H$ is not abelian. By lemma 3.2, we have $J_1 \triangleleft H$. Suppose that $\Psi^*(J_2) \neq J_1$. Let $\Sigma = \Psi^{-1}$ the inverse of $\Psi$. Then we have that $J_2 = \Sigma^*(\Psi^*(J_2))$ is not normal in $H$. But lemma 3.2 applied to the domain $\Omega_2$ gives a contradiction. Hence $\Psi^*(J_2) = J_1$ and Lemma 3.4 is proved.

**Remark 3.5:** Lemma 3.4 gives the proof of Theorem 1.2 in the case $\dim_R G_1 = \dim_R G_2 = 2$.

### 4 The homogeneous case

Now we are going to handle the case when the domains in question are homogeneous, i.e. the group $G$ acts transitively on them.

Assume that $\Omega = \{\Re w + Q(z, \bar{z}) < 0\}$ is a homogeneous complex manifold.

Then by a theorem of Rosay [5] the domain $\Omega$ is biholomorphic to the unit ball $B_2 = \{|w|^2 + |z|^2 < 1\}$. As other “canonical” models for $B_2$ we mention the two realisations $\{\Re w + (\Re z)^2 < 0\}$ and $\{\Re w + |z|^2 < 0\}$, which we use in the sequel. Here the polynomials $(\Re z)^2$ and $|z|^2$ are obviously equivalent.

So we assume that $\Omega_1 = \{\Re w + (\Re z)^2 < 0\}$ and $\Omega_2 = \{\Re w + Q_2(z, \bar{z}) < 0\}$.

**Lemma 4.1** Suppose that $\Omega_1$ and $\Omega_2$ are biholomorphic. Then $Q_2(z, \bar{z}) \sim (\Re z)^2$.

**Proof:** Let $\Psi : \Omega_1 \rightarrow \Omega_2$ denote a biholomorphism. The group $G_1$ is isomorphic to $SU(2, 1)$ and $J_1$ and $\Psi^*(J_2)$ are two closed one-dimensional non-compact subgroups of $SU(2, 1)$. By investigating the structure of $SU(2, 1)$ one can show that the normaliser $N_{G_1}(J_1)$ of $J_1$ in $G_1$ is five-dimensional and closed and that there is an element $g \in G_1$ such that $g\Psi^*(J_2)g^{-1} \subset N_{G_1}(J_1)$. So one can replace the map $\Psi$ by another biholomorphism $\tilde{\Psi}$, such that $J_1$ and $\tilde{\Psi}^*(J_2)$ are contained in a two dimensional subgroup $H$ of $G_1$. But then by lemma 3.4 $\Psi^*(J_2) = J_1$ and $Q_2(z, \bar{z}) = (\Re z)^2$.

**Remark 4.2:** The above mentioned theorem of Rosay and lemma 4.1 prove theorem 1.2 and theorem 1.3 in the homogeneous case.

### 5 The three-dimensional case

We start with the following two useful lemmas.

**Lemma 5.1** Let $H \subset G$ be an at least three-dimensional subgroup of $G = \text{Aut}^0_\varnothing(\Omega)$. Then $H$ is not abelian.
Proof: By assumption $G$ and therefore $H$ act effectively on $\Omega$. The lemma follows from the fact that $\Omega$ is a two-dimensional hyperbolic complex manifold.

Lemma 5.2 Assume that $G = \text{Aut}_0^0(\Omega)$ is not solvable and that $\Omega$ is not homogeneous. Let $G = s \ltimes r$ be a Levi-Malcev decomposition of $G = \text{Lie}(G)$. Then the semisimple part $s$ is isomorphic to $sl_2(\mathbb{R})$, the Lie algebra of $SL_2(\mathbb{R})$.

Proof: Let $\tilde{s}$ be a complex simple Lie algebra admitting a one or two codimensional complex subalgebra. Then $\tilde{s} \simeq sl_2(\mathbb{C})$ or $\tilde{s} = sl_3(\mathbb{C})$.

Hence our real semi-simple algebra $s$ is isomorphic to $sl_2(\mathbb{R})$, $su(2)$, $sl_3(\mathbb{R})$, $su(2,1)$ or $su(3)$.

In the last four cases, $s$ admits a subalgebra, which is isomorphic to $su(2)$. This means that we have an almost effective action of $SU(2,\mathbb{C})$ on $\Omega$. Then the generic orbit of this action is a compact 3-dimensional $CR$-hypersurface isomorphic to a finite quotient of $S^3$. But we have also the non-compact closed subgroup $J \subset G$, which shows that $G$ has an open orbit in $\Omega$. This orbit is isomorphic to the unit ball $B_2$ and for a point $p$ in this orbit the isotropy group $I_G(p)$ is a maximal compact subgroup $K$. Assume that there is a point $q \in \Omega$ such that $\dim_{\mathbb{R}} G(q) < 4$. The $\Omega$ being hyperbolic implies that $I_G(q)$ is compact and of greater dimension that $K$. This is impossible. Hence $\Omega$ is already homogeneous. But this contradicts our assumption. Hence $s \simeq sl_2(\mathbb{R})$ and the lemma is proved.

Now we assume that $\dim_{\mathbb{R}} G \geq 3$ and that there is a three-dimensional subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$ such that $J \subset H$. In view of Lemmas 5.1 and 5.2, we have the following cases:

I. $\mathfrak{h}$ is solvable and not abelian.
II. $\mathfrak{h} \simeq sL_2(\mathbb{R})$.

5.1 Case I:

The Lie algebra $\mathfrak{h}$ is solvable and $\dim_{\mathbb{R}} h = 3$.

In view of lemma 5.1 $\mathfrak{h}$ cannot be abelian.

We use the classification of three-dimensional solvable Lie algebras given in [2]. Let $\mathfrak{h} = \langle a, b, c \rangle_{\mathbb{R}}$. Then there are the following cases:

1. $[a, b] = b$, $[a, c] = [b, c] = 0$;
2. $[a, c] = b$, $[a, b] = [c, b] = 0$, i.e. $\mathfrak{h}$ is nilpotent.
3. $[c, b] = 0$, $[a, b] = ab + \beta c$, $[a, c] = \gamma b + \delta c$, where

$$D := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{R})$$
**Lemma 5.3** Assume that the structure of $\mathfrak{h}$ is given by (1) above. Then $j = \mathfrak{h}'$, the commutator of $\mathfrak{h}$ and $Q \sim P$, where $P(z, \bar{z}) = |z|^{2k}$, $k \geq 2$.

**Proof:** In view of lemma 3.2, we have that $j \subset b, c > \mathbb{R} \subset \mathfrak{h}$. Our first step of the proof will be to prove that the group $H$ cannot be simply connected. So we assume this and produce a contradiction.

Then the group $L$ associated to the Lie algebra $l = \langle b, c > \mathbb{R}$ is isomorphic to $(\mathbb{R}^2, +)$ and contains $J$.

Hence $\Omega = \{\text{Re} w + Q(\text{Re} z) < 0\}$ by lemma 3.1. Since $L$ is normal $H$ we have by [4] that the group $H$ acts as a subgroup of $GL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{C}^2$ and hence on $\Omega$. So we have a one parameter subgroup $\{(A(t), v(t)) \in H \mid t \in \mathbb{R}\}$ with $\{A(t) \mid t \in \mathbb{R}\} \subset GL_2(\mathbb{R})$ being a non-trivial one parameter subgroup of $GL_2(\mathbb{R})$. By considering the Lie algebra structure of $\mathfrak{h}$ and the shape of $\Omega$, it is an easy calculation to see that this is impossible.

Hence $H$ is not simply connected and isomorphic to $N \times S^1$ where $N$ is the non-abelian two-dimensional Lie group. The group $J$ is contained in $N' \times S^1 \simeq \mathbb{R} \times S^1$ and therefore we have that $\Omega = \{\text{Re} w + Q(|z|^2) < 0\}$, the action of $S^1$ being given as the rotations in the $z$-variable.

Now let $\{\sigma^t \mid t \in \mathbb{R}\}$ be the one parameter subgroup of $H$ with Lie algebra $< a > \mathbb{R}$. Since $S^1$ is central in $H$, it follows

$$
\sigma_1^t(w, e^{is} \cdot z) = \sigma_1^t(w, z) \\
\sigma_2^t(w, e^{is} \cdot z) = e^{is} \sigma_2^t(w, z) \quad \text{for all} \quad s, t \in \mathbb{R},
$$

i.e.

$$
\sigma_1^t(w, z) = \sigma_1^t(w) \\
\sigma_2^t(w, z) = g(t, w) \cdot z \quad \text{with} \quad g(t, \cdot) \text{ holomorphic in } w.
$$

Furthermore there is a non-compact one parameter group of the form

$$
\left\{ \begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} w + is \\ e^{i\alpha s} \cdot z \end{pmatrix} \mid \alpha \in \mathbb{R} \text{ fixed}, t \in \mathbb{R} \right\} \triangleleft N
$$

which generates together with $\{\sigma^t\}$ the group $N$ i.e. there is a $\rho \in \mathbb{R}^*$ such that

$$
\sigma_1^t(w + is, e^{i\alpha s} \cdot z) = \sigma_1^t(w, z) + ie^{\rho t} \cdot s
$$

$$
\sigma_2^t(w + is, e^{i\alpha s} \cdot z) = \sigma_1^t(w, z) \cdot e^{i\alpha e^{\rho t} \cdot s}
$$

and so

$$
\sigma_1^t(w, z) = e^{\rho t} \cdot w \\
\sigma_2^t(w, z) = g(t, w) \cdot z
$$
with \( g(t, w) \cdot e^{i\alpha e^t \cdot s} = g(t, w + is) \cdot e^{i\alpha s} \) for all \( s, t \in \mathbb{R} \) i.e. \( g(t, w + is) = e^{i\alpha s(e^t-1)} \cdot g(t, w) \) and so

\[
\frac{\partial g}{\partial w}(t, w) = \alpha(e^t - 1) \cdot g(t, w)
\]

\[
g(t, w) = c(t)e^{(\alpha(e^t - 1))w}.
\]

Hence is a global automorphism of \( \mathbb{C}^2 \) stabilizing \( \Omega \). But this is only possible if \( g(t, w) \) does not depend on \( w \), i.e. \( g(t, w) = g(t) \) and then

\[
\sigma_1^t(w, z) = e^{pt} \cdot w, \quad \text{and} \quad \sigma_2^t(w, z) = g(t) \cdot z, \quad \text{with} \quad g(t + \hat{t}) = g(t) \cdot g(\hat{t}).
\]

This implies \( g(t) = c \cdot e^{\nu t} \), \( \nu \in \mathbb{R} \). Then it is easy to conclude that \( Q(z, \bar{z}) \sim |z|^{2k} \) and it is obvious that \( J = N' \triangleleft H \). The lemma is proved.

\textbf{Remark 5.4} : In the setting of lemma 5.3, i.e. \( \Omega = \{ \text{Re} \ w + |z|^{2k} < 0 \} \), the automorphism group \( G \) of \( \Omega \) is \( S \cdot T \), where \( S \) is a finite covering of \( SL_2(\mathbb{R}) \) and \( T \) is a central subgroup isomorphic to \( S^1 \), i.e. \( \dim G = 4 \). This case will also appear below.

\textbf{Lemma 5.5} Assume that the structure of \( \mathfrak{h} \) is given by (2) above. Then \( \Omega \) is biholomorphic to the unit ball \( B_2 \).

\textbf{Proof} : Here \( \mathfrak{h} \) is isomorphic to the Lie algebra of the three-dimensional Heisenberg group \( H_3 \). First we consider the case that \( H \) is not simply-connected. Then \( H = H_3/\Gamma \), where \( \Gamma \) is a discrete subgroup of \( H_3 \) isomorphic to \( \mathbb{Z} \) lying in the center \( C \) of \( H_3 \). Hence \( H \) contains a central subgroup \( L = C/\Gamma \simeq S^1 \). Then \( J \) and \( L \) generate a two-dimensional subgroup isomorphic to \( \mathbb{R} \times S^1 \) and by lemma 3.1 we may assume that \( \Omega = \{ \text{Re} \ w + Q(|z|^2) < 0 \} \) with the natural \( \mathbb{R} \times S^1 \) action. The polynomial \( Q \) depends only on \( |z|^2 \), is subharmonic and can be assumed to satisfy \( Q(0) = 0 \) and \( Q > 0 \). Then \( \tau(\Omega) = \{ \text{Re} \ w < 0 \} \), where \( \tau : (w, z) \rightarrow z \) from \( \mathbb{C}^2 \) to \( \mathbb{C} \) denotes the projection on the first component.

This map is an equivariant \( \mathcal{H} \)-map since \( L \simeq S^1 \) is central in \( H \) and the \( L \)-action is given by rotations in the \( Z \)-variable. Therefore the two-dimensional group \( H/L \) acts on \( \{ \text{Re} \ w < 0 \} = \tau(\Omega) \). But this action cannot be effective, since there is no two-dimensional abelian subgroup in the automorphism group of the half-plane. Hence a two-dimensional subgroup of \( H \) containing \( L \) stabilizes all fibers of \( \tau \) and acts effectively on the fibers. But the \( \tau \)-fibers in \( \Omega \) are also half-planes and every two-dimensional subgroup of \( H \) is abelian. This is again not possible. So we have proven that \( H \) is isomorphic to the simply-connected Heisenberg group \( H_3 \). Then there is a two-dimensional subgroup \( A \) containing
J which is isomorphic to \((\mathbb{R}^2, +)\). By lemma 3.1, the domain \(\Omega\) is given by \(\{\text{Re} \, w + Q(\text{Re} \, z) < 0\}\) a tube domain.

Let \(\{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbb{R}\}\) be a one-parameter group in \(H\) which together with \(A\) generates \(H\). Since \(A \subset H\) is normal, we have by [4] that \(\{\sigma^t \mid t \in \mathbb{R}\}\) is a subgroup of the affine linear group \(GL_2(\mathbb{R}) \times \mathbb{R}^2\).

So let \(\left(\begin{array}{cc} a(t) & b(t) \\ c(t) & d(t) \end{array}\right), \left(\begin{array}{c} e(t) \\ f(t) \end{array}\right) = \sigma^t \subset GL_2(\mathbb{R}) \times \mathbb{R}^2\) denote this group. The group \(\{A(t) = \left(\begin{array}{cc} a(t) & b(t) \\ c(t) & d(t) \end{array}\right) \mid t \in \mathbb{R}\}\) is not trivial in \(GL_2(\mathbb{R})\). We have

\[
\sigma^t(w, z) = \left(\begin{array}{c} a(t)w + b(t)z + e(t) \\ c(t)w + d(t)z + f(t) \end{array}\right)
\]

and \(\sigma^t\) stabilizing \(\Omega\) implies:

\[
\Omega = \{\text{Re} \, w + Q(\text{Re} \, z) < 0\} = \{a(t) \text{Re} \, w + b(t) \text{Re} \, z + e(t) + Q(c(t) \text{Re} \, w + d(t) \text{Re} \, z + f(t)) < 0\}
\]

It follows immediately that \(c(t) = 0\) for all \(t \in \mathbb{R}\) and that

\[
Q(\text{Re} \, z) = \frac{b(t)}{a(t)} \text{Re} \, z + \frac{e(t)}{a(t)} + \frac{1}{a(t)}Q(d(t) \text{Re} \, z + f(t))
\]

The group \(H\) being nilpotent implies that \(a(t) = d(t) = 1\) for all \(t \in \mathbb{R}\), i.e.

\[
Q(\text{Re} \, z) = b(t) \text{Re} \, z + e(t) + Q(\text{Re} \, z + f(t)).
\]

Since \(b(t)\) is not identically zero, this equation implies that \(\text{deg} \, Q = 2\) and that \(\Omega\) is biholomorphic to \(\mathbb{B}_2\).

\[\text{Lemma 5.6} \quad \text{Assume that the structure of } h \text{ is given by (3) above and that } \Omega \text{ is not homogeneous. Then } \Omega = \{\text{Re} \, w + (\text{Re} \, w)^{2k} < 0\}, \, k \geq 2 \text{ and } G = H.\]

\[\text{Proof : } \text{The structure of } h \text{ implies that } \dim_{\mathbb{R}} h' = 2 \text{ and that the associated group } H' \subset H \text{ is isomorphic to } (\mathbb{R}^2, +). \text{ So } \Omega \text{ as a simply-connected hyperbolic Stein manifold of dimension two with an action of } (\mathbb{R}^2, +), \text{ therefore it is biholomorphic to a tube domain } \Omega' = F + i\mathbb{R}^2, \text{ where } F \text{ is a convex domain in } \mathbb{R}^2 \text{ containing no complex lines (see [7])}. \text{ The group } H' \simeq (\mathbb{R}^2, +) \text{ being normal in } H \text{ implies that } H \text{ acts on } \Omega' \text{ as a subgroup of } GL_2(\mathbb{R}) \times \mathbb{C}^2 \text{ (see [4]).}
\]

Let \(\{\sigma^t = (\sigma_1^t, \sigma_2^t)\}\) be a one-parameter subgroup of \(H\) generating together with \(H'\) the group \(H\). Then

\[
\sigma^t = \left(\begin{array}{cc} a(t) & b(t) \\ c(t) & d(t) \end{array}\right), \left(\begin{array}{c} e(t) \\ f(t) \end{array}\right) \in GL_2(\mathbb{R}) \times \mathbb{R}^2
\]
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with $A(t) = e^{tD}$, where

$$D = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}.$$  

Since $D \in GL_2(\mathbb{R})$, after a conjugation with an element of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \rtimes \mathbb{R}^2,$$

we have that $\vec{v}(t) = 0$ for all $t \in \mathbb{R}$, i.e.

$$\sigma^t = \left( \begin{pmatrix} a(t) b(t) \\ c(t) d(t) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

for all $t \in \mathbb{R}$.

Now assume that $D$ is not triangulisable over $\mathbb{R}$. Then $\{\sigma^t \mid t \in \mathbb{R}\}$ is isomorphic to $S^1$, since any one dimensional subgroup of $GL_2(\mathbb{R})$, which is not compact, is triangulisable over $\mathbb{R}$. So the domain $F \subset \mathbb{R}^2$ is invariant by a linear $S^1$-action and must therefore be bounded.

On the other hand we have that $J$ has to lie in $H'$ because otherwise it would be a compact group. Then $\Omega = \{\text{Re} w + Q(\text{Re} z) < 0\}$ and $H$ acts affinely on $\Omega$. Since the set $\{(y, x) \in \mathbb{R}^2 \mid y + Q(x) < 0\}$ is not bounded we get a contradiction.

So we can assume that the matrix $D$ is triangulisable over $\mathbb{R}$. Hence $H'$ contains a one-dimensional normal subgroup of $H$. If $J \not\subset H'$, then this group and $J$ generate a two-dimensional non-abelian group, which is impossible by lemma 3.2.

So we have that $J \subset H' \cong (\mathbb{R}^2, +)$, $\Omega = \{\text{Re} w + Q(\text{Re} z) < 0\}$ a tube domain and that $H$ acts affinely on $\mathbb{C}^2$ and on $\Omega$ with $H' \subset H$ the group of imaginary translations as a normal subgroup.

We have that

$$\sigma^t = A(t) = \begin{pmatrix} a(t) b(t) \\ c(t) d(t) \end{pmatrix}, \in GL_2(\mathbb{R})$$

$$= e^{tD}, D = \begin{pmatrix} \alpha & \beta \\ \mu & \delta \end{pmatrix}, t \in \mathbb{R}.$$  

Then

$$\Omega = \{\text{Re} w + Q(\text{Re} z) < 0\}$$

$$= \{a(t) \text{Re} w + b(t) \text{Re} z + Q(c(t) \text{Re} w + d(t) \text{Re} z) < 0\}$$

which shows that $c(t) \equiv 0$, i.e.

$$Q(\text{Re} z) = \frac{b(t)}{a(t)} \text{Re} z + \frac{1}{a(t)} Q(d(t) \text{Re} z)$$

for all $t \in \mathbb{R}$.  

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Since we may assume that $Q$ has no harmonic summands we get

$$Q(\Re z) = \frac{1}{a(t)} Q(d(t) \Re z).$$

This implies that $Q(\Re z) = (\Re z)^{2k}$, $k \geq 2$ and that the action of $\sigma^t$ is given by

$$\sigma^t(w, z) = (e^{2kt} \cdot w, e^t \cdot z), t \in \mathbb{R}.$$

Now we prove that $G = H$. First we show that $G$ is solvable. Assume to the contrary that $G$ is not solvable. Then, since $\Omega$ is not homogeneous, the semisimple part of $G$ is isomorphic to a covering of $SL_2(\mathbb{R})$. Then by checking the possibilities for $G$ as an automorphism group of a 2-dimensional hyperbolic manifold (see Case II) it is easy to see that $G'$ does not contain a two-dimensional abelian subgroup. So $G$ is solvable and $\dim_{\mathbb{R}} G' \geq 2$. Furthermore $G'$ is nilpotent and contains $H' \simeq (\mathbb{R}^2, +)$. Then it is easy to see (by checking the possibilities for $G'$) that $H' \triangleleft G'$, which implies that $H' = G'$ (lemma 5.1 and lemma 5.5). Then $H' \triangleleft G$ and by applying again [4] one concludes that $G = H$. 

5.2 Case II : $\mathfrak{h} \sim sl_2(\mathbb{R})$

Here we are going to handle completely the situation where $\Omega$ is not homogeneous and $G$ is not solvable.

By lemma 5.2, there is a three-dimensional subgroup $H$ of $G$ such that the Lie algebra $\mathfrak{h}$ is isomorphic to $sl_2(\mathbb{R})$.

Since $\Omega$ is not homogeneous we have that $3 \leq \dim_{\mathbb{R}} \mathcal{G} \leq 5$, in view of the possibilities of a maximal compact subgroup $K : K = (e), K = S^1, K = (S^1)^2$.

Let $\mathcal{G} = \mathfrak{h} \ltimes r$ be a Levi-Malcev decomposition of $\mathcal{G}$. Here $r$ denotes the radical of $\mathcal{G}$. Hence $\dim_{\mathbb{R}} r = 1$ or 2. If $\dim_{\mathbb{R}} r = 2$, then $r$ is abelian, because otherwise the center of $SL_2(\mathbb{R}) \ltimes R$ is too small to admit a discrete central quotient with maximal compact subgroup $(S^1)^2$. But then $\mathcal{G} = \mathfrak{h} \ltimes r$ is a direct product again because otherwise there is no central subgroup with quotient $(S^1)^2$. The existence of a three-dimensional abelian subgroup excludes this case (Lemma 5.1). If $\dim_{\mathbb{R}} r = 1$, then $\mathcal{G} = h \ltimes r$ a direct product.

Hence we have only two possibilities for $\mathcal{G} :$

$$\mathcal{G} = \mathfrak{h} = sl_2(\mathbb{R}) \text{ or } \mathcal{G} = \mathfrak{h} \ltimes R = sl_2(\mathbb{R}) \times \mathbb{R}.$$

We consider these cases in the following lemmas.

Lemma 5.7 Assume that $j \subset \mathfrak{h} \subset \mathcal{G}$. Then $J$ is contained in a two-dimensional subgroup of $H$. 

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Proof: If $H$ is modulo a finite covering isomorphic to $\text{SL}_2(\mathbb{R})$, then $J$ as a non-compact subgroup of $H$ is contained in a two-dimensional subgroup of $H$. So assume that $H \simeq \text{SL}_2(\mathbb{R})$, the universal covering of $\text{SL}_2(\mathbb{R})$, and let $C$ denote the center of $H$ which is isomorphic to $\mathbb{Z}$. If $J \cap C = (e)$, then $J$ is also contained in a two-dimensional subgroup of $H$. So assume that $J \cap C \neq (e)$, i.e. $J \cap C \simeq \mathbb{Z}$.

First this implies that $H$ is a closed subgroup of $G$. (If $H \simeq \text{SL}_2(\mathbb{R})$ is not closed in $G$, then the maximal compact subgroup $K$ of $G$ is $(S^1)^2$ and contains $C$. But $J \subset G$ is a closed, non-compact subgroup of $G$ and therefore $J \cap C = (e)$, which is a contradiction.)

Hence $H$ acts freely on $\Omega$ and all orbits are closed and isomorphic to $\mathbb{R}^3$. We may assume that $J \cap C = \{(w, z) \mapsto (w + 2\pi ik, z) \mid k \in \mathbb{Z}\}$. This group acts freely and properly discontinuous on $\Omega$ and we can consider the quotient

$$\Omega = \{\text{Re } w + Q(z, \bar{z}) < 0\} \overset{(e^w z)}{\longrightarrow} \{0 < |w|^2 e^{2Q(z, \bar{z})} < 1\} = \Omega'.$$

Then there is an action of a group $S = \text{SL}_2(\mathbb{R})/J \cap C$ on $\Omega'$ and the group $J/J \cap C$ acts as rotations in the $w$-variable. Furthermore the $S$-action is free and all orbits are closed.

Now let $(X_1, X_2, X_3)$ be a basis of the three-dimensional vector space of holomorphic vector fields induced by the $S$-action on $\Omega'$. We take the exterior products $\sigma_1 = X_1 \wedge X_2$, $\sigma_2 = X_1 \wedge X_3$, $\sigma_3 = X_2 \wedge X_3$. The $\sigma_i$ are sections in the anticanonical bundle $\text{det}(T^{1,0}_\Omega) = \kappa^{-1}$ and generate an $S$-invariant subspace of $\Gamma_\sigma(\Omega', \kappa^{-1})$. For every point $p \in \Omega'$, there is $\sigma_i$ such that $\sigma_i(p) \neq 0$. Hence we get an $S$-equivariant holomorphic mapping $\alpha : \Omega' \to \mathbb{P}_2(\mathbb{C})$ defined by

$$\alpha(p) = (\sigma_1(p) : \sigma_2(p) : \sigma_3(p)),$$

where the $S$-action on $\mathbb{P}_2(\mathbb{C})$ is given by the natural $S/\text{C}(S) \simeq \text{PSL}_2(\mathbb{R})$-action which is of course projective-linear.

Since there is no $\text{PSL}_2(\mathbb{R})$-fix-point in $\mathbb{P}_2(\mathbb{C})$ the map $\alpha$ cannot be trivial.

Hence the map $\alpha$ is either locally biholomorphic or the dimension of the fibers is one.

In the latter case, the restriction of $\alpha$ to every $S$-orbit is an $S^1$-principal Cauchy-Riemann bundle (see [5]) and this fact yields that there is an additional holomorphic $S^1$-action on $\Omega'$ which commutes with the $S$-action. Hence $\dim_{\mathbb{R}} G = 4$ and we get a 2-dimensional abelian subgroup of $G$ containing $J$, i.e. by Lemma 3.1, $\Omega = \{\text{Re } w + Q(|z|^2) < 0\}$ or $\Omega = \{\text{Re } w + Q(\text{Re } z) < 0\}$. In both cases, one can assume that $Q(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}$.

But then an automorphism of $\Omega'$ extends to an automorphism of $\Omega' \cup \{w = 0\}$ and we get an $S$-action on $\mathbb{C} \simeq \{w = 0\}$. This is impossible.
So we have to consider the case where the map $\alpha$ is locally biholomorphic. By considering the $PSL_2(\mathbb{R})$-invariant domains in $\mathbb{P}_2$, with the property that all $PSL_2(\mathbb{R})$-orbits are 3-dimensional, one sees that the image of $\Omega'$ by $\alpha$ is contained in a domain biholomorphic to $\Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta)$ with the diagonal $PSL_2(\mathbb{R})$-action. (Here $\Delta = \{ y \in \mathbb{C} \mid |y| < 1 \}$).

Furthermore the associated map of $S$ resp. $PSL_2(\mathbb{R})$-orbits is injective, since they are 3-dimensional in a 2-dimensional complex manifold and $\alpha$ is locally biholomorphic.

So we have a locally biholomorphic, $S$-equivariant map

$$\tilde{\alpha} : \Omega' \to \Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta).$$

Using the $S$-equivariance and the concrete description of $PSL_2(\mathbb{R})$-orbits in $\Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta)$, one can see that this is impossible. The lemma is proved.

**Lemma 5.8** Assume that $j \subseteq h \subseteq G$ and that $J$ is contained in a two-dimensional subgroup of $H$. Then $H$ is a finite covering of $SL_2(\mathbb{R})$ and $Q \sim P$, with $P(z, \bar{z}) = |z|^{2k}$, $k \geq 2$.

**Proof:** We assume that $J$ is contained in a two dimensional subgroup of $H$. We are going to prove $Q \sim P$, with $P(z, \bar{z}) = |z|^{2k}$ directly. Then it follows that $H$ is modulo a finite covering isomorphic to $SL_2(\mathbb{R})$, by an investigation of the automorphism group of $\{ \text{Re } w + |z|^{2k} < 0 \}$.

By lemma 3.2, we have the two holomorphic vector fields $X = i \frac{\partial}{\partial w}$ and $Z = -2w \frac{\partial}{\partial w} - \bar{z} \frac{\partial}{\partial z}$ induced by $J$ and the group $\{(w, z) \mapsto (e^{2kt}w, e^t \cdot z) \mid t \in \mathbb{R}\}$. In view of structure of $H$ there is a third holomorphic vector field $Y$ induced by a one parameter subgroup of $H$ such that

$$[Z, X] = 2X,$$

$$[X, Y] = Z,$$

$$[Z, Y] = -2Y.$$

Furthermore $< \text{Re } X, \text{Re } Y, \text{Re } Z >_{\mathbb{R}}$ is the Lie algebra of real infinitesimal holomorphic transformations induced by $H$ on $\Omega$.

Now let $Y(w, z) = f(w, z) \frac{\partial}{\partial w} + g(w, z) \frac{\partial}{\partial z}$. Using the commutator relations we calculate $f$ and $g$ :

$$[X, Y] = [i \frac{\partial}{\partial w}, f \frac{\partial}{\partial w} + g \frac{\partial}{\partial z}]$$

$$= i \frac{\partial f}{\partial w} \frac{\partial}{\partial w} + i \frac{\partial g}{\partial w} \frac{\partial}{\partial z}$$

$$= -2w \frac{\partial}{\partial w} - \frac{z}{k} \frac{\partial}{\partial z} = Z.$$
Hence $\frac{\partial f}{\partial w} = -2iw, \frac{\partial g}{\partial w} = -\frac{iz}{k}$ and so

$$f(w, z) = -iw^2 + f_1(z) \quad \text{and} \quad g(w, z) = -\frac{izw}{k} + g_1(z).$$

Furthermore:

$$[Z,Y] = \left[-2w \frac{\partial}{\partial w} - \frac{z}{k} \frac{\partial}{\partial z}, f \frac{\partial}{\partial w} + g \frac{\partial}{\partial z}\right]$$

$$= -2w \frac{\partial f}{\partial w} \frac{\partial}{\partial w} - 2w \frac{\partial g}{\partial w} \frac{\partial}{\partial z} - \frac{z}{k} \frac{\partial f}{\partial z}$$

$$- \frac{z}{k} \frac{\partial g}{\partial z} + 2f \frac{\partial}{\partial w} + \frac{g}{k} \frac{\partial}{\partial z}$$

$$= -2f \frac{\partial}{\partial w} - 2g \frac{\partial}{\partial z} = -2Y.$$ 

and therefore

$$-2f = -2w \frac{\partial f}{\partial w} + 2f - \frac{z}{k} \frac{\partial f}{\partial z}$$

$$-2g = -2w \frac{\partial g}{\partial w} - \frac{z}{k} \frac{\partial g}{\partial z} + \frac{g}{k}$$

and finally

$$4f = 2w \frac{\partial f}{\partial w} + \frac{z}{k} \frac{\partial f}{\partial z}, \quad (2k + 1)g = 2kw \frac{\partial g}{\partial w} + \frac{z}{k} \frac{\partial g}{\partial z}.$$ 

It follows that:

$$4(-iw^2 + f_1(z)) = -4iw^2 + \frac{z}{k} f'_1(z)$$

$$(2k + 1)(-\frac{izw}{k} + g_1(z)) = -2izw - \frac{izw}{k} + zg'_1(z), \quad \text{i.e.}$$

$$4f_1(z) = \frac{z}{k} f'_1(z) \text{ and } (2k + 1)g_1(z) = zg'_1(z), \text{ which implies}$$

$$f_1(z) = c \cdot z^{4k}$$

$$g_1(z) = d \cdot z^{2k+1}, \quad c, d \in \mathbb{C}.$$ 

The vector field $Y$ is therefore given by

$$Y = (-iw^2 + cz^{4k}) \frac{\partial}{\partial w} + \left(-\frac{izw}{k} + d \cdot z^{2k+1}\right) \frac{\partial}{\partial z}.$$ 

In particular $Y$ is a global holomorphic vector field on $\mathbb{C}^2$ and $\text{Re}Y$ stabilizes the CR-hypersurface $M = \{\text{Re} w + P_{2k}(z, \bar{z}) = 0\}$, which means that

$$(Y + \bar{Y})(\text{Re} w + P_{2k}(z, \bar{z})) \mid_M \equiv 0.$$
We will compute this expression now:

\[(Y + \bar{Y})(\text{Re } w + P_{2k}(z, \bar{z})) = \frac{1}{2}(-iw^2 + cz^4k) + \frac{1}{2}(i\bar{w}^2 + \bar{c}\bar{z}^4k)\]

\[+ \left(-\frac{izw}{k} + dz^{2k+1}\right)\frac{\partial P_{2k}}{\partial z} + \left(\frac{i\bar{z}\bar{w}}{k} + \bar{d}\bar{z}^{2k+1}\right)\frac{\partial P_{2k}}{\partial \bar{z}}\]

\[= \frac{1}{2}(cz^4k + \bar{c}\bar{z}^4k) + \frac{1}{2}i(-(\text{Re } w + i\text{ Im } w)^2 + (\text{Re } w - i\text{ Im } w)^2)\]

\[+ \left(dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}}\right) - \frac{iz}{k} (\text{Re } w + i\text{ Im } w)\frac{\partial P_{2k}}{\partial z}\]

\[+ \frac{iz}{k} (\text{Re } w - i\text{ Im } w)\frac{\partial P_{2k}}{\partial \bar{z}}\]

\[= \frac{1}{2}(cz^4k + \bar{c}\bar{z}^4k) + (dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}})\]

\[+ (2\text{ Re } w \text{ Im } w + \frac{z}{k}\text{ Im } w\frac{\partial P_{2k}}{\partial z} + \frac{\bar{z}}{k}\text{ Im } w\frac{\partial P_{2k}}{\partial \bar{z}})\]

\[+ (-\frac{iz}{k} \text{ Re } w\frac{\partial P_{2k}}{\partial z} + \frac{iz}{k} \text{ Re } w\frac{\partial P_{2k}}{\partial \bar{z}}).\]

We put \(\text{Re } w = -P_{2k}\) and observe that \(P_{2k}\) being homogeneous implies that \(P_{2k} = \frac{1}{2k}(z\frac{\partial P_{2k}}{\partial z} + \bar{z}\frac{\partial P_{2k}}{\partial \bar{z}})\) to get that the expression

\[\frac{1}{2}(cz^4k + \bar{c}\bar{z}^4k) + (dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}})\]

\[+ \left(\frac{iz}{k} P_{2k} - i\bar{z} P_{2k}\right) = 0\] for all \(z \in \mathbb{C}\).

We may assume that \(P_{2k}\) has no harmonic summands and reduce to

\[dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}} + 
\]

\[iz \frac{\partial P_{2k}}{\partial z} - \bar{z} \frac{\partial P_{2k}}{\partial \bar{z}} \cdot P_{2k} - \frac{iz}{k} \frac{\partial P_{2k}}{\partial \bar{z}} P_{2k} = 0,\]

for all \(z \in \mathbb{C}\), with \(P_{2k}(z, \bar{z}) = \sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j}, a_j = \bar{a}_{2k-j}\) and \(k \geq 2\).

If the constant \(d = 0\), then it follows that

\[z \frac{\partial P_{2k}}{\partial z} = \bar{z} \frac{\partial P_{2k}}{\partial \bar{z}},\]

which forces \(P_{2k}(\bar{z}, z) = a_k |z|^{2k}, a_k \in \mathbb{R})\).

So assume that \(d \neq 0\). Then we have

\[d \cdot \sum_{j=1}^{2k-1} j a_j z^{2k+j} \bar{z}^{2k-j} + \bar{d} \sum_{j=1}^{2k-1} a_j (2k - j) z^j \bar{z}^{4k-j}\]
ON THE AUTOMORPHISM GROUP OF HYPERBOLIC DOMAINS IN \( \mathbb{C}^2 \)

\[
+ \frac{2i}{k} \left[ \left( \sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j} \right) \left( \sum_{j=1}^{2k-1} a_j(j - k) z^j \bar{z}^{2k-j} \right) \right]
\]

\[
= d \sum_{j=1}^{2k-1} a_j (2k - j) z^j \bar{z}^{4k-j} + d \sum_{j=2k+1}^{4k-1} a_j (j - 2k) z^j \bar{z}^{4k-j}
\]

\[
+ \frac{2i}{k} \left[ \left( \sum_{j=2}^{4k-2} \left( \sum_{l+n=j} a_l a_n(n - k) \right) z^j \bar{z}^{k-j} \right) \right] = 0 \text{ for all } z \in \mathbb{C}.
\]

Let \( \tau \in \{1, \ldots, k\} \) be the smallest number such that \( a_\tau \neq 0 \). Then our expression becomes

\[
+ \frac{2i}{k} \left[ \left( \sum_{j=2}^{4k-2} \left( \sum_{l+n=j} a_l a_n(n - k) \right) z^j \bar{z}^{k-j} \right) \right] = 0.
\]

But then \( a_\tau = 0 \), which is a contradiction.

So we have that \( \mathcal{P}(z, \bar{z}) = |z|^{2k}, k \geq 2 \) and the lemma is proved.

\textbf{Lemma 5.9} Assume that \( G = \mathfrak{h} \times r, \dim r = 1 \). Then \( j \subset \mathfrak{h}. \)

\textbf{Proof:} Assume that \( G = \mathfrak{h} \times r \) and \( j \not\subset \mathfrak{h} \). In view of lemma 5.3, we have \( j \neq r \). Let \( \pi : \mathcal{G} \to \mathfrak{h} \) be the projection of \( \mathcal{G} \) onto \( h \) with kernel \( r \). Again in view of lemma 5.3, we have that \( \pi(j) \) is the Lie algebra of a maximal compact subgroup of \( SL_2(\mathbb{R}) \). Let \( L \) be the two-dimensional subgroup of \( G \) whose Lie algebra \( l \) is generated by \( r \) and \( \pi(j) \). It is clear that \( L \) is a two-dimensional Lie group containing \( J \) and the center \( C \) of \( G \). Therefore \( L = S^1 \times \mathbb{R} \), since otherwise \( G = SL_2(\mathbb{R}) \times \mathbb{R} \), which is impossible. Hence \( \Omega = \{ \text{Re } w + Q(|z|^2) < 0 \} \), where we may assume that \( Q(|z|^2) \geq 0 \) for all \( z \in \mathbb{C} \). The action of the connected component of \( C^0 \) the center of \( G \) is given by

\[
(w, z) \mapsto (w + it, e^{ipt} \cdot z), t \in \mathbb{R}, p \in \mathbb{R}^* \text{ fixed}.
\]

We consider the function \( (w, z) \mapsto z \cdot e^{-\rho w} \in \mathbb{C} \), which is invariant under this action. We have

\[
|z \cdot e^{-\rho w}|^2 = |z|^2 \cdot e^{-\rho 2 \text{Re } w} \geq |z|^2 e^{\rho 2 Q(|z|^2)}.
\]

The expression on the right side tends to \(+\infty\) when \( |z| \to +\infty \) and the image of \( f \) is \( S^1 \)-invariant. Hence \( f : \Omega \to \mathbb{C} \) is surjective and has maximal rank
everywhere. Hence we get an $G/C^0$ action on $C$ which is impossible. The lemma is proved.

Remark 5.10    a) The automorphism group of a domain $\Omega = \{\text{Re} \, w + |z|^{2k} < 0\}, \, k \geq 2$ is a product $S \cdot S^1$, where $S$ is modulo a finite group isomorphic to $SL_2(\mathbb{R})$ and $S^1$ is a central one-dimensional group. Hence $G$ is four-dimensional.

b) In the case $\dim_{\mathbb{R}} G = 3$ the lemmas 5.3 to 5.9 prove theorem 1 and theorem 2.

c) We mention that from now on we may assume that $G$ is solvable since the non-solvable case is completely handled by the lemmas 5.2 to 5.9.

6  The case $\dim_{\mathbb{R}} G \geq 4$

Lemma 6.1 Let $\Omega = \{\text{Re} \, w + Q(z, \bar{z}) < 0\}$ and assume that $G = \text{Aut}^0_0(\Omega)$ is solvable. Then $\dim_{\mathbb{R}} G \leq 3$.

Proof: We assume that $\dim_{\mathbb{R}} G \geq 4$ and that $\Omega$ is not homogeneous. So we have that $\dim G = 4$ or 5, since the highest dimensional compact subgroup of $G$ is $(S^1)^2$.

Let $N \subset G$ be the largest nilpotent normal connected subgroup of $G$. Clearly, $N$ contains $(G')^0$, the connected component of the commutator $G'$ of $G$.

We first show that $\dim_{\mathbb{R}} N \leq 3$. Assume the contrary, i.e. $\dim N \geq 4$. Then the maximal compact subgroup of $N$ is not trivial, i.e. isomorphic to $S^1$ or $(S^1)^2$. But compact subgroups of nilpotent Lie groups are always central, in view of the bijectivity of the exponential map. Then $N$ as a subgroup of $G$ does not act effectively, a contradiction. So $\dim_{\mathbb{R}} N \leq 3$. So we have to consider three cases:

i) $n = h_3$ the three-dimensional Heisenberg algebra;

ii) $\dim N = 2$ and $N$ is abelian;

iii) $\dim N = 1$.

Cas i) : $n = h_3$. By similar arguments as above and using the fact that all maximal compact subgroups are conjugate one sees that $N$ is simply connected. Hence all $N$ and therefore all $G$-orbits in $\Omega$ are closed CR-hypersurfaces isomorphic to $\mathbb{R}^3$. Using the results of [4], [7], it is not hard to check that a simply connected hyperbolic Stein manifold acted on by $H_3$ is biholomorphic to the ball; this contradicts our assumption.
Case ii) : \( \dim_{\mathbb{R}} N = 2 \) and \( N \) is abelian.

If \( J \not\subset N \) then \( J \) and \( N \) generate a three-dimensional solvable group. Using the lemmas of Section V, we see that \( G \) cannot be solvable and of dimension four or greater, if \( \Omega \) is not homogeneous.

So we have \( J \subset N \) and we can find a 3-dimensional solvable group containing \( J \). Using again the lemmas of Section V we conclude like above.

Case iii) : \( \dim_{\mathbb{R}} N = 1 \). Then either \( J = N \) or \( J \) and \( N \) generate a two dimensional abelian group. In both cases we can take the complex-analytic quotient of \( \Omega \) by \( N \), which is either the upper half plane or \( \mathbb{C} \). But \( G/N \) is at least 3-dimensional and abelian. This is impossible.

Remark 6.2 Using the same methods as above it can be shown that the number of connected components of \( \text{Aut}_\Omega(\Omega) \) is always finite.

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Bibliographie


Karl Oeljeklaus
Université des Sciences et Technologies de Lille
U.R.A. 751 “GAT” associée au CNRS
UFR de Mathématiques Pures et Appliquées
F-59655 - Villeneuve d’Ascq Cedex (France)

New address (after the 1.10.1993):
U.F.R. de Mathématiques, Informatiques et Mécanique
Université de Provence (Aix-Marseille I)
3, place Victor Hugo
F-13331 - Marseille Cedex (France)