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Introduction

In the text of this article, definitions, propositions, theorems, lemmas, examples, corollaries are numerated in the same sequential order. Formulas follow an independent sequential order.

Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and let M be a smooth manifold on which G acts. Let $\mathcal{A}(M) = \sum_i \mathcal{A}^i(M)$ be the space of smooth differential forms on M and let $\mathcal{A}_{cpt}(M)$ be the subspace of forms with compact support. Let us recall (Cartan ;[10]) that the G-equivariant de Rham complex of M is by definition the differential $\mathbb{Z}_+$-graded algebra

$$\mathcal{A}_G(M) := (S(\mathfrak{g}') \otimes \mathcal{A}(M))^G$$

endowed with the tensor product graded algebra structure (where elements of $\mathfrak{g}'$ are assigned degree 2) together with the equivariant de Rham differential $d_\mathfrak{g}$ of degree 1 (see section 2, Formula 5). Its cohomology denoted $H^*_G(M)$ is called the G- equivariant de Rham cohomology of M. Alternatively, an element $\alpha \in \mathcal{A}_G(M)$ can be thought of as a differential form $\alpha(X)$ on M depending polynomially on $X \in \mathfrak{g}$, such that α is equivariant:

$$\alpha(g \cdot X) = g \cdot \alpha(X),$$

for all $g \in G$.

The complex $\mathcal{A}_G(M)$ admits a subcomplex

$$\mathcal{A}_{cpt,G}(M) := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M))^G,$$

and its cohomology is called the G-equivariant de Rham cohomology with compact support $H^*_{cpt,G}(M)$.

Sometimes, it is natural to consider the space $\mathcal{A}^\infty_G(M)$ of equivariant forms $\alpha(X)$ on M depending smoothly on $X \in \mathfrak{g}$. The differential $d_\mathfrak{g}$ extends to this space and the cohomology of the complex $(\mathcal{A}^\infty_G(M), d_\mathfrak{g})$ is denoted by $H^\infty_G(M)$.

This cohomology $H_G^{\infty}(M)$ is studied in the preceding article of this volume (notation differs slightly from the ones used in the preceding article. In particular, the dual vector space of $\mathfrak{g}$ is denoted here by $\mathfrak{g}'$ instead of $\mathfrak{g}^*$, the space denoted here by $\mathcal{A}_G^{\infty}(M)$ (resp. $H_G^{\infty}(M)$) was denoted by $\mathcal{A}_G^{\infty}(\mathfrak{g}, M)$ (resp.by $\mathcal{H}_G^{\infty}(\mathfrak{g}, M)$)). In some situations, it is also important to consider the space $\mathcal{A}_G^{-\infty}(M)$ of equivariant forms depending in a generalized way on the variable $X \in \mathfrak{g}$ (cf. section 2, Definition 3 for a precise definition). The differential $d_{\mathfrak{g}}$ still has a meaning on $\mathcal{A}_G^{-\infty}(M)$ and the cohomology of the complex $(\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}})$ is denoted by $H_G^{-\infty}(M)$. (The space $\mathcal{A}_G^{-\infty}(M)$ and its cohomology $H_G^{-\infty}(M)$ were introduced in [12].) One similarly defines $H_{cpt,G}^{-\infty}(M)$. When M is a point, $H_G^{-\infty}(point)$ is equal to the space $C^{-\infty}(\mathfrak{g})^G$ of G-invariant generalized functions on $\mathfrak{g}$. There is a natural map $H_G^{\infty}(M) \to H_G^{-\infty}(M)$.

If M is compact and G-oriented, integration over M gives us a map from $H_G^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$. More generally if $p : M \to B$ is a G-equivariant fibration, with G-oriented fibers, then there is defined an integration along the fiber map

$$p_* : H_{cpt,G}^{-\infty}(M) \to H_{cpt,G}^{-\infty}(B)$$

(cf. Formula 8).

If M is non compact, and if $\alpha(X)$ is an equivariant form on M depending smoothly on $X \in \mathfrak{g}$, the integral of $\alpha(X)$ over M may sometimes exist in a generalized sense: after integrating $\alpha(X)$ against a test function Φ on the Lie algebra $\mathfrak{g}$, the form $(\alpha, \Phi) := \int_{\mathfrak{g}} \alpha(X)\Phi(X)dX$ may become integrable over M and we can define $\int_M \alpha \in C^{-\infty}(\mathfrak{g})^G$ by

$$(\int_M \alpha, \Phi) = \int_M (\alpha, \Phi).$$

Many important examples of generalized functions on $\mathfrak{g}$ arise this way. For instance, characters of representations of G attached to a generic coadjoint orbit M are given by the integral of an equivariant form over M (see [21], [12]). If G is compact, the formula of [20] for the index of a G-transversally elliptic operator D on a compact G-manifold B is given by the integration (in the generalized sense) over $M = T^*B$ of a G-equivariant form $\alpha(\sigma)(X)$ on M (depending smoothly on $X \in \mathfrak{g}$) attached to the symbol σ of D.

It will thus be useful to understand the space $H_G^{-\infty}(M)$. The aim of this article is to start a systematic study of the cohomology space $H_G^{-\infty}(M)$.

Now we describe some of the results we prove in this article.

We first prove (in section 2) that for a G-equivariant real vector bundle $p : \mathcal{V} \to B$, the canonical pull-back map $p^* : H_G^{-\infty}(B) \to H_G^{-\infty}(\mathcal{V})$ is an isomorphism (cf. Proposition 8). In particular, for a real representation V of G, $H_G^{-\infty}(V) \cong C^{-\infty}(\mathfrak{g})^G$. Similarly, we prove the Thom isomorphism; asserting that if G is compact and the fibers of p are G-oriented, then the integration along the fiber map

$$p_* : H^{-\infty}_{cpt,G}(\mathcal{V}) \to H^{-\infty}_{cpt,G}(B)$$

is an isomorphism (cf. Proposition 11).

Let $K \subset G$ be a closed subgroup . Let $\chi = \chi_{G/K} : K \to \{\pm 1\}$ be the character of K defined by $\chi(k) = sign \det_{\mathfrak{g}/\mathfrak{k}} k$, for all $k \in K$. Let M be a K-manifold and let $H^{-\infty}_{K,\chi}(M)$ be the cohomology of the complex

$$(\mathcal{A}^{-\infty}_{K,\chi}(M), d_{\mathfrak{k}}) := (C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^{\chi}, d_{\mathfrak{k}})$$

(cf. Definition 49). Fix an orientation o on $\mathfrak{g}/\mathfrak{k}$. Consider the space $G \times_K M$, fibered over G/K with fiber M. In section 5, we define a cochain map

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,\chi}(M) \to \mathcal{A}^{-\infty}_{G}(G \times_K M)$$

(cf. Proposition 50) and prove that if K is compact, $Ind_{G/K,o}$ induces an isomorphism in cohomology (cf. Theorem 52). This is one of the central results of this article. The proof of this result relies on a study of the homology of the perturbed Koszul complexes defined in sections 3 and 4. This technique is already used in [13] for the study of G-equivariant cohomology with smooth coefficients.

Taking $M = point$, we get the isomorphism

$$C^{-\infty}(\mathfrak{k})^{\chi} \cong H^{-\infty}_{G}(G/K)$$

where $C^{-\infty}(\mathfrak{k})^{\chi} := \{f \in C^{-\infty}(\mathfrak{k}); k \cdot f = \chi(k) f, \text{ for all } k \in K\}$.

The explicit description of the isomorphism (cf. Proposition 43) indicates the analogy between $\int_{G/K} Ind_{G/K,o} f$ and characters of induced representations (cf. Proposition 44).

Recall [13] that $H^{\infty}_{G}(G/K)$ is canonically isomorphic to $C^{\infty}(\mathfrak{k})^{K}$. We determine the canonical map

$$C^{\infty}(\mathfrak{k})^{K} \cong H^{\infty}_{G}(G/K) \to C^{-\infty}(\mathfrak{k})^{\chi} \cong H^{-\infty}_{G}(G/K),$$

coming from the natural map $H^{\infty}_{G}(G/K) \to H^{-\infty}_{G}(G/K)$, in Proposition 53.

From now on in the introduction, the notation K will be reserved to denote a compact connected Lie group with maximal torus T and Weyl group W. The Lie algebras of K, T are denoted by $\mathfrak{k}$, $\mathfrak{t}$ respectively.

In section 6, we prove a Künneth theorem: Let D be a compact K-manifold such that $H_K(D)$ is free over $H_K(point)$ (e.g. $D = K/U$, for a closed subgroup $U \subset K$ of the same rank, cf. Lemma 65). Then, for any K-manifold M, the canonical Künneth map

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H^{-\infty}_{K}(M) \to H^{-\infty}_{K}(D \times M)$$

is an isomorphism (cf. Theorem 61). In fact Theorem 61 is true without the assumption on K to be connected, but then we need to add the assumption that the evaluation map $H_K(D) \to H(D)$ is surjective. In particular for any such D, taking M to be a point, we get

$$H_K^{-\infty}(D) \cong C^{-\infty}(\mathfrak{k})^K \otimes_{S(\mathfrak{k}')^K} H_K(D)$$

(cf. Corollary 64).

Using Künneth theorem and the Induction isomorphism, we obtain in Proposition 66 an extension of Chevalley's theorem. Let $C^{-\infty}(\mathfrak{t})^\epsilon$ be the space of all the generalized W-anti-invariant functions on $\mathfrak{t}$. Then the multiplication of generalized functions on $\mathfrak{t}$ by polynomial functions induces an isomorphism

$$C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \cong C^{-\infty}(\mathfrak{t}).$$

By the same technique, we obtain the Reduction Theorem asserting that for any K-manifold M, we have a canonical isomorphism

$$H_T^{-\infty}(M)^W \cong H_K^{-\infty}(M),$$

where $H_T^{-\infty}(M)^W$ refers to the W-invariants under the canonical action of W on $H_T^{-\infty}(M)$ (cf. Theorem 74). The proof of this reduction theorem is inspired by the proof of Theorem 4.2 in Atiyah [1].

Again combining the Künneth theorem and the Induction isomorphism, we obtain an isomorphism of $H_L(point)$-modules

$$H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_{L,\chi_{K/L}}^{-\infty}(M),$$

for any closed subgroup $L \subset K$ of the same rank and any K-manifold M (cf. Theorem 70). In particular, taking $M = K/U$ (for any closed subgroup U of K), we obtain

$$H_{L,\chi_{K/L}}^{-\infty}(K/U) \cong S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} (C^{-\infty}(\mathfrak{u})^{\chi_{K/U}}).$$

If M is a T-manifold, we give a homology spectral sequence (in section 10) with

$$E_p^2 = Tor_p^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$$

converging to the cohomology $H_T^{-\infty}(M)$, where $S(\mathfrak{t}')$ acts by multiplication on $C^{-\infty}(\mathfrak{t})$ (cf. Theorem 102). We show that this spectral sequence degenerates at the E^2-term for any homogeneous space $M = K/U$ (U any closed subgroup of K) (cf. Proposition 106).

In section 9, we study free actions. Let P be a principal G-bundle (for any Lie group G) and let $q : P \to P/G$ be the quotient map. Assume that the fibers of q admit a G-orientation o. Then we prove (cf. Theorem 89) that $H_G^{-\infty}(P)$ is a free module over $H_G(P) \cong H(P/G)$ with a generator γ_o. We determine this generator explicitly (cf. Proposition 80). Consider, for example, the free action of T on $P = K$ by right translations. Then the space $H_T^{-\infty}(P)$ is a vector space of dimension $|W|$ over $\mathbb{R}$. We use the description of $H_T^{-\infty}(K)$ to conclude that the canonical map $H_T^{\infty}(K) \to H_T^{-\infty}(K)$ is identically 0 (cf. Corollary 96).

More generally, we consider the case of a manifold P with a right action of a Lie group G and where we assume that a normal closed subgroup N of G acts principally on P (cf Definition 75). In addition, assume that the principal N-bundle $q_N : P \to P/N$ admits a G-invariant connection and that the fibers of q_N admit a G-orientation o. We then construct a map

$$m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P),$$

and show that m_o is an isomorphism if G is compact (cf. Theorem 91).

In section 11, we prove a Localization theorem for any compact oriented T-manifold M. We first need to take a T-equivariant embedding of M in a representation space V of T. This gives rise to a certain non-zero polynomial $P \in S(\mathfrak{t}')$. Now we determine

$$P(X) \int_M \alpha(X) \in C^{-\infty}(\mathfrak{t}),$$

for any $\alpha \in H_T^{-\infty}(M)$, in terms of the restriction of α to M^T and of the equivariant Euler class of the normal bundle of the submanifold $M^T \subset M$ (cf. Theorem 107). One striking difference from the smooth case is that it is possible to have $\int_M \alpha(X) \neq 0$, for $\alpha \in H_T^{-\infty}(M)$, even though M^T may be empty. In fact, we prove that $\int_K : H_T^{-\infty}(K) \to C^{-\infty}(\mathfrak{t})$ is injective, where T acts on K by right multiplication (cf. Proposition 95, section 9).

Finally in the appendix, we prove that if M is a paracompact manifold, then the de Rham differential d admits a continuous splitting on the space of exact differential forms on M. This result seems to be new and interesting on its own. In a similar way, we prove that the equivariant de Rham differential $d_{\mathfrak{g}}$ for the action of a compact Lie group G on a paracompact manifold M admits a continuous splitting. We were motivated to prove this result, as this enables us to obtain the spectral sequence of section 10 for any paracompact T-manifold.

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1 Notation

By a manifold M, we shall always mean a paracompact C^∞ real manifold without boundary (unless otherwise stated). We denote by $C^\infty(M)$ the space of C^∞- (real-valued) functions on M. We denote by $C^{-\infty}(M)$ the space of generalized (real-valued) functions on M. By definition, $C^{-\infty}(M)$ is the continuous dual of the space of smooth compactly supported densities on M under the C^∞-topology. The space $C^\infty(M)$ is canonically a subspace of $C^{-\infty}(M)$ and $C^{-\infty}(M)$ is a module over $C^\infty(M)$.

We denote the space of smooth differential forms on M (with real coefficients) by $\mathcal{A}^*(M)$. We denote the subspace of compactly supported differential forms by $\mathcal{A}^*_{cpt}(M)$. The exterior derivative is denoted by d_M or simply by d. If ξ is a vector field on M, we denote by $\iota(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ the contraction by the vector field ξ. We denote by $\mathcal{L}(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ the Lie derivative action of ξ. The operators $d, \iota(\xi), \mathcal{L}(\xi)$ on $\mathcal{A}(M)$ satisfy the Cartan relation :

$$d\iota(\xi) + \iota(\xi)d = \mathcal{L}(\xi). \tag{1}$$

If M is oriented, for $\alpha \in \mathcal{A}_{cpt}(M) = \oplus_{i=0}^{\dim M} \mathcal{A}^i_{cpt}(M)$, we note $\int_M \alpha$ the integral of the component of α in $\mathcal{A}^{\dim M}_{cpt}(M)$.

Let G be a real Lie group. By a G-manifold, we mean a manifold on which G acts smoothly. Let $\mathfrak{g}$ be the Lie algebra of G. If $X \in \mathfrak{g}$, we denote by X_M (or simply X, if no confusion is likely) the vector field on M such that

$$(X_M \cdot \varphi)(x) = \frac{d}{d\varepsilon}\varphi((\exp -\varepsilon X)x)|_{\varepsilon=0}$$

for $\varphi \in C^\infty(M)$, $x \in M$.

Unless otherwise stated, vector spaces are over $\mathbb{R}$, and the dual $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of a vector space V is denoted by V'. If $E^i, 1 \leq i \leq n$ is a basis of a n-dimensional vector space V, E_i denotes the dual basis of V'.

Tensor products without subscripts will mean over $\mathbb{R}$. Unless otherwise indicated, cohomology of a manifold is taken to be the de Rham cohomology (with real coefficients).

In this article, $\mathbb{Z}/2$-graded objects will carry a superscript $\bullet$, while $\mathbb{Z}$-graded objects will carry a superscript $*$. A vector space with a $\mathbb{Z}/2$-grading will often be called a superspace. In defining actions on the tensor product $V \otimes W$ of two $\mathbb{Z}/2$-graded vector spaces V, W, we will respect usual rules of signs. For example, an odd endomorphism A of the $\mathbb{Z}/2$-graded vector space W is extended to an endomorphism still denoted by A of $V \otimes W$ by defining

(2) $$A(v \otimes w) = v \otimes Aw \qquad \text{if } v \in V^{even}, w \in W.$$

(3) $$A(v \otimes w) = -v \otimes Aw \qquad \text{if } v \in V^{odd}, w \in W.$$

Any $\mathbb{Z}$-graded object C^* can of course be thought of as a $\mathbb{Z}/2$-graded object $C^\bullet$ by defining
$$C^{even} = \sum_{n\in\mathbb{Z}} C^{2n}, \qquad C^{odd} = \sum_{n\in\mathbb{Z}} C^{2n+1}.$$

Lie algebra of any real Lie group will be denoted by the same lower case German letter.

If a group G acts on a set E, we denote by E^G the subset of invariants.

2 G-equivariant cohomology with generalized coefficients - Basic definitions

Let G be a Lie group and let M be a G-manifold. Let us recall the definition of the G-equivariant de Rham cohomology $H^*_G(M)$ of M:

Let $S(\mathfrak{g}')$ be the symmetric algebra of $\mathfrak{g}'$. Consider the $\mathbb{Z}_+$-graded space $S(\mathfrak{g}') \otimes \mathcal{A}(M)$, where the degree of an element $P \otimes \alpha, P \in S^p(\mathfrak{g}'), \alpha \in \mathcal{A}^q(M)$ is defined by

(4) $$\deg(P \otimes \alpha) = 2p + q.$$

We refer to this degree as being the total degree.

Let E^i be a basis of $\mathfrak{g}$ and let $E_i \in \mathfrak{g}'$ be the dual basis. Define the operator $d_\mathfrak{g}$ of degree 1 on $S(\mathfrak{g}') \otimes \mathcal{A}(M)$ by:

$$d_\mathfrak{g}(P \otimes \alpha) = P \otimes d_M\alpha - \sum_i E_iP \otimes \iota(E^i_M)\alpha$$

for $P \in S(\mathfrak{g}'), \alpha \in \mathcal{A}(M)$. This expression is independent of the choice of the basis E^i, as the element $\sum_i E_i \otimes E^i \in \mathfrak{g}' \otimes \mathfrak{g}$ is the canonical element $I \in \mathrm{End}(\mathfrak{g})$, where I is the identity element of $\mathrm{End}(\mathfrak{g})$.

We often will identify $S(\mathfrak{g}')$ with the space of polynomial functions on $\mathfrak{g}$. Writing $X \in \mathfrak{g}$ as $X = \sum x_iE^i$, we identify E_i with the linear coordinate function x_i. An element α of the space $S(\mathfrak{g}') \otimes \mathcal{A}(M)$ can be viewed as a polynomial map $X \mapsto \alpha(X)$ from $\mathfrak{g}$ to $\mathcal{A}(M)$ and then the operator $d_{\mathfrak{g}}$ is given by the formula:

$$(5) \qquad (d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \iota(X_M)(\alpha(X))$$

or by

$$(6) \qquad (d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \sum_i x_i \iota(E^i_M)(\alpha(X)).$$

Consider the action of G on $S(\mathfrak{g}')$ induced from the adjoint representation of G on $\mathfrak{g}$ and the action of G on $\mathcal{A}(M)$ induced from the action of G on M. Let

$$\mathcal{A}_G(M) := (S(\mathfrak{g}') \otimes \mathcal{A}(M))^G$$

be the space of G-invariants in $S(\mathfrak{g}') \otimes \mathcal{A}(M)$. In other words, an element α of $\mathcal{A}_G(M)$ is an equivariant polynomial map (i.e.$\alpha(g \cdot X) = g \cdot (\alpha(X))$) from $\mathfrak{g}$ to $\mathcal{A}(M)$. The operator $d_{\mathfrak{g}}$ commutes with the tensor product action of G on $S(\mathfrak{g}') \otimes \mathcal{A}(M)$, thus $d_{\mathfrak{g}}$ preserves $\mathcal{A}_G(M)$. The Cartan relation (1)

$$\mathcal{L}(X_M) = d\iota(X_M) + \iota(X_M)d$$

implies $(d_{\mathfrak{g}}^2\alpha)(X) = -\mathcal{L}(X_M)(\alpha(X))$. Thus $(d_{\mathfrak{g}}^2\alpha)(X) = 0$ for $\alpha \in \mathcal{A}_G(M)$ and thus $(\mathcal{A}_G(M), d_{\mathfrak{g}})$ is a complex.

Definition 1 *Define:*

$$Z_G(M) = \{\alpha \in \mathcal{A}_G(M), d_{\mathfrak{g}}\alpha = 0\},$$

$$B_G(M) = \{\alpha \in \mathcal{A}_G(M), \alpha = d_{\mathfrak{g}}\beta, \text{for some} \beta \in \mathcal{A}_G(M)\}$$

and

$$H_G(M) = Z_G(M)/B_G(M).$$

The space $H_G(M)$ is called the *G-equivariant de Rham cohomology of M.* The cohomology $H_G(M)$ inherits the $\mathbb{Z}_+$-grading from $\mathcal{A}^*_G(M)$. The graded algebra $S(\mathfrak{g}')^G$ of invariant polynomial functions on $\mathfrak{g}$ acts by multiplication on $\mathcal{A}_G(M)$. This action commutes with the differential $d_{\mathfrak{g}}$. Thus $H^*_G(M)$ is a $\mathbb{Z}$-graded $S(\mathfrak{g}')^G$-module.

In particular for G reduced to the identity element, the space $Z_G(M) \subset \mathcal{A}(M)$ is the subspace $Z(M)$ of closed differential forms on M, the space $B_G(M) \subset \mathcal{A}(M)$ is the space $B(M)$ of exact differential forms and $H^*_G(M)$ is the usual de Rham cohomology $H^*(M)$ with real coefficients.

If K is a closed subgroup of G, the restriction to $\mathfrak{k}$ of a function defined on $\mathfrak{g}$ induces a map from $H^*_G(M)$ to $H^*_K(M)$. In particular, evaluation at $0 \in \mathfrak{g}$: $\alpha \mapsto \alpha(0)$ induces a map from $H^*_G(M)$ to $H^*(M)$.

The complex $\mathcal{A}_G(M)$ has a subcomplex

$$\mathcal{A}_{cpt,G}(M) := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M))^G.$$

The *compactly supported G-equivariant de Rham cohomology* $H^*_{cpt,G}(M)$ of M is defined as the cohomology of the complex $(\mathcal{A}^*_{cpt,G}, d_\mathfrak{g})$.

We may also consider the space $C^\infty(\mathfrak{g}, \mathcal{A}(M))$ of C^∞-maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. The group G acts naturally on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$. The operator $d_\mathfrak{g}$ is defined by the same formula (5) on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$. The space $C^\infty(\mathfrak{g}, \mathcal{A}(M))$ has a $\mathbb{Z}/2$-grading given by parity of differential forms. The operator $d_\mathfrak{g}$ is an odd operator on this superspace. However it is impossible to define a $\mathbb{Z}_+$-grading on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$, such that $d_\mathfrak{g}$ would be of degree 1.

We denote by

$$\mathcal{A}^\infty_G(M) = C^\infty(\mathfrak{g}, \mathcal{A}(M))^G,$$

the space of G-equivariant C^∞ maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. The Cartan relation implies again $d_\mathfrak{g}^2 = 0$ on $\mathcal{A}^\infty_G(M)$.

Definition 2 *Define:*

$$Z^\infty_G(M) = \{\alpha \in \mathcal{A}^\infty_G(M), d_\mathfrak{g}\alpha = 0\},$$

$$B^\infty_G(M) = \{\alpha \in \mathcal{A}^\infty_G(M), \alpha = d_\mathfrak{g}\beta \quad \text{for some}\, \beta \in \mathcal{A}^\infty_G(M)\}$$

and

$$H^\infty_G(M) = Z^\infty_G(M)/B^\infty_G(M).$$

Introduce (as in [12]) the space $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ of generalized functions on $\mathfrak{g}$ with values in the space $\mathcal{A}(M)$. This is, by definition, the space of continuous $\mathbb{R}$-linear maps $\mathrm{Hom}(\mathcal{D}(\mathfrak{g}), \mathcal{A}(M))$ from the space of smooth compactly supported densities $\mathcal{D}(\mathfrak{g})$ on $\mathfrak{g}$ to the space $\mathcal{A}(M)$, where $\mathcal{D}(\mathfrak{g})$ and $\mathcal{A}(M)$ are both endowed with the C^∞-topologies. Thus, if α is an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ and if Φ is a smooth compactly supported density on $\mathfrak{g}$, then (α, Φ) is a differential form on M denoted by $\int_\mathfrak{g} \alpha(X) d\Phi(X)$. A compactly supported C^∞ density on $\mathfrak{g}$ will be called a *test density* (on $\mathfrak{g}$). A compactly supported C^∞ function on $\mathfrak{g}$ will be called a *test function*. We write d_M for the operator on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ defined by

$$(d_M\alpha, \Phi) = d_M(\alpha, \Phi), \qquad \text{for}\, \Phi\, \text{a test density},$$

and ι for the operator defined by

$$(\iota\alpha, \Phi) = \sum_i \iota(E^i_M)(\alpha, x_i\Phi).$$

Then define the operator $d_\mathfrak{g}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ by

$$d_\mathfrak{g}\alpha = d_M\alpha - \iota\alpha.$$

Observe that for $\alpha \in C^{\infty}(\mathfrak{g}, \mathcal{A}(M)) \subset C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$, the operator $d_{\mathfrak{g}}$ coincides with the operator $d_{\mathfrak{g}}$ introduced above. We thus will also write informally

$$(d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \sum_i x_i \iota(E_M^i)(\alpha(X))$$

for $\alpha \in C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$.

The group G acts naturally on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$:

$$(g\alpha, \Phi) = g \cdot (\alpha, g^{-1} \cdot \Phi).$$

It can be easily seen that the operators d and ι commute with the action of G. Define

$$\mathcal{A}_G^{-\infty}(M) = C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))^G$$

as the space of G -equivariant $C^{-\infty}$-maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. An element of the space $\mathcal{A}_G^{-\infty}(M)$ will be called a *G-equivariant form with generalized coefficients*, or simply *an equivariant form*. If Φ is a test function on $\mathfrak{g}$, we denote by Φ^g the function $\Phi^g(X) = \Phi(gX)$. Let dX be an Euclidean measure on $\mathfrak{g}$. For $\alpha \in \mathcal{A}_G^{-\infty}(M)$ and $g \in G$, we have

$$|\det{}_{\mathfrak{g}}(g)|(\int_{\mathfrak{g}} \alpha(X)\Phi^g(X)dX) = g^{-1} \cdot (\int_{\mathfrak{g}} \alpha(X)\Phi(X)dX). \tag{7}$$

The operator $d_{\mathfrak{g}}$ preserves $\mathcal{A}_G^{-\infty}(M)$ and the Cartan relation (1) implies again $d_{\mathfrak{g}}^2 = 0$ on $\mathcal{A}_G^{-\infty}(M)$.

Definition 3 *Define:*

$$Z_G^{-\infty}(M) = \{\alpha \in \mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}}\alpha = 0\},$$

$$B_G^{-\infty}(M) = \{\alpha \in \mathcal{A}_G^{-\infty}(M), \alpha = d_{\mathfrak{g}}\beta \qquad \text{for some } \beta \in \mathcal{A}_G^{-\infty}(M)\}$$

and

$$H_G^{-\infty}(M) = Z_G^{-\infty}(M)/B_G^{-\infty}(M).$$

An equivariant form in $Z_G^{-\infty}(M)$ (resp. $B_G^{-\infty}(M)$) is said to be *closed* (resp. *exact*).

Observe that the parity of the exterior degree on $\mathcal{A}(M)$ induces a $\mathbb{Z}/2$-degree on the preceding spaces. We denote them by

$$Z_G^{-\infty}(M)^\bullet, B_G^{-\infty}(M)^\bullet, H_G^{-\infty}(M)^\bullet.$$

The ring $S(\mathfrak{g}')^G$ of invariant polynomial functions on $\mathfrak{g}$ acts by multiplication on $\mathcal{A}_G^{-\infty}(M)$. This action commutes with the differential $d_{\mathfrak{g}}$. Thus $H_G^{-\infty}(M)^\bullet$

is a $S(\mathfrak{g}')^G$-module. In fact $H_G^{-\infty}(M)$ is a module for $H_G^{\infty}(M)$ under left multiplication.

If M is a point, then, from the definition, it is clear that $H_G(point) = S(\mathfrak{g}')^G$, $H_G^{\infty}(point) = C^{\infty}(\mathfrak{g})^G$ and $H_G^{-\infty}(point) = C^{-\infty}(\mathfrak{g})^G$.

There is a natural map

$$H_G^{\infty}(M) \to H_G^{-\infty}(M).$$

Define similarly $(\mathcal{A}_{cpt,G}^{-\infty}(M), d_{\mathfrak{g}})$ as the subcomplex of $(\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}})$, consisting of all $\alpha \in \mathcal{A}_G^{-\infty}(M)$ such that $(\alpha, \Phi) \in \mathcal{A}_{cpt}(M)$, for all test densities Φ, and define $H_{cpt,G}^{-\infty}(M)$ as the cohomology of this subcomplex.

If $\phi : N \to M$ is a G- equivariant map between two G-manifolds, then the pull-back of differential forms induces a cochain map

$$\phi^* : (\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}}) \to (\mathcal{A}_G^{-\infty}(N), d_{\mathfrak{g}}),$$

in particular a map in cohomology (again denoted by)

$$\phi^* : H_G^{-\infty}(M) \to H_G^{-\infty}(N).$$

Thus the correspondence $M \mapsto H_G^{-\infty}(M)$ is a contravariant functor from the category of G-manifolds and G-equivariant maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g}')^G$-modules.

Similarly the correspondence $M \mapsto H_{cpt,G}^{-\infty}(M)$ is a contravariant functor from the category of G-manifolds and G-equivariant proper maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g}')^G$-modules.

Definition 4 *Let $p : M \to B$ be a G-equivariant fibration of G-manifolds. Then the map p is said to have* G-oriented *fibers if the fibers of p are oriented with an orientation varying continuously and if the G-action on M preserves the orientation of all the fibers.*

If $p : \mathcal{V} \to B$ is a G-equivariant real vector bundle and if p has G-oriented fibers, we will just say that the *vector bundle $\mathcal{V}$ is G-oriented.* We say that a *G-manifold B* is *G-oriented* if the tangent bundle of B is G-oriented.

If M is G-oriented, integration over M defines a map $\int_M$ from $\mathcal{A}_{cpt,G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$:

$$(\int_M \alpha, \Phi) := \int_M (\alpha, \Phi), \qquad \text{for any test density } \Phi \text{ on } \mathfrak{g}.$$

This map induces a map from $H_{cpt,G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$.

If $p : M \to B$ is a G-equivariant fibration of G-manifolds with G-oriented fibers, then the integration over the fibers gives a cochain map denoted by $\int_{M/B}$ or by p_* from $\mathcal{A}^{-\infty}_{cpt,G}(M)$ to $\mathcal{A}^{-\infty}_{cpt,G}(B)$:

$$(8) \qquad (\int_{M/B} \alpha, \Phi) := \int_{M/B} (\alpha, \Phi), \qquad \text{for any test density } \Phi \text{ on } \mathfrak{g}.$$

In particular, we get an induced map in cohomology (again denoted by) p_* or $\int_{M/B}$ (depending on the choice of a G-orientation):

$$p_* : H^{-\infty}_{cpt,G}(M) \to H^{-\infty}_{cpt,G}(B).$$

Similarly if, in addition, p is a proper map, we get the integration map

$$p_* : H^{-\infty}_G(M) \to H^{-\infty}_G(B).$$

Observe that if $\alpha \in H^{\infty}_{cpt,G}(M), \beta \in H^{-\infty}_G(B)$, then $\alpha \wedge p^*\beta \in H^{-\infty}_{cpt,G}(M)$ and we have

$$(9) \qquad p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(Our sign convention for p_* is as in [3], chapter 1.)

If $\mathcal{E} \to M$ is a G-equivariant real vector bundle, we introduce the manifold $M_{\mathcal{E}}$: An element of $M_{\mathcal{E}}$ is a couple (m, o), where $m \in M$ and o is an orientation of the fiber $\mathcal{E}_m$. Then $M_{\mathcal{E}}$ has a canonical G-manifold structure. Define a G-equivariant diffeomorphism of $M_{\mathcal{E}}$ by $\epsilon(m, o) = (m, -o)$. As in section 5 of [13], we may also consider the $\mathcal{E}$-twisted cohomology group $H_G(M)_{\mathcal{E}}$, which is by definition the cohomology of the complex

$$\mathcal{A}_G(M)_{\mathcal{E}} := \{\alpha \in \mathcal{A}_G(M_{\mathcal{E}}); \epsilon \cdot \alpha = -\alpha\}.$$

We define similarly the $\mathcal{E}$-twisted groups $H^{\pm\infty}_G(M)_{\mathcal{E}}$ and $H^{\pm\infty}_{cpt,G}(M)_{\mathcal{E}}$. If $p : M \to B$ is a G-equivariant fibration with vertical tangent bundle $\mathcal{V}$, then we get the integration map

$$p_* : H^{\pm\infty}_{cpt,G}(M)_{\mathcal{V}} \to H^{\pm\infty}_{cpt,G}(B).$$

If $TM \to M$ is the tangent bundle, the manifold M_{TM} is denoted by M_t, the TM-twisted group $H_G(M)_{TM}$ is denoted by $H_G(M)_t$ and the compactly supported twisted group by $H_{cpt,G}(M)_t$. If M is G-oriented, the space $H_G(M)_t$ is canonically isomorphic with $H_G(M)$. In general $H_G(M)_t$ is a module over $H_G(M)$. Taking the fibration $p : M \to point$, we get

$$p_* = \int_M : H_{cpt,G}(M)_t \to S(\mathfrak{g}')^G$$

even if M is not oriented. We thus can form the bilinear map $(,) : H_G(M) \times H_{cpt,G}(M)_t \to S(\mathfrak{g}')^G$ defined by: for $\alpha \in H_G(M)$ and $\beta \in H_{cpt,G}(M)_t$

$$(\alpha, \beta) = \int_M \alpha\beta.$$

If $G = e$, the bilinear form above (over $\mathbb{R}$) is non-degenerate. For lack of reference we include a proof of the following proposition.

Proposition 5 *Let G be a compact Lie group and let M be a G-manifold. If the evaluation map $H_G(M) \to H(M)$ is surjective, then*

1. *the evaluation map $H_{cpt,G}(M)_t \to H_{cpt}(M)_t$ is surjective*

2. *the spaces $H_G(M)$ and $H_{cpt,G}(M)_t$ are free modules over $S(\mathfrak{g}')^G$*

3. *if M is compact, the bilinear form $(.,.)$ induces an isomorphism of $H_G(M)_t$ with $Hom_{S(\mathfrak{g}')^G}(H_G(M), S(\mathfrak{g}')^G)$.*

Remark 6 *Let G be a compact connected Lie group and let M be a G-manifold such that $H_G(M)$ is a free $H_G(point)$-module. Then the Eilenberg-Moore spectral sequence (see [16], chapter 3, section 1) degenerates at the E^2-term. In particular, the evaluation map $H_G(M) \to H(M)$ is surjective. Observe that the assumption that G is connected can not be dropped here. Consider for example $G = O(3)$ and $M = O(3)/O(2)$.*

Proof: Let $n = \dim M$. For $\alpha \in S(\mathfrak{g}') \otimes \mathcal{A}(M)$ we define the exterior degree of α to be the smallest integer k such that $\alpha \in S(\mathfrak{g}') \otimes \left(\sum_{i=0}^k \mathcal{A}^i(M)\right)$ and we write $\alpha = \sum_{i=0}^k \alpha_{[i]}$ with $\alpha_{[i]} \in S(\mathfrak{g}') \otimes \mathcal{A}^i(M)$.

Let us prove (1). As $H_G(M)$ surjects on $H(M)$ under evaluation at 0, the group G acts trivially on $H(M)$ and (by Poincaré duality) also on $H_{cpt}(M)_t$. Let $[\alpha] \in H_{cpt}(M)_t$ and let α be an element of $Z_{cpt}(M)_t^G$ representing the (G-invariant) cohomology class $[\alpha]$. Assume α to be homogeneous. Let $\tilde{\alpha}$ be a homogeneous (for the total degree given by formula 4) element of $\mathcal{A}_{cpt,G}(M)_t := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M)_t)^G$ such that $\tilde{\alpha}(0) = \alpha$. Let $\beta = d_{\mathfrak{g}}\tilde{\alpha}$. As $d\alpha = 0$, $\beta(0) = 0$. Let k be the exterior degree of β. Let $\tilde{\nu} \in Z_G(M)$ be homogeneous of total degree $(n - k)$. Since $\tilde{\nu}$ is $d_{\mathfrak{g}}$-closed,

$$(\tilde{\nu}, \beta) = \int_M \tilde{\nu}\beta = \int_M \tilde{\nu}(d_{\mathfrak{g}}\tilde{\alpha}) = 0.$$

As only the terms of exterior degree n of $\tilde{\nu}\beta$ contribute to the integral, we obtain that the polynomial $(\tilde{\nu}(0), \beta_{[k]})$ is 0. If $\beta_{[k]} = \sum P_i\beta_i$, where $P_i \in S(\mathfrak{g}')$

are linearly independent and $\beta_i \in \mathcal{A}^k(M)$, we obtain $\sum_i P_i \int_M \tilde{\nu}(0)\beta_i = 0$. Thus $\int_M \tilde{\nu}(0)\beta_i = 0$ for all homogeneous elements $\tilde{\nu} \in Z_G(M)$ of total degree $n-k$. Since the evaluation map $H_G(M) \to H(M)$ is surjective, this implies that $\beta_i \in B_{cpt}(M)_t$ (by Poincaré duality). Thus $\beta_{[k]} \in (S(\mathfrak{g}') \otimes B_{cpt}(M)_t)^G$. As G is compact, we can choose $\beta_{[k]} = d\omega$ with homogeneous $\omega \in (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M)_t)^G$ of exterior degree $(k-1)$ and such that $\omega(0) = 0$. Thus $\gamma := \tilde{\alpha} - \omega$ is still such that $\gamma(0) = \alpha$ and $d_{\mathfrak{g}}(\gamma)$ is of exterior degree strictly less than k. By induction on k, we can construct a $d_{\mathfrak{g}}$-closed form κ such that $\kappa(0) = \alpha$. Thus the evaluation map at 0 gives a surjective map from $H_{cpt,G}(M)_t$ to $H_{cpt}(M)_t$.

Let us prove (2): Let x_a be a homogeneous basis of $H(M)$ over $\mathbb{R}$ and let α_a be any homogeneous elements of $Z_G(M)$ such that $[\alpha_a(0)] = x_a$. Let us prove that the cohomology classes of the elements α_a form a system of generators for $H_G(M)$ over $S(\mathfrak{g}')^G$. Let $\mathcal{H} \subset Z(M)^G$ be the subspace generated by the elements $\alpha_a(0)$. We have $Z(M) = \mathcal{H} \oplus B(M)$. Take $\alpha \in Z_G(M)$ (say of exterior degree k). Then

$$\alpha_{[k]} \in (S(\mathfrak{g}') \otimes Z(M))^G = S(\mathfrak{g}')^G \otimes \mathcal{H} \oplus (S(\mathfrak{g}') \otimes B(M))^G.$$

Arguing as in the proof of (1), we see that the cohomology class of α is congruent to an element $\beta + \sum_a P_a \alpha_a$ with $P_a \in S(\mathfrak{g}')^G$ and $\beta \in Z_G(M)$ of exterior degree strictly less than k. By induction on k, this proves that the equivariant cohomology classes of the elements α_a form a system of generators for $H_G(M)$. Let us prove that they are independent over $S(\mathfrak{g}')^G$. Let $\beta = \sum_a P_a \alpha_a \in Z_G(M)$ be such that β is 0 in $H_G(M)$. Let k be the maximum of the exterior degrees of those α_a's such that $P_a \neq 0$. Then for every $\nu \in Z_{cpt,G}(M)_t$ homogeneous of total degree $(n-k)$, the polynomial

$$(\beta, \nu) = \sum_a P_a \int_M (\alpha_a(0))_{[k]} \nu(0)$$

is equal to zero. By (1) the evaluation map $H_{cpt,G}(M)_t \to H_{cpt}(M)_t$ is surjective. Thus the polynomial map $\sum_{deg x_a = k} P_a x_a$ from $\mathfrak{g}$ to $H(M)$ is identically 0. But the elements x_a being linearly independent, this implies that the elements P_a for which x_a is of exterior degree k are identically 0. This is in contradiction with the definition of k. This prove that $\{\alpha_a\}$ are linearly independent over $S(\mathfrak{g}')^G$. One can similarly prove that $H_{cpt,G}(M)_t$ is a free module over $S(\mathfrak{g}')^G$.

Proceeding in a similar way and using Poincaré duality we can construct (if M is compact) a basis $\alpha^b \in H_G(M)_t$ such that $(\alpha_a, \alpha^b) = \delta_a^b$. This proves (3). ∎

Definition 7 *If Q is a continuous operator on $\mathcal{A}(M)$, we still denote by Q the operator on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ defined by*

$$(Q \cdot \alpha, \Phi) = Q \cdot (\alpha, \Phi), \qquad \text{for all test densities } \Phi \text{ on } \mathfrak{g}.$$

We will often write the above as $(Q\alpha)(X) = Q \cdot (\alpha(X))$ *and say that* Q *is the* pointwise extension *of* Q *to* $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$.

We start to compute the space $H_G^{-\infty}$ in some elementary situations.

Let $p : \mathcal{V} \to B$ be a G-equivariant real vector bundle over B. Let $i : B \to \mathcal{V}$ be the inclusion from B to $\mathcal{V}$ as the zero section.

Proposition 8 *The canonical maps*

$$i^* : H_G^{-\infty}(\mathcal{V}) \to H_G^{-\infty}(B)$$

and

$$p^* : H_G^{-\infty}(B) \to H_G^{-\infty}(\mathcal{V})$$

are inverses to each other. In particular, both of them are isomorphisms.

Proof: The proof is similar to the Poincaré lemma for the de Rham complex. Let $\mathcal{R}$ be the vertical Euler vector field on $\mathcal{V}$. Extend the operator $\mathcal{L}(\mathcal{R})$ pointwise to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. Extend similarly the operator $\iota(\mathcal{R})$ to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $\iota(E^i)\iota(\mathcal{R}) + \iota(\mathcal{R})\iota(E^i) = 0$, for all i, Cartan relation implies

$$\mathcal{L}(\mathcal{R}) = d_{\mathfrak{g}}\iota(\mathcal{R}) + \iota(\mathcal{R})d_{\mathfrak{g}}$$

on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $\mathcal{R}$ commutes with the action of G, the operators $\mathcal{L}(\mathcal{R})$, $\iota(\mathcal{R})$ preserve $\mathcal{A}_G^{-\infty}(\mathcal{V})$. Let $h_t(v) = tv$. Let $\beta \in \mathcal{A}(\mathcal{V})$. Then $h_t^*\beta = \beta$ for $t = 1$, while $h_t^*\beta = p^*i^*\beta$ for $t = 0$. We compute

$$\frac{d}{dt}h_t^*\beta = t^{-1}\mathcal{L}(\mathcal{R})h_t^*\beta.$$

(As $\mathcal{R}$ vanishes at 0, the right hand side depends smoothly on $t \in \mathbb{R}$). Clearly, the same relation persists for differential forms with parameters. Thus for $\beta \in \mathcal{A}_G^{-\infty}(\mathcal{V})$, we have

$$\frac{d}{dt}h_t^*\beta = t^{-1}\mathcal{L}(\mathcal{R})h_t^*\beta = t^{-1}(\iota(\mathcal{R})d_{\mathfrak{g}} + d_{\mathfrak{g}}\iota(\mathcal{R}))h_t^*\beta.$$

Define $H : \mathcal{A}_G^{-\infty}(\mathcal{V}) \to \mathcal{A}_G^{-\infty}(\mathcal{V})$ by

$$H\beta = \int_0^1 h_t^*(\iota(\mathcal{R})\beta)t^{-1}dt.$$

Then we obtain

$$\beta - p^*i^*\beta = \int_0^1 \frac{d}{dt}h_t^*\beta dt = (d_{\mathfrak{g}}H + Hd_{\mathfrak{g}})\beta,$$

for all $\beta \in A_G^{-\infty}(\mathcal{V})$. Thus we see that $p^*i^* = I$ in cohomology, where I is the identity operator. Of course, $pi = I$, in particular $i^*p^* = I$. This proves the proposition. ∎

Considering the case of a vector space V, we have

Corollary 9 *Let G be a Lie group and V be a finite dimensional real representation space for G, then*

$$H_G^{-\infty}(V) = C^{-\infty}(\mathfrak{g})^G.$$

We now prove the Thom isomorphism for compactly supported cohomology of vector bundles.

Definition 10 *Let G be a compact Lie group and let $p : \mathcal{V} \to B$ be a G-equivariant G-oriented vector bundle over a compact base B.*

An element $u \in H_{cpt,G}(\mathcal{V})$ such that $p_* u = 1$ in $H_G(B)$ will be called a Thom class.

Given a G-orientation o on $\mathcal{V}$, recall [18] that there exists a unique Thom class $u_o \in H_{cpt,G}(\mathcal{V})$. Multiplication by u_o induces a map m_o: $m_o(\alpha) = u_o \wedge p^*\alpha$ from $H_G(B)$ to $H_{cpt,G}(\mathcal{V})$ and the map m_o is an isomorphism. Similarly, as $u_o \in H_{cpt,G}(\mathcal{V})$, we can define the map $m_o(\alpha) = u_o \wedge p^*\alpha$ from $H_G^{-\infty}(B)$ to $H_{cpt,G}^{-\infty}(\mathcal{V})$.

Proposition 11 . *Let G be a compact Lie group and let $\mathcal{V} \to B$ be a G-equivariant G-oriented real vector bundle over a compact base B. Then the maps*

$$m_o : H_G^{-\infty}(B) \to H_{cpt,G}^{-\infty}(\mathcal{V})$$

and

$$p_* : H_{cpt,G}^{-\infty}(\mathcal{V}) \to H_G^{-\infty}(B)$$

are inverses to each other. In particular both of them are isomorphisms.

Remark 12 *In the above proposition, we have assumed the base B to be compact, just in order to simplify notation. If B is not necessarily compact, the same proof will lead to isomorphism of $H_{cpt,G}^{-\infty}(B)$ with $H_{cpt,G}^{-\infty}(\mathcal{V})$. It is also clear that a similar Thom isomorphism will hold between $H_G^{-\infty}(B)$ and the cohomology $H_{vcpt,G}^{-\infty}(\mathcal{V})$ of the complex of equivariant forms on $\mathcal{V}$ with compact support along the fibers.*

Proof: The proof is similar to the proof of the Thom isomorphism in equivariant cohomology given in [19]. For $\alpha \in \mathcal{A}(\mathcal{V})$, we denote by $\tilde{\alpha}$ the image of α under the automorphism $x \mapsto -x$ of $\mathcal{V}$. Let us consider the bundle $\mathcal{V} \oplus \mathcal{V}$ over B and let $\sigma(x, y) = (y, -x)$ be the automorphism of $\mathcal{V} \oplus \mathcal{V}$. Denote by $\sigma_t, t \in \mathbb{R}$ the transformation

$$\sigma_t(x, y) = ((\cos t)x + (\sin t)y, -(\sin t)x + (\cos t)y)$$

of the fibers of $\mathcal{V} \oplus \mathcal{V}$. Then σ_0 is the identity, while $\sigma_{\pi/2}$ is equal to σ. Let

$$S = (\frac{d}{dt}\sigma_t)$$

be the vector field on $\mathcal{V} \oplus \mathcal{V}$ induced by the group of transformations σ_t. We have

$$\mathcal{L}(S) = d\iota(S) + \iota(S)d$$

on $\mathcal{A}_{cpt}(\mathcal{V} \oplus \mathcal{V})$.

Extending these transformations pointwise to $C^{-\infty}(\mathfrak{g}, \mathcal{A}_{cpt}(\mathcal{V}\oplus\mathcal{V}))$ (cf. Definition 7) we obtain the relation:

$$\mathcal{L}(S) = d_{\mathfrak{g}}\iota(S) + \iota(S)d_{\mathfrak{g}}$$

on $C^{-\infty}(\mathfrak{g}, \mathcal{A}_{cpt}(\mathcal{V} \oplus \mathcal{V}))$. The transformations σ_t commute with the action of G. Thus $\iota(S)$ and $\mathcal{L}(S)$ preserve $\mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$.

Define $H : \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V}) \to \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$ by

$$H\nu = \int_0^{\pi/2} (\sigma_t^*\iota(S)\nu)dt.$$

We obtain, as in the proof of the preceding proposition (8)

$$\sigma^*\nu - \nu = (d_{\mathfrak{g}}H + Hd_{\mathfrak{g}})\nu$$

for any $\nu \in \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$.

Let $p_i : \mathcal{V} \oplus \mathcal{V} \to \mathcal{V}, i = 1,2$ be the natural maps obtained by projections on the first or second component respectively. Consider $\alpha \in \mathcal{A}^{\infty}_{cpt,G}(\mathcal{V})$ and $\beta \in \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V})$. Then $p_1^*\alpha \wedge p_2^*\beta$ is a well defined element of $\mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$. If α, β are closed equivariant forms, then $p_1^*\alpha \wedge p_2^*\beta$ is closed and is in the same cohomology class as $\sigma^*(p_1^*\alpha \wedge p_2^*\beta) = p_2^*\alpha \wedge p_1^*\tilde{\beta}$. Let us integrate over the fibers of p_2. It is clear that $(p_2)_*(p_1^*\alpha \wedge p_2^*\beta) = p^*(p_*\alpha)\beta$. Thus the equality in cohomology

$$p_1^*\alpha \wedge p_2^*\beta \cong p_2^*\alpha \wedge p_1^*\tilde{\beta}$$

implies (if N is the rank of $\mathcal{V}$, and $|\alpha| \in \{0,1\}$ is the parity of α):

$$p^*(p_*\alpha) \wedge \beta \cong (-1)^{N(|\alpha|-1)}\alpha \wedge p^*(p_*\beta)$$

in cohomology. In particular let $\alpha = u_o$ be the equivariant Thom class of $\mathcal{V} \to B$, which is of parity N. We obtain the relation

$$\beta \cong u_o \wedge p^*p_*\beta$$

in $H^{-\infty}_{cpt,G}(\mathcal{V})$. Thus, we see that $m_o p_* = I$ in cohomology. Of course $p_* m_o = I$. This proves the proposition. ∎

We finish this section by giving an example of a non-trivial closed equivariant form with generalized coefficients.

We need a notation.

Let V be a real vector space of dimension n. Let ν' be a non-zero element in $\Lambda^n V'$. We denote by $|\nu'|^{-1}\delta_V$ the element of $C^{-\infty}(V)$ defined by

$$(10) \qquad \int_V |\nu'|^{-1}\delta_V(X)\Phi(X)dX = \Phi(0), \qquad \text{for any test function } \Phi \text{ on } V,$$

where dX is the Euclidean density on V determined by ν'.

We denote by $\delta_{V,o} \in C^{-\infty}(V) \otimes \Lambda^n V'$ the element:

$$(11) \qquad \delta_{V,o} = |\nu'|^{-1}\delta_V \otimes \nu'.$$

The element $\delta_{V,o}$ depends only on the orientation o of V determined by ν'.

Let G be a Lie group. Consider the action of G on itself by left translations. Let $n = \dim G$. Fix an orientation o on $\mathfrak{g}$. Let $\nu' \in \Lambda^n \mathfrak{g}'$ be a positive element. Let dg be the unique left invariant form of maximal degree on G such that $(dg)_e = \nu'$, where e is the identity element of G.

Lemma 13 *The form*

$$\alpha_{G,o}(X) := |\nu'|^{-1}\delta_{\mathfrak{g}}(X) \otimes |\det{}_{\mathfrak{g}} g| dg$$

is a closed equivariant form on G, which depends only on the choice of o.

Proof: The form $\alpha_{G,o}$ is equivariant, as $g_0 \cdot (|\nu'|^{-1}\delta_{\mathfrak{g}}) = |\det{}_{\mathfrak{g}} g_0|(|\nu'|^{-1}\delta_{\mathfrak{g}})$ and $|\det{}_{\mathfrak{g}}(g_0^{-1}g)| = |\det{}_{\mathfrak{g}} g_0|^{-1}|\det{}_{\mathfrak{g}} g|$. It is immediate to see that $\alpha_{G,o}$ depends only on o. As $\alpha_{G,o}$ is of maximal degree, $d\alpha_{G,o} = 0$. Also, as the generalized function $|\nu'|^{-1}\delta_{\mathfrak{g}}$ is annihilated by multiplication by all the coordinates functions on $\mathfrak{g}$, we see that $\iota\alpha_{G,o} = 0$. ∎

We will prove in section 5 that $H_G^{-\infty}(G) = \mathbb{R}\alpha_{G,o}$

3 Koszul complexes

Our main aim in sections 4 and 5 will be the study of the cohomology of "perturbed" Koszul complexes. Thus, in this section, we recall some well-known facts on Koszul complexes.

Let V be a finite dimensional real vector space of dimension n with basis $e^i, 1 \leq i \leq n$ and dual basis $e_i \in V'$. Let $S(V')$ be the ring of polynomial functions on V. Let L be a $S(V')$-module. We still denote by e_i the action of $e_i \in V'$ on L. We denote by $\iota(e^i) : \Lambda^* V' \to \Lambda^{*-1} V'$ the contraction by the vector $e^i \in V$.

Consider the space

$$L \otimes \Lambda^* V'$$

which is $\mathbb{Z}_+$-graded by the exterior degree. On this space, the operator

$$j_L = \sum_{i=1}^{n} e_i \otimes \iota(e^i) \tag{12}$$

is an operator of degree -1 and its square is zero. We denote by $H(j_L)$ the homology space of j_L. It is a $\mathbb{Z}_+$-graded vector space. If $L = S(V')$ (considered as a $S(V')$-module under multiplication), we denote the operator j_L by j_V. We denote:

$$A^* = S(V') \otimes \Lambda V'.$$

The following proposition is basic

Proposition 14 *Consider the operator j_V on A^*. We have:*

1. *If $i > 0, H_i(j_V) = 0$.*

2. *If $i = 0$, the map $\phi \mapsto \phi(0)$ from $A^0 = S(V')$ to $\mathbb{R}$ induces an isomorphism from $H_0(j_V)$ with $\mathbb{R}$.*

Even though this proposition is well known, we give a proof as we will use the explicit homotopy given below in the rest of the article.

Proof: We identify the space $A^* = S(V') \otimes \Lambda^* V'$ with the space of differential forms with polynomial coefficients on V. We write an element $x \in V$ as $x = \sum_i x_i e^i$, so that $e_i(x) = x_i$. If $I = (i_1, i_2, \ldots, i_k)$ is a multi-index: $1 \leq i_1 < \cdots < i_k \leq n$, we identify $e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k} \in \Lambda V'$ with the k-form $dx_I = dx_{i_1} dx_{i_2} \cdots dx_{i_k}$.

Consider the partial derivative ∂^i in the direction of $e^i \in V$. Let E_V be the Euler vector field $E_V = \sum_i x_i \partial^i$. We denote the contraction operator $\iota(e^i)$ on $\Lambda V'$ by ι^i. Thus

$$j_V = \sum_i x_i \iota^i = \iota(E_V).$$

We denote by ϵ_i the multiplication by dx_i. Let $d_V = \sum_i \partial^i \otimes \epsilon_i$ be the de Rham differential on A^*. Let $\mathcal{L}_V$ be the Euler operator on $S(V') \otimes \Lambda V'$ given by the Lie derivative action of E_V.

$$\mathcal{L}_V(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + kf) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

for $f \in S(V')$.

Then, Cartan relation implies $\mathcal{L}_V = d_V j_V + j_V d_V$.

Consider the subcomplex A_0^* of A^* such that $A_0^k = A^k$, if $k > 0$, while $A_0^0 = \{\phi \in S(V'), \phi(0) = 0\}$. The operator $\mathcal{L}_V$ keeps A_0^* stable and induces an invertible operator of degree 0 on A_0^*. We can give an integral formula for the inverse F_V of the Euler operator $\mathcal{L}_V$: Define F_V on A_0^* by

1. If $f dx_I \in A^k$, $k > 0$,

$$F_V(f dx_I) = (\int_0^1 f(tx)t^{k-1}dt)dx_I.$$

2. If $f \in A_0^0$

$$F_V f = \int_0^1 f(tx)t^{-1}dt.$$

It is well defined as $f(tx)$ vanishes for $t = 0$.

It is immediate to see that $F_V \mathcal{L}_V \phi = \phi$ for every $\phi \in A_0^*$. The operator $F_V = \mathcal{L}_V^{-1}$ commutes with j_V and d_V.

Let $h_V := F_V d_V$, then, if $\phi \in A_0^*$,

$$\phi = (h_V j_V + j_V h_V)\phi.$$

This formula clearly implies the proposition. ∎

Let L, N be two $S(V')$-modules. The tensor product space (over $\mathbb{R}$) $L \otimes N$ is given a structure of $S(V')$-module by defining the action of an element $f \in V'$ to be $f \cdot (m \otimes n) = fm \otimes n - m \otimes fn$.

Consider the operator $j_{L\otimes N}$ on $L \otimes N \otimes \Lambda V'$. The homology space $H_0(j_{L\otimes N})$ in degree 0 is the quotient of $L \otimes N$ by the subspace spanned by elements of the form $fm \otimes n - m \otimes fn$, for $f \in V'$. This quotient is by definition $L \otimes_{S(V')} N$:

$$H_0(j_{L\otimes N}) = L \otimes_{S(V')} N.$$

If N is a $S(V')$-module, denote by N^0 the space N with the trivial action of V'.

Lemma 15 *Let N be a $S(V')$-module. The operator $R := \exp \sum_i \partial^i \otimes e_i$ gives an isomorphism of the $S(V')$-module $S(V') \otimes N$ with the $S(V')$-module $S(V') \otimes N^0$.*

Proof: It is sufficient to check this assertion when V is a 1-dimensional vector space, where it is checked easily. ∎

Corollary 16 *If L is a free $S(V')$-module, then $L \otimes N$ is also free; hence $H_i(j_{L\otimes N}) = 0$ if $i > 0$.*

Lemma 17 *For any $S(V')$-modules L and N, the homology space of the operator $j_{L\otimes N}$ on the complex*

$$L \otimes N \otimes \Lambda V'$$

is equal to the torsion group $Tor^{S(V')}(L, N)$, i.e.,

$$H_i(j_{L\otimes N}) = Tor_i^{S(V')}(L, N).$$

Proof: Let us consider the complex

$$0 \to S(V') \otimes N \otimes \Lambda^n V' \overset{j_{S(V')\otimes N}}{\longrightarrow} \cdots \overset{j_{S(V')\otimes N}}{\longrightarrow} S(V') \otimes N \otimes V' \overset{j_{S(V')\otimes N}}{\longrightarrow} S(V') \otimes N \overset{m}{\to} N \to 0$$

where the last map is the surjective map $m : S(V') \otimes N \to N : m(\phi \otimes n) = \phi n$. By the preceding corollary, this complex is exact. Furthermore if we endow the space $S(V') \otimes N \otimes \Lambda V'$ with the $S(V')$ module structure $S(V') \otimes (N \otimes \Lambda V')^0$, the homomorphisms $j_{S(V')\otimes N}$ and m are $S(V')$-module morphisms. Thus the complex above is a free resolution of N as a $S(V')$-module. We may calculate the torsion group $Tor^{S(V')}(L, N)$ using this resolution. The space $L \otimes_{S(V')} S(V') \otimes (N \otimes \Lambda^* V')^0$ is isomorphic with $L \otimes N \otimes \Lambda^* V'$. The operator $I \otimes_{S(V')} (j_{S(V')\otimes N})$ under this isomorphism becomes the operator $j_{L\otimes N}$. This proves the lemma. ∎

We now introduce another vector space P considered as a parameter space. Let $W = V \oplus P$. We write an element $w \in W$ as $w = x + y$, with $x \in V$, $y \in P$.

Let us consider the space

$$A^{\infty,*} = C^\infty(W) \otimes \Lambda^* V'.$$

The multiplication by the coordinate function x_i is an operator on $C^\infty(W)$. Thus the operator $j_V^\infty := \sum_{i=1}^n x_i \otimes \iota(e^i)$ is an operator of degree -1 on $A^{\infty,*}$ and $(j_V^\infty)^2 = 0$. Let $r_P : C^\infty(W) \to C^\infty(P)$ be the restriction map.

Proposition 18 *Consider the operator j_V^∞ on the complex $A^{\infty,*}$. We have*

1. *If $i > 0, H_i(j_V^\infty) = 0$.*

2. *If $i = 0$, the map r_P from $A^0 = C^\infty(W)$ to $C^\infty(P)$ induces an isomorphism from $H_0(j_V^\infty)$ with $C^\infty(P)$.*

Proof: The method of proof is identical to the proof of Proposition 14. For simplicity, we denote j_V^∞ by j_V. Let

$$d_V = \sum_i \partial^i \otimes \epsilon_i \tag{13}$$

be the partial de Rham differential in the direction of V on the complex $A^{\infty,*}$. Let $\mathcal{L}_V$ be the Euler operator on $A^{\infty,*}$ with respect to the variables x_i, dx_i:

$$\mathcal{L}_V(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + kf) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

if $f \in C^\infty(W)$.

Then, $\mathcal{L}_V = d_V j_V + j_V d_V$.

Let $A_0^{\infty,*}$ be the subcomplex of $A^{\infty,*}$ defined by $A_0^{\infty,k} = A^{\infty,k}$ if $k > 0$ and $A_0^{\infty,0} = \{\phi \in C^\infty(W); r_P\phi = 0\}$.

The Euler operator $\mathcal{L}_V$ induces an operator of degree 0 on $A_0^{\infty,*}$, which is invertible. Its inverse F_V is given explicitly by an integral formula as in the proof of Proposition 14:

Definition 19 *Let us consider the operator F_V of degree 0 on $A_0^{\infty,*}$ defined by*

1. *If $f dx_I \in A_0^{\infty,k}$, $k > 0$,*

$$F_V(f dx_I) = (\int_0^1 f(tx + y) t^{k-1} dt) dx_I.$$

2. *If $f \in A_0^{\infty,0}$,*

$$F_V f = \int_0^1 f(tx + y) t^{-1} dt.$$

It is well defined as $f(tx + y)$ vanishes for $t = 0$.

The operator F_V commutes with d_V, j_V. It is easy to prove

$$F_V \mathcal{L}_V \phi = \phi$$

for every $\phi \in A_0^{\infty,*}$. Thus if

$$h_V = F_V d_V, \tag{14}$$

$$\phi = (h_V j_V + j_V h_V)\phi, \qquad \text{for } \phi \in A_0^{\infty,*}.$$

Thus h_V is a homotopy for the complex $A_0^{\infty,*}$. The existence of h_V implies that the subcomplex $A_0^{\infty,*}$ is exact. This in turn implies the proposition. ∎

Observe that if $f \in C^\infty_{cpt}(W) \otimes \Lambda V'$ is compactly supported, then $F_V f$ is not necessarily compactly supported.

Remark 20 *If a Lie group G acts linearly on V and P, the operator j_V^∞ commutes with the action of G. Thus $(A^{\infty,*})^G$ is a subcomplex of $A^{\infty,*}$. The homotopy h_V, that we have constructed above, commutes with the action of G. It results that the homology of the subcomplex $(A^{\infty,*})^G$ of $A^{\infty,*}$ exists only in degree 0 and in degree 0 is isomorphic to $C^\infty(P)^G$.*

Now, consider the space:

$$A^{-\infty,*} = C^{-\infty}(W) \otimes \Lambda^* V'.$$

The operator $j_V^{-\infty}$ is similarly defined on $A^{-\infty,*}$. The complex $j_V^{-\infty} : A^{-\infty,*} \to A^{-\infty,*-1}$ is called the *Koszul complex with $C^{-\infty}$-coefficients* (with the space P as a parameter space).

Choose an orientation o on V. By Formula (11) of section 2, this determines an element $\delta_{V,o}$ of $C^{-\infty}(V) \otimes \Lambda^n V'$. If $f \in C^{-\infty}(P)$ is a generalized function on P, the product $\delta_{V,o}(x) f(y)$ is in $C^{-\infty}(W) \otimes \Lambda^n V'$.

It is easy to identify the homology of $j_V^{-\infty}$ in top degree.

Lemma 21 *The kernel of $j_V^{-\infty}$ on $A^{-\infty,n} = C^{-\infty}(W) \otimes \Lambda^n V'$ is equal to the space $\delta_{V,o} \otimes C^{-\infty}(P)$.*

Proof: As $x_i \delta_{V,o}(x) = 0$ for all i, the subspace $\delta_{V,o}(x) \otimes C^{-\infty}(P)$ of $C^{-\infty}(W) \otimes \Lambda^n V'$ is in $Ker(j_V^{-\infty})$. Reciprocally if $f \otimes \nu' \in C^{-\infty}(W) \otimes \Lambda^n V'$ is such that $j_V^{-\infty}(f \otimes \nu') = 0$, we see that $x_i f(x+y) = 0$ for all i. Thus f is the product of the δ- function on the transverse subspace V with a generalized function on P. ∎

Proposition 22 *We have*

1. *If $i \neq n$, $H_i(j_V^{-\infty}) = 0$.*

2. *$H_n(j_V^{-\infty}) = \delta_{V,o} \otimes C^{-\infty}(P)$*

Remark 23 *Let K be a compact group acting linearly on W and preserving the direct sum decomposition $W = V \oplus P$. Thus the group K acts on $A^{-\infty,*}$ and the operator j_V commutes with the action of K. Hence $(A^{-\infty,*})^K$ is a subcomplex of $A^{-\infty,*}$. Let $\chi(k) := det_V(k)$. Then, K being compact, $\chi(k) = \pm 1$ and moreover $k \cdot \delta_{V,o} = \chi(k)\delta_{V,o}$. By averaging over K the equation $\alpha = j_V^{-\infty}\beta$, we see that the homology of the subcomplex $(A^{-\infty,*})^K$ of $A^{-\infty,*}$ is also equal to 0, except in top degree n, while in top degree $H_n((A^{-\infty,*})^K) = \delta_{V,o} \otimes C^{-\infty}(P)^\chi$, where $C^{-\infty}(P)^\chi = \{f \in C^{-\infty}(P); k \cdot f = \chi(k) f, \text{for all } k \in K\}$.*

We now give a proof of Proposition 22.

Proof: Let us consider the subcomplex $L^* = C^{-\infty}_{cpt,V}(W) \otimes \Lambda^* V'$ of $A^{-\infty,*}$, where $C^{-\infty}_{cpt,V}(W)$ denotes the space of generalized functions ϕ on W, with support contained in a set of the form $F \oplus P$ where F is a compact subset of V (F depending upon the generalized function ϕ) .

Lemma 24 *The inclusion $L^* \to A^{-\infty,*}$ induces an isomorphism in homology.*

Proof: Let us denote the operator $j_V^{-\infty}$ by j_V. Choose a scalar product on V and an orthonormal basis e^i of V. Let $e_i \in V'$ be the dual basis and let us denote by ϵ_i the exterior multiplication by e_i on $\Lambda V'$. Let ϵ_V be the operator of degree $+1$ on $A^{-\infty,*}$ defined by $\epsilon_V = \sum_i x_i \epsilon_i$. It is easily verified that $\epsilon_V j_V + j_V \epsilon_V = |x|^2 I$.

Let χ be a smooth function on V such that $\chi(x) = 1$ for $|x| < 1/2$ and $\chi(x) = 0$ for $|x| > 1$. Extend χ to a smooth function on W by setting $\chi(x+y) = \chi(x)$. The multiplication by the function χ on $A^{-\infty}$ commutes with j_V. It sends $A^{-\infty}$ to L. The function $(1-\chi(x))(|x|^2)^{-1}$ is a C^∞ function on V, thus on W. We write for $\phi \in C^{-\infty}(W) \otimes \Lambda V'$

$$\phi = \phi_0 + \phi_1$$

with $\phi_0 = \chi\phi$ and

$$\phi_1 = \frac{1-\chi(x)}{|x|^2}|x|^2\phi = \frac{1-\chi(x)}{|x|^2}(j_V\epsilon_V + \epsilon_V j_V)\phi.$$

If $j_V\phi = 0$, both elements ϕ_0 and ϕ_1 are annihilated by j_V. Furthermore ϕ_1 is in the image of j_V. Thus each element of $H(A^{-\infty})$ has a representative in the subcomplex L and hence the natural map $H(L) \to H(A^{-\infty})$ is surjective. Now let $\phi \in L$ be such that $\phi = j_V\alpha$ with $\alpha \in A^{-\infty}$. We can find a C^∞ function θ on W such that it is equal to 1 on the support of ϕ and such that $\theta\alpha \in L$. Thus $\phi = \theta\phi = j_V(\theta\alpha)$ and the homology class of ϕ in $H(L)$ is zero. Thus the natural map $H(L) \to H(A^{-\infty})$ is injective. ∎

If $f \otimes \nu' \in L^n$, i.e $f \in C^{-\infty}_{cpt,V}(W)$, we can define $I(f \otimes \nu') \in C^{-\infty}(P)$ by

$$\int_P I(f \otimes \nu')(y)\phi(y)dy = \int_W f(x,y)\phi(y)dxdy$$

where dx is the positive density on V associated to ν'. If $\ell = \delta_{V,o}(x)g(y)$, then $I(\ell) = g$. We denote by L_0^* the subcomplex of L^* such that $L_0^k = L^k$ if $k \neq n$, while $L_0^n = \{\phi \in L^n, I(\phi) = 0\}$. We will construct an explicit homotopy for $j_V^{-\infty}$ on the subcomplex L_0^*.

Consider the complex $R^* = C^\infty_{cpt,P}(W) \otimes \Lambda^* V'$, where $C^\infty_{cpt,P}(W)$ denotes the space of smooth functions on W, with support contained in a set of the form $V \oplus F$ where F is a compact subset of P. Define the subcomplex R_0^* by $R_0^k := R^k$, if $k > 0$, while $R_0^0 := \{\phi \in R^0; r_P\phi = 0\}$, where r_P is the restriction map from $C^\infty(W)$ to $C^\infty(P)$. The operator F_V given in Definition 19 preserves R_0^*. Thus the homotopy $h_V = F_V d_V$ of $A_0^{\infty,*}$ is also a homotopy for R_0^*.

Let us choose a Lebesgue measure dy on P. The pairing $(,)$ between the complexes L^* and R^{n-*} defined by

$$(\phi dx_I, f dx_J) = \int_W \phi(x+y)f(x+y)(dx_I \wedge dx_J)dy$$

is a non degenerate pairing. The space L_0^n is the orthogonal of the subspace $1 \otimes C^\infty_{cpt}(P)$ of functions on W constant in the $x \in V$ variables. Thus $(,)$ induces a non degenerate pairing between L_0^* and R_0^{n-*}.

The operator j_V satisfies

$$(j_V\alpha, \beta) + (-1)^{|\alpha|}(\alpha, j_V\beta) = 0$$

for $\alpha \in L^*, \beta \in R^{n-*}$.

Thus we can transpose the homotopy for R_0^* and obtain a homotopy for L_0^*. More explicitly, define the operator U_V of degre 0 on L_0^* by

$$(U_V\alpha, \beta) = (\alpha, F_V\beta)$$

for $\alpha \in L_0^*, \beta \in R_0^{n-*}$. We extend the partial de Rham differential d_V from $A^{\infty,*}$ to $A^{-\infty,*}$ (again denoted by d_V) by the same formula (13). The operator d_V also satisfies

$$(d_V\alpha, \beta) + (-1)^{|\alpha|}(\alpha, d_V\beta) = 0.$$

Thus $d_V L^{n-1} \subset L_0^n$ and d_V is an operator of degree 1 on L_0^*. The operator U_V commutes with d_V. Let $k_V = U_V d_V = d_V U_V$. Then, for $\alpha \in L_0$, $\beta \in R_0$,

$$(k_V\alpha, \beta) + (-1)^{|\alpha|}(\alpha, h_V\beta) = 0.$$

Thus, for $\alpha \in L_0$,

$$\alpha = (k_V j_V + j_V k_V)\alpha$$

as follows from the transpose relation $\beta = (h_V j_V + j_V h_V)\beta$.

The complex L_0^* is thus exact and this implies the proposition. ∎

Let us now consider a Lie group G with Lie algebra $\mathfrak{g}$. Recall the definition of the Koszul differential c_L on the space $T^* = L \otimes \Lambda^* \mathfrak{g}'$ calculating the cohomology of a $\mathfrak{g}$-module L: For $\alpha \in L \otimes \Lambda^p \mathfrak{g}'$, $c_L\alpha \in L \otimes \Lambda^{p+1}\mathfrak{g}'$ is defined by

$$(15) \qquad \begin{aligned}(c_L\alpha)(X_1, ..., X_{p+1}) = \\ \sum_i (-1)^{i+1} X_i \cdot \alpha(X_1, ..., \hat{X}_i, ..., X_{p+1}) \\ + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1}),\end{aligned}$$

where X_1,...,X_{p+1} are elements of $\mathfrak{g}$.

Consider $\iota(X) : T^* \to T^{*-1}$, the contraction by an element $X \in \mathfrak{g}$ and $\mathcal{L}(X)$ the action of $\mathfrak{g}$ by tensor product on T. It is not difficult to verify the relation $\mathcal{L}(X) = c_L\iota(X) + \iota(X)c_L$. The space $\Lambda\mathfrak{g}'$ acts by exterior multiplication on T and c_L satisfies the Leibniz's rule: $c_L(\alpha\xi) = c(\alpha)\xi + (-1)^{|\alpha|}\alpha c_L(\xi)$, where c in this formula denotes the Koszul differential of the complex $\Lambda\mathfrak{g}'$ (corresponding to the trivial one dimensional representation L).

Let K be a Lie subgroup of G. Assume that L is a $(\mathfrak{g}, K)$-module. Consider the subspace

$$T_K^* = (L \otimes \Lambda^*(\mathfrak{g}/\mathfrak{k})')^K$$

of $L \otimes \Lambda\mathfrak{g}'$. From the relation $\mathcal{L}(X) = c_L\iota(X) + \iota(X)c_L$, it is easy to see that (T_K^*, c_L) is a subcomplex of $(L \otimes \Lambda^*\mathfrak{g}', c_L)$. The cohomology of the subcomplex (T_K^*, c_L) is by definition the relative Lie algebra cohomology $H^*(\mathfrak{g}, K, L)$ of the $(\mathfrak{g}, K)$-module L.

Consider the algebra $D(\mathfrak{g})$ of differential operators on $\mathfrak{g}$ with polynomial coefficients. Then the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ determines a Lie algebra homomorphism τ from $\mathfrak{g}$ into $D(\mathfrak{g})$. If L is a $D(\mathfrak{g})$-module, then L is a $\mathfrak{g}$-module, via the adjoint action. Furthermore, as $D(\mathfrak{g})$ contains $S(\mathfrak{g}')$, the module L is a $S(\mathfrak{g}')$-module.

Let e^i be a basis of $\mathfrak{g}$, $e_i \in \mathfrak{g}'$ the dual basis. We consider the element $\Omega = \sum_i e_i\tau(e^i)$ of $D(\mathfrak{g})$

Lemma 25 *The element $\Omega \in D(\mathfrak{g})$ is identically* 0.

Proof: For $X \in \mathfrak{g}$, denote by ∂_X the constant coefficient vector field on $\mathfrak{g}$ equal to X. The adjoint vector field $\tau(e^i)$ is given by: $\tau(e^i) = -\sum_j x_j\partial_{[e^i, e^j]}$ and hence

$$\Omega = -\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$$

which is equal to zero, as the vector field $-\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$ is the vector field equal at the point $X \in \mathfrak{g}$ to $[X, X] = 0$. ∎

Let L be a $D(\mathfrak{g})$-module. Consider the space

$$A^* := L \otimes \Lambda\mathfrak{g}'.$$

As L is a $S(\mathfrak{g}')$-module, we can consider the operator $j = j_L : A^* \to A^{*-1}$ given (as in Formula (12)) by $j = \sum_i e_i \otimes \iota(e^i)$.

On the other hand, the $\mathfrak{g}$-module structure on L gives rise to the Koszul differential $c : L^* \to L^{*+1}$.

Lemma 26 . *Let L be a $D(\mathfrak{g})$-module. The operator $j + c$ satisfies*

$$(j+c)^2 = 0$$

Proof: As $j^2 = 0$, $c^2 = 0$, we have to verify that $jc + cj = 0$. We have, (setting $\iota^i = \iota(e^i)$) and using the Leibniz's rule:

$$jc+cj = \sum_i (e_i\iota^i c + ce_i\iota^i) = \sum_i e_i(\iota^i c + c\iota^i) + \sum_i c(e_i)\iota^i = \sum_i e_i\mathcal{L}(e^i) + \sum_i c(e_i)\iota^i.$$

The action $\mathcal{L}(e^i)$ is by the tensor product action $\tau(e^i) \otimes I + I \otimes \mathcal{L}_\Lambda(e^i)$, where $\mathcal{L}_\Lambda(e^i)$ is the action of $\mathfrak{g}$ on $\Lambda\mathfrak{g}'$ induced from the adjoint representation. Thus, as $\sum_i e_i\tau(e^i) = 0$ from the preceding lemma, it remains to see that $\sum_i e_i\mathcal{L}_\Lambda(e^i)\xi + \sum_i c(e_i)\iota^i\xi = 0$ for $\xi \in \Lambda\mathfrak{g}'$. Writing ξ as a product of elements $\alpha \in \mathfrak{g}'$, it is sufficient to prove this relation for $\xi \in \mathfrak{g}'$ where this is checked easily. ∎

The spaces $L = C^{\pm\infty}(\mathfrak{g})$ have natural $D(\mathfrak{g})$-module structures. Thus on $L \otimes \Lambda\mathfrak{g}'$, the operators j and c are defined and satisfy $(j+c)^2 = 0$. We will see in section 5 that we obtain this example of perturbed Koszul complex when computing the G-equivariant cohomology of a Lie group G provided with the free action of G on itself given by left translation. We compute the cohomology of slightly more complicated complexes in the next section.

4 Induction of equivariant differential complexes

Let (A, d) be a differential complex, i.e. a $\mathbb{Z}/2$-graded vector space over $\mathbb{R}$, with a differential d of odd degree. We will assume that A is a Fréchet space and that d is continuous. (In most of the applications, A will be the space of smooth differential forms on a G-manifold M.) Let G be a Lie group acting on A. We assume that the action of G on A is differentiable. As in H.Cartan [10], we say that (A, d) is a *G-differential complex*, if

1. The action of G preserves the $\mathbb{Z}/2$-grading and commutes with d.

2. There are given continuous contraction operators $\iota(X), X \in \mathfrak{g}$ of odd degree satisfying $\iota(X)\iota(Y) + \iota(Y)\iota(X) = 0$ for all $X, Y \in \mathfrak{g}$ and $g\iota(X)g^{-1} = \iota(gX)$, for all $g \in G$, $X \in \mathfrak{g}$.

3. The Lie derivative action $\mathcal{L}(X)$ of the action of G on A satisfies $\mathcal{L}(X) = d\iota(X) + \iota(X)d$, for all $X \in \mathfrak{g}$.

If A is a G-differential complex, define the space A_{horG} of *G-horizontal* elements to be

$$A_{horG} = \{\alpha \in A; \iota(Y)\alpha = 0, \quad \text{for all Y} \in \mathfrak{g}\}.$$

The space A_{basG} of *G-basic* elements is by definition the space of elements of A which are G-invariant and horizontal:

$$A_{basG} = A^G_{horG}.$$

Remark that the differential d leaves the space A_{basG} stable.

Let (A, d) be a G-differential complex. We call (A, d) a *G-differential algebra* if, in addition, A has a $\mathbb{Z}/2$-graded algebra structure satisfying the following:

1. The action of any $g \in G$ on A is by algebra automorphisms.

2. The operator d and the operators $\iota(X), X \in \mathfrak{g}$, are odd derivations of the algebra A.

If G acts smoothly on a manifold M, then $\mathcal{A}^\bullet(M) = \mathcal{A}^{even}(M) \oplus \mathcal{A}^{odd}(M)$ is a G-differential algebra. If L is a differentiable G-module, the tensor product G-module $L \otimes \Lambda\mathfrak{g}'$ together with the Koszul differential c_L and the contraction operators $I \otimes \iota(X)$ is a G-differential complex. In particular, taking L to be the trivial one dimensional G-module, $\Lambda\mathfrak{g}'$ is a G-differential complex, in fact a G-differential algebra.

The tensor product (over $\mathbb{R}$) of two G-differential complexes is canonically a G-differential complex. Thus, for any G- differential complex A, we can form the G-differential complex $A \otimes \Lambda\mathfrak{g}'$. We write an element $\alpha \in A \otimes \Lambda\mathfrak{g}'$ as $\alpha = \sum_k \alpha_{[k]}$ with $\alpha_{[k]} \in A \otimes \Lambda^k\mathfrak{g}'$. We denote by $r : A \otimes \Lambda\mathfrak{g}' \to A$ the projection of an element $\alpha \in A \otimes \Lambda\mathfrak{g}'$ on its component $\alpha_{[0]}$ of exterior degree 0.

Lemma 27 *Let A be a G-differential complex. The map $r : A \otimes \Lambda\mathfrak{g}' \to A$ induces an isomorphism from $(A \otimes \Lambda\mathfrak{g}')_{horG}$ to A.*

Proof: For $X \in \mathfrak{g}$, we denote by $\iota_t(X)$ the tensor product contraction on $A\otimes\Lambda\mathfrak{g}'$. If $\alpha \in (A\otimes\Lambda\mathfrak{g}')_{horG}$ is such that $\alpha_{[0]} = 0$, it is easy to see by induction on the exterior degree that $\alpha = 0$. Let us prove that r is surjective: Let E^i be a basis of $\mathfrak{g}$ with dual basis $E_i \in \mathfrak{g}'$. Denote by ϵ_i the exterior multiplication by E_i on $A\otimes\Lambda\mathfrak{g}'$ from the left. Let $h_i = I - \epsilon_i\iota_t(E^i)$. It is easy to see that the operators

h_i commute with each other. In particular, the operator $h = \prod_i (I - \epsilon_i \iota_t(E^i))$ acting on $A \otimes \Lambda\mathfrak{g}'$ is well defined, i.e. it does not depend upon the order in which the product is taken. We verify : $\iota_t(E^i)h_i = 0, \iota_t(E^j)h_i = h_i\iota_t(E^j)$ for $i \neq j$. Thus h is a projector from $A \otimes \Lambda\mathfrak{g}'$ to the set $(A \otimes \Lambda\mathfrak{g}')_{hor_G}$. If $a \in A = A \otimes \Lambda^0\mathfrak{g}'$, the element $h(a) = a - (-1)^{|a|-1} \sum_i \iota(E^i)a \otimes E_i + \cdots$ is a horizontal element of $A \otimes \Lambda\mathfrak{g}'$ with component of exterior degree 0 equal to a. Thus r is surjective. ∎

Definition 28 *The operator $h := \prod_i (I - \epsilon_i \iota_t(E^i))$ on $A \otimes \Lambda\mathfrak{g}'$ is called the horizontal projection.*

By the preceding lemma, if $\alpha \in A \otimes \Lambda\mathfrak{g}'$, the element $h(\alpha)$ is the unique horizontal element of $A \otimes \Lambda\mathfrak{g}'$ whose component of exterior degree 0 is $\alpha_{[0]}$.

If (A, d_A) is a G-differential complex, we can define the spaces $\mathcal{A}^{\pm\infty} := C^{\pm\infty}(\mathfrak{g}, A)$. We denote both of them by $\mathcal{A}$ when there is no need to indicate precisely the smoothness properties of a function $f : \mathfrak{g} \to A$ that we assume. The space $\mathcal{A}$ inherits a $\mathbb{Z}/2$-graded structure from that of A.

For any $E \in \mathfrak{g}$, the contraction operator $\iota(E)$ is extended to $\mathcal{A}$ pointwise:

$$(\iota(E)f)(X) = \iota(E)(f(X)).$$

Similarly the differential d_A is extended on $\mathcal{A}$ by

$$(d_A f)(X) = d_A(f(X)).$$

Thus we can define on $\mathcal{A}$ the operator

$$\iota_{\mathfrak{g}} = \sum x_i \iota(E^i)$$

i.e. $(\iota_{\mathfrak{g}} f)(X) = \sum_i x_i(\iota(E^i)f(X))$ and the operator

$$d_{\mathfrak{g}} = d_A - \iota_{\mathfrak{g}}.$$

When $\mathfrak{g}$ is understood, we will just write ι for $\iota_{\mathfrak{g}}$. If we take $A = \mathcal{A}(M)$, for a G-manifold M, then $\mathcal{A}^{\pm\infty}$ was introduced in section 2 and $d_{\mathfrak{g}}$ here coincides with the operator $d_{\mathfrak{g}}$ of section 2.

Lemma 29 *The operator $d_{\mathfrak{g}}$ is odd and satisfies $d_{\mathfrak{g}}^2 = 0$ on*

$$\mathcal{A}_G^{\pm\infty} := C^{\pm\infty}(\mathfrak{g}, A)^G.$$

More generally if χ is a character of G trivial on its connected component, then

$$(\mathcal{A}_{G,\chi}^{\pm\infty}, d_{\mathfrak{g}}) := (C^{\pm\infty}(\mathfrak{g}, A)^\chi, d_{\mathfrak{g}})$$

is a complex, where

$$C^{\pm\infty}(\mathfrak{g}, A)^\chi := \{f \in C^{\pm\infty}(\mathfrak{g}, A) : g \cdot f = \chi(g)f, \text{ for all } g \in G\}.$$

Proof: It is easy to see that $\iota^2 = 0$. So to prove that $d_{\mathfrak{g}}^2 f = 0$ for $f \in \mathcal{A}_{G,X}^{\pm\infty}$, it suffices to show that $(d_A\iota + \iota d_A)f = 0$. Now $((d_A\iota + \iota d_A)f)(X) = \sum_i x_i \mathcal{L}_A(E^i)(f(X))$, where $\mathcal{L}_A(E^i)$ is the Lie derivative action on A of the element $E^i \in \mathfrak{g}$. The invariance condition on an element $f \in C^{\pm\infty}(\mathfrak{g}, A)^X$ implies that $\mathcal{L}_A(E^i)(f(X)) = \frac{d}{d\epsilon} f(X + \epsilon[E^i, X])$. Thus, for $f \in C^{\pm\infty}(\mathfrak{g}, A)^X$, we obtain in the notation of Lemma 25 (section 3)

$$((d_A\iota + \iota d_A)f)(X) = \sum_i x_i \frac{d}{d\epsilon} f(X + \epsilon[E^i, X]) = \sum_{i,j} x_i x_j \partial_{[E^i, E^j]} f(X) = 0.$$

∎

Let G be a Lie group and let K be a closed subgroup of G. Let (A, d_A) be a K-differential complex. Let

$$L^{\pm\infty} = C^{\pm\infty}(\mathfrak{g}, A).$$

As for $\mathcal{A}$, we denote both of these by L, when there is no need to indicate the precise smoothness assumption. The $\mathbb{Z}/2$-graded structure on A induces a $\mathbb{Z}/2$-structure on L. Consider on L the structure of G-module, defined by $(g \cdot f)(X) = f(g^{-1} \cdot X)$, for $f \in C^{\pm\infty}(\mathfrak{g}, A)$. This structure of G-module of course induces a structure of $\mathfrak{g}$-module on differentiation. Thus on $L \otimes \Lambda\mathfrak{g}'$, we can define the Koszul differential c (associated to this $\mathfrak{g}$-module structure on L). However, we take in account sign rules in defining c: c coincides with the Koszul differential $c_{L^{even}}$ (see Formula 15 of section 3) on $L^{even} \otimes \Lambda\mathfrak{g}'$, while we define $c = -c_{L^{odd}}$ on $L^{odd} \otimes \Lambda\mathfrak{g}'$.

We extend the operator d_A pointwise on L: $(d_A f)(X) = d_A(f(X))$. We still denote by d_A the operator $d_A \otimes I$ on $L \otimes \Lambda\mathfrak{g}'$.

Let E^i be a basis of $\mathfrak{g}$. We extend the operator $\iota_\Lambda(E^i)$ to $L \otimes \Lambda\mathfrak{g}'$ following the sign rules (2), (3) given in section 1.

Consider the operator

$$j = \sum x_i \iota_\Lambda(E^i)$$

on $L \otimes \Lambda\mathfrak{g}'$.

Lemma 30 *The operator $c_{\mathfrak{g}} := d_A + c + j$ satisfies $c_{\mathfrak{g}}^2 = 0$.*

Proof: The $\mathfrak{g}$-module structure on L is induced from its $D(\mathfrak{g})$-module structure via the adjoint representation. It follows from Lemma 26 of section 3 that we have $(c + j)^2 = 0$. Furthermore, as can be easily seen, we have $(c + j)d_A + d_A(c + j) = 0$. ∎

The space L is equipped with the operators still denoted by $\iota_A(E), E \in \mathfrak{k}$, defined pointwise by their action on A.

Total contraction operators $\iota_t(Y), Y \in \mathfrak{k}$, are defined by the tensor product contraction on $L \otimes \Lambda\mathfrak{g}'$. Let us consider the action of K on L by

$$(k \cdot f)(X) = k \cdot f(k^{-1}X)$$

and the action on $L \otimes \Lambda\mathfrak{g}'$ by tensor product. We denote by $\mathcal{L}_t(Y), Y \in \mathfrak{k}$, the corresponding infinitesimal action of $Y \in \mathfrak{k}$.

We define $(L \otimes \Lambda\mathfrak{g}')_{horK} = \{\alpha \in L \otimes \Lambda\mathfrak{g}'; \iota_t(Y)\alpha = 0, \text{for all } Y \in \mathfrak{k}\}$. The subspace $(L\otimes\Lambda\mathfrak{g}')_{basK}$ of K-basic elements is the subspace of elements of $L\otimes\Lambda\mathfrak{g}'$ which are horizontal and invariant under K:

$$(L \otimes \Lambda\mathfrak{g}')_{basK} = ((L \otimes \Lambda\mathfrak{g}')_{horK})^K.$$

Lemma 31 *The operators $c_A := c + d_A$ and j preserve the subspace of K-basic elements.*

Proof: The operator j clearly preserves the space of horizontal elements. It commutes with the action of K, thus it preserves the space of K basic elements.

Using the relation $\iota_t(Y)c_A + c_A\iota_t(Y) = \mathcal{L}_t(Y)$, we see that the space of K-basic elements is stable under c_A. ∎

Definition 32 *Let A be a K-differential complex. Define the induced complex $Ind^{\pm\infty}_{G/K}A$ from K to G of the K-differential complex A to be the space*

$$Ind^{\pm\infty}_{G/K}A = (C^{\pm\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}$$

with the differential $c_\mathfrak{g} = c + d_A + j$.

If $(A, d_A) = (\mathbb{R}, 0)$, then $Ind^{\pm\infty}_{G/K}\mathbb{R} = (C^{\pm\infty}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}/\mathfrak{k})')^K$ with differential $c_\mathfrak{g} = c + j$, where c is the Koszul differential of the $(\mathfrak{g}, K)$-module $C^{\pm\infty}(\mathfrak{g})$.

Our aim is to compute the cohomology of the complex $(Ind^{\pm\infty}_{G/K}A, c_\mathfrak{g})$ in terms of the cohomology of the complex $(C^{\pm\infty}(\mathfrak{k}, A)^K, d_\mathfrak{k})$.

The cohomology of the complex $Ind^{\infty}_{G/K}A$ is determined in [13]. Recall the results:

Theorem 33 *The restriction map*

$$r_\mathfrak{k} : C^\infty(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}' \to C^\infty(\mathfrak{k}, A)$$

given by

$$(r_\mathfrak{k}\alpha)(Y) = \alpha(Y) \qquad \text{for } \alpha \in \mathrm{C}^\infty(\mathfrak{g}, \mathrm{A}), \mathrm{Y} \in \mathfrak{k}$$
$$(r_\mathfrak{k}\alpha) = 0 \qquad \text{if } \alpha \in \mathrm{C}^\infty(\mathfrak{g}, \mathrm{A}) \otimes \Lambda^{\geq 1}\mathfrak{g}'$$

defines a cochain map from $(Ind^{\infty}_{G/K}(A), c_\mathfrak{g})$ to $(C^\infty(\mathfrak{k}, A)^K, d_\mathfrak{k})$. Furthermore if the principal bundle $G \to G/K$ possesses a G-invariant connection, then the restriction map $r_\mathfrak{k}$ induces an isomorphism in cohomology.

In particular, if $A = \mathbb{R}$, we obtain a map $r_{\mathfrak{k}} : Ind_{G/K}\mathbb{R} \to C^\infty(\mathfrak{k})^K$ and this map is an isomorphism in cohomology when K is compact.

Remark here that the restriction map does not extend to $Ind^{-\infty}_{G/K}(A)$ as generalized functions cannot usually be restricted to a subspace.

The assumption that $G \to G/K$ possesses a G-invariant connection is satisfied for example when K is a reductive Lie group, in particular when K is a compact subgroup.

We now consider $Ind^{-\infty}_{G/K}A$. Let $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$. We identify $\mathfrak{r}'$ with $\mathfrak{k}^\perp = \{f \in \mathfrak{g}', f|\mathfrak{k} = 0\}$. Let $n = \dim \mathfrak{r}$. Consider the character $\chi_{G/K}(k) = sign(det_{\mathfrak{r}} k)$ of K. As G and K are fixed, we denote $\chi_{G/K}$ simply by χ. Let $\nu' \in \Lambda^n\mathfrak{r}'$ be a nonzero element. The element ν' determines an Euclidean measure $|d\nu'|$ and an orientation o on $\mathfrak{r}$. If dX is an Euclidean measure on $\mathfrak{g}$, we denote by dY the Euclidean measure on $\mathfrak{k}$ such that $dX = |d\nu'|dY$.

Remark that, as $n = \dim \mathfrak{g}/\mathfrak{k}$, the space $\Lambda\mathfrak{k}' \otimes \Lambda^n\mathfrak{r}'$ is naturally embedded as a subspace of $\Lambda\mathfrak{g}'$. Consider the horizontal projection operator (see Definition 28) $h_K : A \to (A \otimes \Lambda\mathfrak{k}')_{horK}$. Thus, for $a \in A$, the element $h_K(a) \wedge \nu'$ belongs to $(A \otimes \Lambda\mathfrak{g}')_{horK}$.

Definition 34 *Let $f \in \mathcal{A}^{-\infty}_{K,\chi} := C^{-\infty}(\mathfrak{k}, A)^\chi$. Choose ν' a nonzero element of $\Lambda^n\mathfrak{r}'$. We define $Ind_{G/K,\nu'}f \in Ind^{-\infty}_{G/K}A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}$ by*

$$(Ind_{G/K,\nu'}f, \Phi dX) = h_K\Big(\int_{\mathfrak{k}} f(Y)\Phi(Y)dY\Big) \wedge \nu', \text{ for any test function } \Phi \text{ on } \mathfrak{g}$$

(with $dX = dY|d\nu'|$).

It is easy to see that the map $Ind_{G/K,\nu'}$ depends only on the orientation o of $\mathfrak{g}/\mathfrak{k}$ determined from ν'. Thus we write $Ind_{G/K,o}$ for $Ind_{G/K,\nu'}$. Remark that the map $Ind_{G/K,o}$ is injective.

Proposition 35 *The map $Ind_{G/K,o}$ is a cochain map of parity degree equal to $\dim G/K$ from the $\mathbb{Z}/2$-cochain complex $(\mathcal{A}^{-\infty}_{K,\chi}, d_{\mathfrak{k}})$ to the $\mathbb{Z}/2$-cochain complex $(Ind^{-\infty}_{G/K}A, c_{\mathfrak{g}})$.*

Proof: Let $L = C^{-\infty}(\mathfrak{g}, A)$ and let $L^{\det} \subset L$ be the subset of elements $F \in L$ satisfying $k \cdot F = (\det_{\mathfrak{r}} k)F$. If $F \in L$, define $h_K(F) \in L \otimes \Lambda\mathfrak{k}'$ by $h_K(F)(X) = h_K(F(X))$. The map $F \to h_K(F)\wedge\nu'$ sends L to $(L\otimes\Lambda\mathfrak{k}')_{horK}\wedge\nu'$. It sends $L^{\det}$ to $(L \otimes \Lambda\mathfrak{g}')_{basK} = Ind^{-\infty}_{G/K}A$.

Lemma 36 *For every $F \in L^{\det}$, we have $c_A(h_K F \wedge \nu') = h_K(d_A F) \wedge \nu'$, where c_A is the operator on $Ind^{-\infty}_{G/K}A$ defined in Lemma 31.*

Proof: We compute $c_A(h_K(F) \wedge \nu')$ for $F \in L^{\det}$. Let K^i be a basis of $\mathfrak{k}$ with dual basis K_i. It is easy to see that $c(\nu') = -\sum_i (Tr\,(ad_{\mathfrak{r}} K^i))K_i \wedge \nu'$. Thus, by Leibniz's rule,

$$c_A(h_K(F) \wedge \nu') \in L \otimes (\Lambda\mathfrak{g}' \wedge \nu') = L \otimes (\Lambda\mathfrak{k}' \wedge \nu').$$

As c_A preserves K-basic elements (cf. Lemma 31), $c_A(h_K(F) \wedge \nu')$ is K-basic, in particular is horizontal. The space $(L \otimes (\Lambda\mathfrak{g}' \wedge \nu'))_{horK}$ is isomorphic with L by the map $\alpha \wedge \nu' \to \alpha_{[0]} \in L$ for $\alpha \in L \otimes \Lambda\mathfrak{g}'$, where $\alpha_{[0]}$ is the component of α in the zeroth exterior degree. Thus, if $c_A(h_K(F) \wedge \nu') = G \wedge \nu'$, then $c_A(h_K(F) \wedge \nu') = h_K(G_{[0]}) \wedge \nu'$.

We have $G = d_A(h_K(F)) + c(h_K F) + (-1)^{|F|+1} h_K F \wedge \sum_i (Trad_{\mathfrak{r}} K^i)K_i$. Looking at the term of zeroth-exterior degree, it follows that $G_{[0]} = d_A F$. This proves the lemma. ∎

For $f \in C^{-\infty}(\mathfrak{k}, A)^\chi$, the element $F \in C^{-\infty}(\mathfrak{g}, A)$ defined by $(F, \Phi dX) = \int_{\mathfrak{k}} f(Y)\Phi(Y)dY$ belongs to $L^{\det}$ and $Ind_{G/K,o} f = h_K(F) \wedge \nu'$. The preceding lemma implies $c_A Ind_{G/K,o} f = Ind_{G/K,o} d_A f$. Proposition 35 is now a consequence of the following

Lemma 37

$$j(Ind_{G/K,o} f) = -Ind_{G/K,o}(\iota_{\mathfrak{k}} f).$$

Proof: Let E^i be a basis of $\mathfrak{g}$ such that the first elements form a basis of $\mathfrak{k}$. Let E_i be the dual basis. Then the last n coordinates x_i vanish on $\mathfrak{k}$. We denote by $y_i, 1 \le i \le \dim \mathfrak{k}$ the coordinates on $\mathfrak{k}$ corresponding to the basis of $\mathfrak{k}'$ dual to the basis E^i of $\mathfrak{k}$. We have then

$$(j(Ind_{G/K,o} f), \Phi dX) = \sum_{i=1}^{\dim \mathfrak{k}} (\iota_\Lambda(E^i) h_K(\int_{\mathfrak{k}} f(Y) y_i \Phi(Y) dY)) \wedge \nu'.$$

As h_K is a projector on the K-horizontal elements for the tensor product contraction, it satisfies for $E^i \in \mathfrak{k}$ and $a \in A$, $\iota_\Lambda(E^i)(h_K a) + \iota_A(E^i)(h_K a) = 0$. But $\iota_A(E^i) h_K = h_K \iota_A(E^i)$, and we obtain the lemma. ∎

The proof of this lemma completes the proof of Proposition 35. ∎

If A is a K-differential algebra, then $C^{-\infty}(\mathfrak{k}, A)^\chi$ is a module over $C^\infty(\mathfrak{k}, A)^K$. Similarly $Ind_{G/K}^{-\infty} A$ is a module over $Ind_{G/K}^{\infty} A$. Remark the following relation between the maps $r_{\mathfrak{k}}$ and $Ind_{G/K,o}$

Lemma 38 *If $\alpha \in Ind_{G/K}^{\infty} A$ and $s \in C^{-\infty}(\mathfrak{k}, A)^\chi$, then*

$$\alpha Ind_{G/K,o} s = Ind_{G/K,o}((r_{\mathfrak{k}} \alpha) s).$$

The main result of this section is the following

Theorem 39 *Assume that the group K is compact. Then the map $Ind_{G/K,o}$: $(\mathcal{A}^{-\infty}_{K,X}, d_{\mathfrak{k}}) \to (Ind^{-\infty}_{G/K}A, c_{\mathfrak{g}})$ induces an isomorphism in cohomology.*

Proof: We can choose a K-invariant decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}.$$

To this direct sum decomposition is associated the tensor product decomposition $\Lambda\mathfrak{g}' = \Lambda\mathfrak{k}' \otimes \Lambda\mathfrak{r}'$.

Let $h_K : L \to L \otimes \Lambda\mathfrak{k}'$ be the projection on K-horizontal vectors for the tensor product contraction (see definition 28). The map

$$W(v \otimes \xi) = h_K(v) \wedge \xi$$

for $v \in L$ and $\xi \in \Lambda\mathfrak{r}'$ is an isomorphism from the space $L \otimes \Lambda\mathfrak{r}'$ to the space $(L \otimes \Lambda\mathfrak{g}')_{horK}$. The map W commutes with the action of K and allows us to identify the space

$$T_K := (L \otimes \Lambda\mathfrak{r}')^K$$

with the space

$$Ind^{-\infty}_{G/K}A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}.$$

On the space T_K, we will use the $\mathbb{Z}_+$-gradation given by the exterior degree $T_K = \oplus_{p=0}^{n} T_K^p = \oplus_{p=0}^{n} (L \otimes \Lambda^p\mathfrak{r}')^K$.

Let R^i be a basis of $\mathfrak{r}$ with dual basis R_i and let K^j be a basis of $\mathfrak{k}$ with dual basis K_j. We write $X \in \mathfrak{g}$ as $X = Y + R$, with $R = \sum_i x_i R^i$, $Y = \sum_j y_j K^j$.

The operator $\iota_{\mathfrak{k}} = \sum_j y_j \iota_A(K^j)$ is defined on $L = C^{-\infty}(\mathfrak{g}, A)$. Let $j_{\mathfrak{r}}$: $L \otimes \Lambda^*\mathfrak{r}' \to L \otimes \Lambda^{*-1}\mathfrak{r}'$ be given by $j_{\mathfrak{r}} = \sum_i x_i \iota_\Lambda(R^i)$. It is easy to see that $\iota_{\mathfrak{k}}$ and $j_{\mathfrak{r}}$ commute with the action of K on $L \otimes \Lambda\mathfrak{r}'$.

Lemma 40 *For all $\alpha \in L \otimes \Lambda\mathfrak{r}'$,*

$$jW(\alpha) = W(j_{\mathfrak{r}}\alpha - \iota_{\mathfrak{k}}\alpha).$$

Proof: We have, for $v \in L$ and $\xi \in \Lambda\mathfrak{r}'$,

$$j(h_K v \wedge \xi) = \sum_i x_i \iota_\Lambda(R^i)(h_K v \wedge \xi) + \sum_j y_j (\iota_\Lambda(K^j) h_K(v)) \wedge \xi.$$

Further, for $K^j \in \mathfrak{k}$, $\iota_\Lambda(K^j) h_K(v) + h_K(\iota_A(K^j)v) = 0$, and we obtain the lemma. ∎

Lemma 41 *If $\alpha \in T_K^p$, then $W^{-1} c_A W(\alpha) \in T_K^p \oplus T_K^{p+1} \oplus T_K^{p+2}$.*

Proof: This is obvious since, $\mathfrak{k}$ being a subalgebra, $c(\Lambda^q\mathfrak{k}'\otimes\Lambda^p\mathfrak{r}')\subset(\Lambda^{q+1}\mathfrak{k}'\otimes\Lambda^p\mathfrak{r}')\oplus(\Lambda^q\mathfrak{k}'\otimes\Lambda^{p+1}\mathfrak{r}')\oplus(\Lambda^{q-1}\mathfrak{k}'\otimes\Lambda^{p+2}\mathfrak{r}')$. ∎

Let us now prove Theorem 39. By the map W, we identify $Ind^{-\infty}_{G/K}A$ with T_K and write still $c_\mathfrak{g}$ for the operator $W^{-1}c_\mathfrak{g}W$ on T_K. As we have seen in Proposition 22 and Remark 23 (of section 3), the homology groups of the operator $j_\mathfrak{r}: T_K^* \to T_K^{*-1}$ are equal to zero except in maximal degree $n=\dim\mathfrak{r}$. Furthermore, if $\alpha\in T_K^n$ is such that $j_\mathfrak{r}\alpha=0$, then $\alpha=\delta_{\mathfrak{r},o}(x)\otimes f(y)$ for some $f\in C^{-\infty}(\mathfrak{k},A)^\chi$ by Lemma 21 and Remark 23 (of section 3). But then, by definition of $Ind_{G/K,o}$, we obtain $\alpha=Ind_{G/K,o}f$.

Now, if $\alpha\in T_K^n$ is such that $c_\mathfrak{g}\alpha=0$, writing this equation componentwise, we see that α satisfies the relation $j_\mathfrak{r}\alpha=0$. Thus α is of the form $Ind_{G/K,o}f$. As $c_\mathfrak{g}Ind_{G/K,o}f=Ind_{G/K,o}d_\mathfrak{k}f$ (by Proposition 35), we see that $f\in Ker d_\mathfrak{k}$, since $Ind_{G/K,o}$ is an injective map. Consider now an element $\alpha=\sum_{k\geq k_0}\alpha_{[k]}$ in the kernel of $c_\mathfrak{g}$. From the degree consideration, we see that its component of minimal exterior degree k_0 is annihilated by $j_\mathfrak{r}$. If k_0 is less than n, there exists an element $\beta\in T_K^{k_0+1}$ such that $\alpha_{[k_0]}=j_\mathfrak{r}\beta$. The element $\alpha-c_\mathfrak{g}\beta$ is in the same cohomology class as α and all its non-zero exterior degree components are of degree strictly greater than k_0. By induction, α has a representative in T_K^n and we see that the map $Ind_{G/K,o}$ is surjective in cohomology.

Suppose now that $Ind_{G/K,o}f=c_\mathfrak{g}\beta$ with $\beta=\sum_{k\geq k_0}\beta_{[k]}$. Writing this equation component wise, we see that $j_\mathfrak{r}\beta_{[k_0]}=0$. If $k_0<n$, changing β to $\beta'=\beta-c_\mathfrak{g}\gamma$ with $\gamma\in T_K^{k_0+1}$ and $j_\mathfrak{r}\gamma=\beta_{[k_0]}$, we still have $Ind_{G/K,o}f=c_\mathfrak{g}\beta'$. By choice, $\beta'=\sum_{k>k_0}\beta'_{[k]}$. Hence, by induction, we obtain an element $\tilde\beta\in T_K^n$ such that $Ind_{G/K,o}f=c_\mathfrak{g}\tilde\beta$. From degree consideration, $j_\mathfrak{r}\tilde\beta=0$. But then $\tilde\beta=Ind_{G/K,o}g$ for some $g\in C^{-\infty}(\mathfrak{k},A)^\chi$. The equation $c_\mathfrak{g}Ind_{G/K,o}g=Ind_{G/K,o}f$ reads as $Ind_{G/K,o}d_\mathfrak{k}g=Ind_{G/K,o}f$. But $Ind_{G/K,o}$ being an injective map, we get $d_\mathfrak{k}g=f$. This proves that the map $Ind_{G/K,o}$ is injective in cohomology, thereby completing the proof of Theorem 39. ∎

As a particular case, if K is compact and if $(A,d_A)=(\mathbb{R},0)$, we obtain that the map $Ind_{G/K,o}$ induces an isomorphism from $C^{-\infty}(\mathfrak{k})^\chi$ to the cohomology of the complex $((C^{-\infty}(\mathfrak{g})\otimes\Lambda(\mathfrak{g}/\mathfrak{k})')^K, j+c)$.

We apply these calculations in the next section to the calculation of the equivariant cohomology of fiber bundles over homogeneous spaces.

5 Equivariant cohomology of homogeneous spaces

Let G be a Lie group and let K be a closed subgroup of G. Let $D=G/K$. Let $e\in D$ be the base point of D. We identify the tangent space of G at a point $g\in G$ with $\mathfrak{g}$ by sending $X\in\mathfrak{g}$ to the tangent vector to the curve

$g \exp \epsilon X$ in G. Let $n = \dim D$. Let $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$. The tangent space $T_e D$ at $e \in D$ is identified with $\mathfrak{r}$. If $g \in G$, we again denote by L_g the map $T_e D \to T_{g \cdot e} D$ induced from the left multiplication map $L_g : D \to D$. For $\alpha \in C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(D)) := \sum_{p=0}^{\dim D} C^{\pm\infty}(\mathfrak{g}, \mathcal{A}^p(D))$, $\alpha_{[p]}$ is the p-th component of α. If $x \in D$ and $\xi_1, \cdots, \xi_p \in T_x D$, then $(\alpha_{[p]})_x(\xi_1, \cdots, \xi_p)$ is a function (maybe generalized) on $\mathfrak{g}$ given by $X \mapsto (\alpha_{[p]}(X))_x(\xi_1, \cdots, \xi_p)$.

Let $\alpha \in \mathcal{A}_G^{\pm\infty}(D) = C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(D))^G$. Let $R^1, .., R^p \in \mathfrak{r}$, $X \in \mathfrak{g}$ and $g \in G$. As α iz an equivariant form

$$\alpha_{[p]}(Ad(g)(X))_{g \cdot e}(L_g R^1, \ldots, L_g R^p) = \alpha_{[p]}(X)_e(R^1, ..., R^p).$$

Let $\tilde{\alpha}(X) = \alpha(X)_e$. Thus $\tilde{\alpha}$ is a function on $\mathfrak{g}$ with values in $\Lambda\mathfrak{r}'$ and the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism from the space $\mathcal{A}_G^{\pm\infty}(D)$ to the space $T_K = (C^{\pm\infty}(\mathfrak{g}) \otimes \Lambda\mathfrak{r}')^K$, where the action of K on both $C^{\pm\infty}(\mathfrak{g})$ and $\Lambda\mathfrak{r}'$ is induced from the adjoint representation.

Let R^i be a basis of $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$, with dual basis $R_i \in \mathfrak{r}'$. Let $x_i = R_i(X)$. Let $j_\mathfrak{r}$ be the operator on T_K given by

$$j_\mathfrak{r} = \sum_i x_i \iota_\Lambda(R^i).$$

Let c be the Koszul differential on T_K (see Formula 15 of section 3). From [13], we have

Lemma 42 *For $\alpha \in \mathcal{A}_G^{\pm\infty}(D)$, we have $(\widetilde{d_\mathfrak{g}\alpha}) = (c + j_\mathfrak{r})\tilde{\alpha}$.*

For example, the complex $\mathcal{A}_G^{-\infty}(G)$ becomes isomorphic, under evaluation at e, to $C^{-\infty}(\mathfrak{g}) \otimes \Lambda\mathfrak{g}'$ and the differential $d_\mathfrak{g}$ becomes the perturbed differential $c + j$.

In the notation of the preceeding section, if $(A, d_A) = (\mathbb{R}, 0)$, we have $\mathcal{A}_G^{\pm\infty}(D) \cong Ind_{G/K}^{\pm\infty}(\mathbb{R})$, as cochain complexes.

Let $\chi_{G/K}(k) := sign(det_\mathfrak{r} k)$. As G and K are fixed in this section, we denote $\chi_{G/K}$ by χ. Recall the definition (see Theorem 33 and Definition 34 of section 4) of the maps

$$r_\mathfrak{k} : Ind_{G/K}^{\infty}(\mathbb{R}) \to C^\infty(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to Ind_{G/K}^{-\infty}(\mathbb{R}).$$

Using the identification $\alpha \mapsto \tilde{\alpha}$, we get maps again denoted by $r_\mathfrak{k}$ and $Ind_{G/K,o}$:

$$r_\mathfrak{k} : \mathcal{A}_G^{\infty}(D) \to C^\infty(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_G^{-\infty}(D).$$

Let us describe explicitly these maps.

The point e is a K-stable submanifold of D. The successive restriction maps $\mathcal{A}_G^\infty(D) \to \mathcal{A}_K^\infty(D) \to \mathcal{A}_K^\infty(e)$ are well defined. The composed map $\mathcal{A}_G^\infty(D) \to \mathcal{A}_K^\infty(e) = C^\infty(\mathfrak{k})^K$ coincides obviously with the map $r_\mathfrak{k}$:

$$(r_\mathfrak{k}\alpha)(Y) = (\alpha_{[0]})_e(Y)$$

for $Y \in \mathfrak{k}$.

We now describe the map $Ind_{G/K,o}$ from $C^{-\infty}(\mathfrak{k})^\chi$ to $\mathcal{A}_G^{-\infty}(G/K)$. Let $\nu' \in \Lambda^n\mathfrak{r}'$ be a non-zero element. The element ν' determines an orientation o and an Euclidean measure $|d\nu'|$ on $\mathfrak{g}/\mathfrak{k}$. If dX is an Euclidean measure on $\mathfrak{g}$, we denote by dY the Euclidean measure on $\mathfrak{k}$ such that $dX = |d\nu'|dY$.

We identify the space $\mathcal{A}(G/K)$ with $C^\infty(G, \Lambda\mathfrak{r}')^K$.

Proposition 43 *The map $Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_G^{-\infty}(G/K)$ is given, for $f \in C^{-\infty}(\mathfrak{k})^\chi$, by*

$$(16) \qquad (\int_\mathfrak{g}(Ind_{G/K,o}f)(X)\phi(X)dX)(g) = |det_\mathfrak{g}(g)|(\int_\mathfrak{k} f(Y)\phi(gY)dY)\nu',$$

where ϕ is any test function on $\mathfrak{g}$ and $g \in G$.

Moreover, for any $f \in C^{-\infty}(\mathfrak{k})^\chi$, the equivariant form $Ind_{G/K,o}(f)$ is $d_\mathfrak{g}$-closed.

Proof: It is easy to verify that $Ind_{G/K,o}f$ defined by the Formula 16 is indeed an element of $\mathcal{A}_G^{-\infty}(D)$. It obviously coincides with the map given in Definition 34 of section 4 (denoted also by $Ind_{G/K,o}$) at $g = e$. The fact that $Ind_{G/K,o}f$ is $d_\mathfrak{g}$ closed follows from Proposition 35 of section 4. It is also easy to check it directly. ∎

Assume that G/K is compact and G-oriented. Thus $\chi = 1$. We can integrate over G/K an equivariant cohomology class and we obtain then a G-invariant generalized function on $\mathfrak{g}$. The next formula is just the integration over G/K of the formula given in Proposition 43 for $Ind_{G/K,o}f$. However, it indicates the analogy between $\int_{G/K} Ind_{G/K,o}f$ and characters of induced representations.

Proposition 44 *Assume G/K compact and G-oriented. Let $f \in C^{-\infty}(\mathfrak{k})^K$. Then*

$$\int_{G/K,o}(Ind_{G/K,o}f, \Phi dX) = \int_{G/K,o} |\det{}_\mathfrak{g}(g)|(\int_\mathfrak{k} f(Y)\Phi(gY)dY)dg/dk.$$

for any test function Φ on $\mathfrak{g}$ and compatible choices of the left-invariant Haar measure dg on G, of the G-invariant measure dg/dk on G/K and of the Euclidean measure dY on $\mathfrak{k}$.

We still denote by $r_{\mathfrak{k}}$ the map $H_G^{\infty}(G/K) \to C^{\infty}(\mathfrak{k})^K$ induced from the map $r_{\mathfrak{k}}$ at the cohomology level and by $Ind_{G/K,o}$ the map $C^{-\infty}(\mathfrak{k})^{\chi} \to H_G^{-\infty}(G/K)$ induced from the map $Ind_{G/K,o}$ at the cohomology level.

As a particular case of [13] (see Theorem 33 of section 4), we have the following

Proposition 45 *Assume that K is compact. Then the map $r_{\mathfrak{k}}$ gives an isomorphism from $H_G^{\infty}(G/K)$ with $C^{\infty}(\mathfrak{k})^K$.*

Further Theorem 39 of section 4 gives as an immediate corollary the following

Theorem 46 *Assume that K is compact. Then the map $Ind_{G/K,o}$ gives an isomorphism from $C^{-\infty}(\mathfrak{k})^{\chi}$ with $H_G^{-\infty}(G/K)$. The map $Ind_{G/K,o}$ is of even (resp. odd) degree if $\dim G/K$ is even (resp. odd).*

When K is compact, let us give a formula for the map $Ind_{G/K,o}$ in terms of generalized functions.

Choose a K-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$$

and let $pr_{\mathfrak{k}}$ (resp. $pr_{\mathfrak{r}}$) be the projection of $\mathfrak{g}$ on $\mathfrak{k}$ (resp. on $\mathfrak{r}$) determined by this decomposition.

With the notation of Formula 10 of section 2, we have (by Proposition 43), for $f \in C^{-\infty}(\mathfrak{k})^{\chi}$,

$$(Ind_{G/K,o} f(X))(e) = |\nu'|^{-1} \delta_{\mathfrak{r}}(pr_{\mathfrak{r}} X) f(pr_{\mathfrak{k}} X) \nu', \tag{17}$$

where we have identified the space $\mathcal{A}(G/K)$ with $C^{\infty}(G, \Lambda \mathfrak{r}')^K$.

Consider the case where K is the trivial subgroup. Recall that we have defined the element

$$\alpha_{G,o}(X) = |\nu'|^{-1} \delta_{\mathfrak{g}}(X) \otimes |\det{}_{\mathfrak{g}} g| dg$$

in Lemma 13 of section 2. From Formula 17, we see that it is also equal to the element $Ind_{G,o} 1$. Thus we obtain from Theorem 46:

Lemma 47 *Let G be a Lie group acting on itself by left translations, then*

$$H_G^{-\infty}(G) \cong \mathbb{R} \alpha_{G,o}.$$

A more general result is proved in Theorem 89 in section 9.

Now we are going to generalize Proposition 45 and Theorem 46 as follows. Let M be a K-manifold. Consider the product manifold $G \times M$. The group K acts freely on the right on $G \times M$ by $(g,m)k = (gk, k^{-1}m)$. Consider the fiber space $\mathcal{M} = G \times_K M$ of orbits of the K-action. The group G acts on the left on $\mathcal{M}$. When M is a point, the space $\mathcal{M}$ is $D = G/K$. The projection $(g,m) \mapsto g$ induces a map $\mathcal{M} \to D$. Thus the space $\mathcal{M}$ is a fiber space over D with fiber M.

If $\alpha \in \mathcal{A}(\mathcal{M}) \subset \mathcal{A}(G \times M)$, and $g \in G$, then $\alpha(g)$ is an element of $(\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK}$, where $\mathfrak{g}'$ is identified with left invariant 1-forms on G.

Thus

$$\mathcal{A}(\mathcal{M}) = C^\infty(G, (\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK})^K$$

where K-invariants are taken with respect to the action of K by right multiplication on G, left action on M and adjoint action on $\Lambda\mathfrak{g}'$.

If $\alpha(X) \in \mathcal{A}(\mathcal{M})$, then $\tilde{\alpha}(X) := \alpha(X)_e \in (\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK}$.

By G-invariance, the space $\mathcal{A}_G^{\pm\infty}(\mathcal{M})$ is thus identified with

$$Ind_{G/K}^{\pm\infty}(\mathcal{A}(M)) := (C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(M)) \otimes \Lambda\mathfrak{g}')_{basK}.$$

Let $A = \mathcal{A}(M)$ be our K-differential complex, then $Ind_{G/K}^{\pm\infty}(A)$ is provided with a differential $c_\mathfrak{g}$. As before, the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism from $\mathcal{A}_G^{\pm\infty}(G \times_K M)$ to $Ind_{G/K}^{\pm\infty}(A)$ and the following lemma ([13]) iz a generalization of Lemma 42.

Lemma 48 *For any $\alpha \in \mathcal{A}_G^{-\infty}(\mathcal{M})$,*

$$\widetilde{d_\mathfrak{g}\alpha} = c_\mathfrak{g}\tilde{\alpha}.$$

Thus $(\mathcal{A}_G^{\pm\infty}(\mathcal{M}), d_\mathfrak{g})$ is identified with the complex $(Ind_{G/K}^{\pm\infty}(\mathcal{A}(M)), c_\mathfrak{g})$ of section 4. The complex $(C^\infty(\mathfrak{k}, A)^K, d_\mathfrak{k})$ defined in section 4 is the complex $(\mathcal{A}_K^\infty(M), d_\mathfrak{k})$ of the K-equivariant cohomology of M. As χ is trivial on the connected component of K, the operator $d_\mathfrak{k}$ on $C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi$ still satisfies $d_\mathfrak{k}^2 = 0$, as follows from Lemma 29 of section 4.

Definition 49 *Let us denote by $\mathcal{A}_{K,\chi}^{-\infty}(M)$ the space $C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi$. We define $H_{K,\chi}^{-\infty}(M)$ to be the cohomology of the complex $(C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi, d_\mathfrak{k})$.*

Recall the definition of the maps

$$r_\mathfrak{k} : Ind_{G/K}^\infty(\mathcal{A}(M)) \to \mathcal{A}_K^\infty(M)$$

and

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,X}(M) \to Ind^{-\infty}_{G/K}(\mathcal{A}(M))$$

from section 4. Using the identification $\alpha \mapsto \tilde{\alpha}$, we get maps again denoted by $r_{\mathfrak{k}}$ and $Ind_{G/K,o}$:

$$r_{\mathfrak{k}} : \mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$$

and

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,X}(M) \to \mathcal{A}^{-\infty}_{G}(\mathcal{M}).$$

Let us describe explicitly these maps.

The fiber of $\mathcal{M} \to D$ over the point $e \in D$ is canonically identified with M and is a K-stable submanifold of $\mathcal{M}$. The successive restriction maps $\mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$ are well defined. Obviously, the composed map $\mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$ coincides with the map $r_{\mathfrak{k}}$.

The following proposition gives an explicit description of the map $Ind_{G/K,o}$ generalizing Proposition 43, whose proof is identical (and hence is omitted).

Proposition 50 *For any $f \in \mathcal{A}^{-\infty}_{K,X}(M)$, $Ind_{G/K,o}f \in \mathcal{A}^{-\infty}_{G}(\mathcal{M})$ is given by*

$$(Ind_{G/K,o}f, \Phi dX)(g) = |det_{\mathfrak{g}} g| h_K\left(\int_{\mathfrak{k}} f(Y)\Phi(gY)dY\right) \wedge \nu'$$

where Φ is any test function on $\mathfrak{g}$ and $dX = dY|d\nu'|$.

We still denote by $r_{\mathfrak{k}}$ the map from $H^{\infty}_{G}(\mathcal{M})$ to $H^{\infty}_{K}(M)$ induced from the map $r_{\mathfrak{k}}$ at the cohomology level. As a generalization of Proposition 45, we have the following Theorem ([13]), (got from Theorem 33 of section 4):

Theorem 51 *Let K be a compact subgroup of a Lie group G and let M be a K-manifold. Then the map $r_{\mathfrak{k}}$ gives an isomorphism from $H^{\infty}_{G}(\mathcal{M})$ with $H^{\infty}_{K}(M)$.*

We still denote by $Ind_{G/K,o}$ the map from $H^{-\infty}_{K,X}(M)$ to $H^{-\infty}_{G}(G \times_K M)$ induced from the map $Ind_{G/K,o}$ at the cohomology level. As a corollary of Theorem 39 of section 4, we get the main result of this section which generalizes Theorem 46.

Theorem 52 *Let K be a compact subgroup of a Lie group G and let M be a K-manifold. Then the map $Ind_{G/K,o}$ gives an isomorphism in cohomology from $H^{-\infty}_{K,X}(M)$ to $H^{-\infty}_{G}(G \times_K M)$. This map is of even (resp. odd) degree, if $\dim(G/K)$ is even (resp. odd).*

When K is compact, let us write the map $Ind_{G/K,o}$ in terms of generalized functions. We identify $\mathcal{A}(\mathcal{M})$ with the subspace of K-basic elements of $\mathcal{A}(G \times M)$.

Choose a K-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$$

and let $pr_{\mathfrak{k}}$ (resp. $pr_{\mathfrak{r}}$) be the projection of $\mathfrak{g}$ on $\mathfrak{k}$ (resp. on $\mathfrak{r}$) determined by this decomposition. Let $\nu' \in \Lambda^n \mathfrak{r}'$ be a positive element. With the help of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$, we consider ν' as an element of $\Lambda^n \mathfrak{g}'$. With the notation of Formula 10 of section 2, we have by Proposition 50, for any $m \in M$ and $f \in \mathcal{A}^{-\infty}_{K,\chi}(M)$,

$$((Ind_{G/K,o} f)(X))_{(e,m)} = |\nu'|^{-1} \delta_{\mathfrak{r}}(pr_{\mathfrak{r}} X)(h_K(f(pr_{\mathfrak{k}} X)))_m \wedge \nu'. \tag{18}$$

Assume that K is compact. Choose an orientation o on $\mathfrak{r}$. Then there are canonical isomorphisms

$$r_{\mathfrak{k}} : H^{\infty}_G(G/K) \cong C^{\infty}(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^{\chi} \cong H^{-\infty}_G(G/K)$$

guaranteed by Proposition 45 and Theorem 46 respectively.

The natural map $H^{\infty}_G(G/K) \to H^{-\infty}_G(G/K)$ thus gives rise (using the above two identifications) to a map

$$M_o : C^{\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{k})^{\chi}.$$

Comparing the $\mathbb{Z}/2$ degree of the maps, we see that this map is identically zero if G/K is not of even dimension. The orientation o determines a polynomial square root $Y \mapsto det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(Y)$ of the polynomial function $Y \mapsto det_{\mathfrak{g}/\mathfrak{k}}(Y)$ on $\mathfrak{k}$. The normalization of this square root is as in ([12], Formula 12).

Proposition 53 *If $f \in C^{\infty}(\mathfrak{k})^K$, then*

$$M_o(f)(Y) = (-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(Y) f(Y)$$

for $Y \in \mathfrak{k}$.

(If $\dim G/K$ is odd, then the function $det_{\mathfrak{g}/\mathfrak{k}}$ is identically 0.)

Proof: The map M_o is a morphism of $C^{\infty}(\mathfrak{k})^K$-modules by Lemma 38 of section 4. It is thus sufficient to prove that

$$1 \cong Ind_{G/K,o}((-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})$$

in $H_G^{-\infty}(G/K)$.

We will prove this relation using the method of the proof of Proposition 31 of [12]. As K is compact, we can choose a G-invariant metric $< .,. >$ on $D = G/K$. Let $X \in \mathfrak{g}$. Let $\omega(X)$ be the 1-form on D given by $\omega(X)(\xi) =< X_D, \xi >$ for any vector field ξ on D. We denote by ω the G-equivariant form on D given by $X \mapsto \omega(X)$. We have

$$\frac{d}{dt}(e^{td_{\mathfrak{g}}\omega}) = d_{\mathfrak{g}}(\omega e^{td_{\mathfrak{g}}\omega}).$$

Integrating this relation from $t = 0$ to T, we obtain

$$e^{Td_{\mathfrak{g}}\omega} - 1 = d_{\mathfrak{g}}(\int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt).$$

Let $n = \dim G/K$. Let us show that, when $T \to \infty$, the form $e^{Td_{\mathfrak{g}}\omega}$ tends to $Ind_{G/K,o}((-2\pi)^{n/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})$ in $\mathcal{A}_G^{-\infty}(D)$ while $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt$ also has a limit in the space $\mathcal{A}_G^{-\infty}(D)$.

By G-invariance, it is sufficient to compute at the base point $e \in G/K$. Choose a K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$ (as in the proof of Theorem 52). The elements $e^{td_{\mathfrak{g}}\omega}$ and $\omega e^{td_{\mathfrak{g}}\omega}$ evaluated at the base point e are elements of $C^\infty(\mathfrak{g}) \otimes \Lambda \mathfrak{r}'$. The G-invariant Riemannian metric on D gives us a K-invariant scalar product $< .,. >$ on $\mathfrak{r}$. We choose an oriented orthonormal basis E^a of $\mathfrak{r}$ with dual basis $E_a \in \mathfrak{r}'$. Let us write any $X \in \mathfrak{g}$ as $X = X_0 + X_1$ with $X_0 \in \mathfrak{k}$ and $X_1 \in \mathfrak{r}$. We write $X_1 \in \mathfrak{r}$ as $X_1 = \sum_a x_a(X) E^a$. The function $X \mapsto x_a(X)$ is a linear function on $\mathfrak{g}$. Let E_I be the homogeneous basis of $\Lambda \mathfrak{r}'$ indexed by subsets I of $\{1, 2, ..., n\}$. In particular, $E_{\{1,2,\dots,n\}} = E_1 \wedge \cdots \wedge E_n$. Then $\omega_e \in C^\infty(\mathfrak{g}) \otimes \Lambda \mathfrak{r}'$ is given by the formula $\omega_e = -\sum_a x_a \otimes E_a$. We compute $d_{\mathfrak{g}}\omega$ using Lemma 42 of Section 5. For any $X \in \mathfrak{g}$, we have $j_{\mathfrak{r}}(\omega_e)(X) = \|X_1\|^2$ and $c(\omega_e)(X) = \kappa(X)$ where $\kappa(X)$ is the element of $\Lambda^2 \mathfrak{r}'$ given by

$$\kappa(X)(v, v') =< [v, X]_1, v' > - < [v', X]_1, v > + < X_1, [v, v'] > .$$

In particular, if $X_0 \in \mathfrak{k}$, $\kappa(X_0)(v, v') = -2 < (ad_{\mathfrak{r}} X_0) \cdot v, v' >$. We write $f(t) = (e^{td_{\mathfrak{g}}\omega})_e$. We have

$$f(t, X_0, X_1) = e^{-t\|X_1\|^2} \sum_I P_I(X_0, X_1) t^{|I|/2} E_I$$

where P_I is a homogeneous polynomial of degree $|I|/2$ on $\mathfrak{g}$. It is easy to see that $P_{\{1,2,\dots,n\}}(X_0) = (-2)^{n/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(X_0)$. Let us show that $f(t, X_0, X_1)$ has a limit when $t \to \infty$, as a generalised function on $\mathfrak{g}$ with values in $\Lambda \mathfrak{r}'$. For a test function ϕ on $\mathfrak{g}$

$$\int_{\mathfrak{g}} f(t, X)\phi(X) dX = \sum_I \left(\int_{\mathfrak{g}} e^{-t\|X_1\|^2} t^{|I|/2} P_I(X_0, X_1) \phi(X_0, X_1) dX_0 dX_1 \right) E_I.$$

The change of variable $X_1 \mapsto t^{-1/2}X_1$ shows that this is also equal to:

$$\sum_I t^{|I|/2-n/2}\left(\int_{\mathfrak{g}} e^{-\|X_1\|^2} P_I(X_0, t^{-1/2}X_1)\phi(X_0, t^{-1/2}X_1)dX_0dX_1\right) E_I.$$

Thus the limit when $t \to \infty$ of $\int_{\mathfrak{g}} f(t,X)\phi(X)dX \in \Lambda\mathfrak{r}'$ exists. Furthermore, when $t \to \infty$, the coefficient of E_I tends to zero except when $I = \{1, 2, ...n\}$, and the coefficient of $E_{\{1,2,\ldots,n\}}$ tends to $\int_{\mathfrak{g}} e^{-\|X_1\|^2} P_{\{1,2,\ldots,n\}}(X_0, 0)\phi(X_0, 0)dX_0dX_1$. Computing the constants, we obtain that the form $e^{td_{\mathfrak{g}}\omega}$ tends to

$$Ind_{G/K,o}((-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})$$

when $t \to \infty$.

It remains to show that $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt$ also has a limit in the space $\mathcal{A}_G^{-\infty}(D)$.

Remark that $\omega(X)$ is homogeneous of degree 1 in the variable X_1. Let $g(t) = (\omega e^{td_{\mathfrak{g}}\omega})_e$. We have

$$g(t, X) = \sum_{a,I} \left(e^{-t\|X_1\|^2} t^{|I|/2} x_a P_I(X_0, X_1)\right) E_a \wedge E_I.$$

Here all the subsets I occuring in this sum are such that $|I| \le n - 2$. Now

$$\int_{\mathfrak{g}} g(t, X)\phi(X)dX = \sum_{a,I} \left(\int_{\mathfrak{g}} e^{-t\|X_1\|^2} t^{|I|/2} x_a P_I(X_0, X_1)\phi(X_0, X_1)dX_0dX_1\right) E_a \wedge E_I.$$

The same change of variable (as earlier) shows that this is equal to

$$\sum_{a,I} t^{|I|/2-1/2-n/2}\left(\int_{\mathfrak{g}} e^{-\|X_1\|^2} x_a P_I(X_0, t^{-1/2}X_1)\phi(X_0, t^{-1/2}X_1)dX_0dX_1\right) E_a \wedge E_I.$$

In particular, as $|I| \le (n - 2)$, this function of t is uniformly bounded when $t \to \infty$ by $O(t^{-3/2})$. Thus the function $t \mapsto \int_{\mathfrak{g}} g(t, X)\phi(X)dX$ is integrable on $[0, \infty]$. This shows that $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt$ has a limit in $\mathcal{A}_G^{-\infty}(G/K)$ when $T \to \infty$. This concludes the proof. ∎

We now turn to the question of the explicit determination of the inverse of the map $Ind_{G/K,o}$.

Definition 54 *Define the Chern-Weil map*

$$w_{G/K} : C^{\infty}(\mathfrak{k})^K \to H_G^{\infty}(G/K)$$

as the inverse of the map $r_{\mathfrak{k}}$.

Define the map

$$S_o : H_G^{-\infty}(G/K) \to C^{-\infty}(\mathfrak{k})^X$$

as the inverse of the map $Ind_{G/K,o}$.

We denote $w_{G/K}$ simply by w if G and K are understood. An explicit formula for $w_{G/K}$ can be given in terms of the curvature of the principal bundle $G \to G/K$, see [21]. On the other hand we are not able to give an explicit expression for S_o in general. We are able to do it only under some restrictive assumptions or on some particular subspaces. Lemma 38 of section 4 implies that, if $\alpha \in H_G^{-\infty}(G/K)$ and $p \in C^\infty(\mathfrak{k})^K$,

$$S_o(w(p)\alpha) = pS_o(\alpha). \tag{19}$$

It follows from proposition 53 that $S_o\alpha = (-2\pi)^{\dim(G/K)/2}(det_{\mathfrak{g}/\mathfrak{k},o}^{1/2})\, r_{\mathfrak{k}}\alpha$ for $\alpha \in H_G^\infty(G/K)$.

We can improve this result as follows. Let $\alpha \in \mathcal{A}_G^{-\infty}(G/K)$. Then $\alpha_e \in C^{-\infty}(\mathfrak{g}) \otimes \Lambda\mathfrak{r}'$. We say that α_e admits a restriction to $\mathfrak{k}$ if each component of α_e admits a restriction to $\mathfrak{k}$ (see [12]). We can then define $r_{\mathfrak{k}}\alpha = (\alpha_e)_{[0]}|_{\mathfrak{k}}$. Then $r_{\mathfrak{k}}\alpha \in C^{-\infty}(\mathfrak{k})^K$ is a generalized function on $\mathfrak{k}$. This definition extends the definition of $r_{\mathfrak{k}} : \mathcal{A}_G^\infty(G/K) \to C^\infty(\mathfrak{k})^K$.

Proposition 55 *Let $\alpha \in \mathcal{A}_G^{-\infty}(G/K)$ be a closed equivariant differential form on G/K. Assume that α_e admits a restriction to $\mathfrak{k}$. Then*

$$S_o\alpha = (-2\pi)^{\dim(G/K)/2}(det_{\mathfrak{g}/\mathfrak{k},o}^{1/2})\, r_{\mathfrak{k}}\alpha.$$

Proof: Using notation of the proof of Proposition 53, we have $\alpha \cong \alpha e^{td_{\mathfrak{g}}\omega}$. Then using the same arguments as in the proof of Proposition 53, we obtain the desired result. Indeed, if f is a generalized function on $\mathfrak{g}$ admitting a restriction $r_{\mathfrak{k}}f$ to $\mathfrak{k}$, then for any test function ϕ on $\mathfrak{g}$, the limit when $t \to \infty$ of

$$t^{|I|/2}\int_{\mathfrak{g}} e^{-t\|X_1\|^2} f(X_0, X_1)\phi(X_0, X_1)dX_0dX_1$$

is 0 if $|I| < n$ while for $|I| = n$ this is $(\pi)^{n/2}\int_{\mathfrak{k}}(r_{\mathfrak{k}}f)(X_0)\phi(X_0)dX_0$. ∎

Remark 56 *Proposition 31 of [12] becomes then a consequence of Proposition 44 and Proposition 55 above.*

Let $q^* : C^{-\infty}(\mathfrak{g})^G \to H_G^{-\infty}(G/K)$ be induced from the map $q : G/K \to$ *point*. Thus we get a map (still denoted by) S_o

$$S_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^\chi$$

taking f to $S_o(q^*f)$. The map S_o exists under the only assumption that K is compact, and extends the map $f \mapsto (-2\pi)^{\dim(G/K)/2}(det_{\mathfrak{g}/\mathfrak{k},o}^{1/2})\, r_{\mathfrak{k}}f$, defined on generalised functions f admitting a restriction $r_{\mathfrak{k}}f$ to $\mathfrak{k}$. On the open set where $\det_{\mathfrak{g}/\mathfrak{k}}Y \neq 0$, the G-orbits are transverse to $\mathfrak{k}$. Thus the restriction to $\mathfrak{k}$ of an invariant generalized function on $\mathfrak{g}$ has a meaning on this open set. By the same proof as Proposition 36, we see that $S_o(f)$ coincides with

$$(-2\pi)^{\dim(G/K)/2}(det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})\, r_{\mathfrak{k}} f$$

on this open set, for any $f \in C^{-\infty}(\mathfrak{g})^G$. However, we are able to compute the map $S_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^\chi$ on arbitrary invariant generalised functions and on the full space $\mathfrak{k}$ explicitly only when G itself is compact.

Let G be a compact connected Lie group. We also assume that $\chi = 1$ so that G/K is orientable (this is only for convenience). We choose the orientation on G/K given by o.

Definition 57 *If $\Phi \in C^\infty(\mathfrak{k})^K$, define $C_o(\Phi) \in C^\infty(\mathfrak{g})^G$ by*

$$C_o(\Phi)(X) = \int_{G/K,o} (w\Phi)(X)$$

where w is the Chern-Weil homomorphism.

Define

$$F_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^K$$

as the transpose of the map C_o:

$$vol(G/K, dg/dk) \int_{\mathfrak{k}} F_o(f)(Y)\Phi(Y)dY = \int_{\mathfrak{g}} f(X)C_o(\Phi)(X)dX$$

for any $\Phi \in C^\infty_{cpt}(\mathfrak{k})^K$, and where the measures dX on $\mathfrak{g}$, dY on $\mathfrak{k}$ and dg/dk on G/K are chosen in a compatible way.

It is easy to see that C_o sends invariant compactly supported functions on $\mathfrak{k}$ to invariant compactly supported functions on $\mathfrak{g}$, hence the map F_o is well defined.

Lemma 58 *If $f \in C^\infty(\mathfrak{g})^G$, then $F_o(f)(Y) = (-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(Y)f(Y)$, for $Y \in \mathfrak{k}$.*

Proof: The integral formula ([12], page 43) for equivariant cohomology classes gives the lemma. ∎

Proposition 59 *Assume that G is a compact connected Lie group and K a closed subgroup of G such that G/K is oriented. Then for every $f \in C^{-\infty}(\mathfrak{g})^G$ and $p \in C^\infty(\mathfrak{k})^K$,*

$$fw(p) \sim Ind_{G/K,o}(F_o(f)p)$$

in $H^{-\infty}_G(G/K)$.

Proof: Using Formula 19, it is sufficient to prove the formula of this proposition when $p = 1$.

Let Φ be a G-invariant test function on $\mathfrak{g}$ and let $p' \in C^\infty(\mathfrak{k})^K$. Let us compute

$$\int_{\mathfrak{g}} \int_{G/K,o} \Phi(X) f(X) w(p')(X) dX = \int_{\mathfrak{g}} \Phi(X) f(X) C_o(p')(X) dX.$$

If $q^* f \cong Ind_{G/K,o} u$, for $u \in C^{-\infty}(\mathfrak{k})^K$, then $f w(p') \cong Ind_{G/K,o}(p'u)$ by Lemma 38 of section 4. Thus, using Proposition 44, we have:

$$\begin{aligned}
\int_{\mathfrak{g}} \int_{G/K,o} \Phi(X) f(X) w(p')(X) dX &= \int_{\mathfrak{g}} \int_{G/K,o} \Phi(X)(Ind_{G/K,o}(p'u))(X) dX \\
&= \int_{G/K,o} \left(\int_{\mathfrak{g}} \Phi(X) Ind_{G/K,o}(p'u)(X) dX \right) \\
&= \int_{G/K,o} \left(\int_{\mathfrak{k}} \Phi(gY) p'(Y) u(Y) dY \right) dg/dk
\end{aligned}$$

Let p' be compactly supported. Taking Φ with sufficiently large support, we obtain

$$\int_{\mathfrak{g}} f(X) C_o(p')(X) dX = vol(G/K, dg/dk) \int_{\mathfrak{k}} p'(Y) u(Y) dY$$

which is what we needed to prove. ∎

The preceding proposition determines the inverse S_o of the map $Ind_{G/K,o}$ on the subspace of $H_G^{-\infty}(G/K)$ spanned by elements of the form $f\alpha$ where $f \in C^{-\infty}(\mathfrak{g})^G$ and $\alpha \in H_G^\infty(G/K)$. We will see in section 6 that this space is equal to $H_G^{-\infty}(G/K)$ provided that G and K have equal rank.

We compute even more explicitly the map F_o when G is a compact connected Lie group and $K = T$ is a maximal torus of G.

Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Let $C^{\pm\infty}(\mathfrak{t})^\epsilon$ be the space of W-anti-invariant smooth functions (resp. anti-invariant generalized functions) on $\mathfrak{t}$. Recall the definitions of the maps C_o and F_o from Definition 57.

Lemma 60 *The restriction of the map C_o to $C^\infty(\mathfrak{t})^\epsilon$ gives an isomorphism between $C^\infty(\mathfrak{t})^\epsilon$ and $C^\infty(\mathfrak{g})^G$.*

Furthermore the image of F_o is contained in $C^{-\infty}(\mathfrak{t})^\epsilon$ and the map F_o gives an isomorphism between $C^{-\infty}(\mathfrak{g})^G$ and $C^{-\infty}(\mathfrak{t})^\epsilon$.

Proof: If $\phi \in C^\infty(\mathfrak{t})^\epsilon$, then ϕ is divisible by $\det_{\mathfrak{g}/\mathfrak{t},o}(Y)^{1/2}$ and the restriction of $C_o\phi$ to $\mathfrak{t}$ is equal to $|W|(-2\pi)^{n/2} \det_{\mathfrak{g}/\mathfrak{t},o}(Y)^{-1/2}\phi(Y)$, where $n := \dim(G/T)$. Thus the first assertion follows from Chevalley's theorem for C^∞-functions, see for example [12]. The second is a consequence of the first, as C_o preserves the subspace of compactly supported functions. ∎

Let us describe $F_o f$, when f is the δ-function: Let $\delta_{\mathfrak{g}}(X)$ (resp. $\delta_{\mathfrak{t}}(Y)$) be the δ function on $\mathfrak{g}$ (resp. on $\mathfrak{t}$), given with respect to the Euclidean measure on $\mathfrak{g}$ (resp. $\mathfrak{t}$) associated to the Killing form. Let $\alpha \in i\mathfrak{t}'$ be a root . Using the identification of $\mathfrak{t}$ with $\mathfrak{t}'$ determined by the Killing form, we can consider the differential operator $\prod_{\alpha>0} \partial_{i\alpha}$ on $\mathfrak{t}$, where the product is taken over all the positive roots of $\mathfrak{g}$ for an order compatible with the orientation o as defined in [12]. Then $F_o\delta_{\mathfrak{g}} = \prod_{\alpha>0} \partial_{i\alpha}\delta_{\mathfrak{t}}$.

6 Künneth formula and applications

Let K be a Lie group. Let D and M be K-manifolds. Consider the complex $\mathcal{A}_K(D)$ of K-equivariant forms on D with polynomial coefficients and its cohomology $H_K(D) = Z_K(D)/B_K(D)$ defined in section 2, Definition 1. Recall that these spaces are $\mathbb{Z}_+$-graded. The evaluation map E at zero taking $\alpha \mapsto \alpha(0)$ gives a map from $H_K(D)$ to the usual De Rham cohomology $H(D)$ of D. Consider the map m from $\mathcal{A}_K(D) \otimes \mathcal{A}_K(M)$ to $\mathcal{A}_K(D \times M)$ given by $m(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$. It induces a map (still denoted by m) from $H_K(D) \otimes H_K(M)$ to $H_K(D \times M)$.

Similarly, we can also consider the map $m^{-\infty}$ from $\mathcal{A}_K(D) \otimes \mathcal{A}_K^{-\infty}(M)$ to $\mathcal{A}_K^{-\infty}(D \times M)$ given by $m^{-\infty}(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$: this is well defined as we can multiply a generalized function by a polynomial function. It induces a map (still denoted by $m^{-\infty}$) from $H_K(D) \otimes H_K^{-\infty}(M)$ to $H_K^{-\infty}(D \times M)$.

Theorem 61 *Let K be a compact Lie group. Let D be a compact K-manifold. Assume that the evaluation map $E : H_K(D) \to H(D)$ is surjective. Then, for any K-manifold M, the multiplication map $m^{-\infty}$ induces an isomorphism*

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_K^{-\infty}(D \times M).$$

Remark 62 *Proposition 5 implies that $H_K(D)$ is free over $H_K(point)$. As is well known, from the Künneth spectral sequence (see [16]; Proposition 6.1, page 50), the map m induces an isomorphism*

$$\hat{m} : H_K(D) \otimes_{H_K(point)} H_K(M) \cong H_K(D \times M)$$

under the hypothesis of the theorem. (This also follows from the same argument as that for $\hat{m}^{-\infty}$ given below.)

Proof: We can assume D to be oriented. Indeed let us consider the two-fold cover D_t of D defined in section 2 before Proposition 5. An element of D_t is a couple (m, o), where $m \in D$ and o is an orientation of $T_m D$. Thus the manifold D_t is canonically oriented. Consider the map $\epsilon(m, o) = (m, -o)$ and its action on $H_K(D_t)$. We have $H_K(D_t) = H_K(D) \oplus H_K(D)_t$, where $H_K(D)$ is isomorphic to the eigenspace with eigenvalue 1 for the action of ϵ, while $H_K(D)_t$ is by definition the eigenspace of eigenvalue -1 for the action of ϵ. Furthermore by Proposition 5, the manifold D_t also satisfies the hypothesis that the evaluation map $H_K(D_t) \to H(D_t)$ is surjective. The map

$$\hat{m}^{-\infty} : H_K(D_t) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D_t \times M)$$

clearly commutes with the action of ϵ. So if we show that $\hat{m}^{-\infty}$ is an isomorphism for the oriented manifold D_t, by taking the eigenspace of eigenvalue 1 for ϵ we will obtain the desired isomorphism

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_K^{-\infty}(D \times M).$$

Let $\mathcal{A}^{(p)}(D \times M) = \Gamma(D \times M, \Lambda^p T'D \otimes \Lambda T'M)$, where Γ denotes the space of smooth sections and $T'M$ denotes the cotangent bundle of M.

We write $\mathcal{A}(D \times M) = \oplus_{p=0}^{\dim D} \mathcal{A}^{(p)}(D \times M)$. The total exterior differential $d_{D\times M}$ on $\mathcal{A}(D \times M)$ breaks up into the sum $d_D + d_M$ of partial exterior differential d_D along D

$$d_D : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p+1)}(D \times M)$$

and partial exterior differential d_M along M

$$d_M : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p)}(D \times M).$$

Let us consider the complex $\mathcal{A}_K^{-\infty}(D \times M) = C^{-\infty}(\mathfrak{k}, \mathcal{A}(D \times M))^K$. We write

$$B^p = C^{-\infty}(\mathfrak{k}, \mathcal{A}^{(p)}(D \times M))^K.$$

The operator $d_\mathfrak{k}$ can be written as a sum of homogeneous operators $d_\mathfrak{k} = d_1 + r_0 + r_{-1}$, with

$$d_1 : B^p \to B^{p+1}, r_0 : B^p \to B^p, r_{-1} : B^p \to B^{p-1}.$$

We have $d_1 = d_D$, $r_0 = d_M - \sum_i x_i \iota(E_M^i)$, and $r_{-1} = -\sum_i x_i \iota(E_D^i)$, where E^i is a basis of $\mathfrak{k}$. We write $d = d_D$.

Let us choose a K-invariant metric on D and consider D as a Riemannian manifold. This endows the space $\mathcal{A}(D)$ with an inner product. Let $d^* : \mathcal{A}^p(D) \to \mathcal{A}^{p-1}(D)$ be the adjoint operator to $d = d_D$. Let $\mathcal{H}(D) = Ker(d) \cap Ker(d^*)$ be the space of harmonic forms on D. It is a K-invariant finite dimensional space of d-closed forms on D. The map $\mathcal{H}(D) \to H(D)$ is an isomorphism. Recall that we do not necessarily assume K to be connected. However as the evaluation map at 0 is surjective, K acts trivially on $H(D)$ (this would be automatic if K were connected). Thus every element of $\mathcal{H}(D)$ is K-invariant.

Lemma 63 *For any $\alpha_0 \in \mathcal{H}^p(D)$, there exists $\alpha \in Z_K(D)$ such that*

$$\alpha_0 - \alpha(X) \in \sum_{j<p} \mathcal{A}^j(D), \qquad \text{for every } X \in \mathfrak{k}.$$

Proof: Our hypothesis implies that if $\alpha_0 \in \mathcal{A}^p(D)$ is d-closed, we can find $\gamma \in \mathcal{A}^{p-1}(D)$ and $\lambda \in Z_K(D)$ such that $\alpha_0 - d\gamma = \lambda(0)$. If α_0 is K-invariant, we may assume (eventually after averaging this equation by the action of K) that γ is K-invariant. Take $\alpha = \lambda + d_\mathfrak{k}\gamma$, then $\alpha \in Z_K(D)$ and $\alpha(0) = \alpha_0$. The complex $\mathcal{A}_K(D)$ is $\mathbb{Z}_+$-graded by its total equivariant degree. We may thus assume that α is of total degree p. Thus $\alpha(X) - \alpha(0) \in \mathcal{A}^{p-2}(D) \oplus \mathcal{A}^{p-4}(D) \oplus \cdots$. This proves the lemma. ∎

We continue with the proof of Theorem 61.

Let P be the orthogonal projection of $\mathcal{A}(D)$ onto $\mathcal{H}(D)$. We have $Pd = dP = 0$. Let $G : \mathcal{A}^p(D) \to \mathcal{A}^{p-1}(D)$ be the Green kernel. It satisfies $Gd + dG = I - P$. We can extend the operator P, by the formula $P(\alpha)(X) = P(\alpha(X))$, to an operator still denoted by P,

$$P : B^p \to \mathcal{H}^p(D) \otimes \mathcal{A}_\mathfrak{k}^{-\infty}(M).$$

Similarly we can extend pointwise the operator G

$$G : B^p \to B^{p-1}.$$

Let $r = r_0 + r_{-1}$ and let $N = Gr + rG$. The operator N decreases strictly the exterior degree in D . Let $\nu \in B = \sum_j B^j$. The equation $Gd + dG = I - P$ gives the perturbed equation

$$Gd_\mathfrak{k}\nu + d_\mathfrak{k}G\nu = \nu - (P - N)\nu.$$

Assume $d_\mathfrak{k}\nu = 0$. Let us write $\nu = \sum_{j\leq p} \nu_j$, with $\nu_j \in B^j$. We will prove by induction on p that ν has a representative in $Z_K(D) \otimes Z_K^{-\infty}(M)$. The equation above implies that $\nu \cong \nu' := (P-N)\nu$. We have $\nu'_p = P\nu_p \in \mathcal{H}^p(D) \otimes \mathcal{A}_K^{-\infty}(M)$. Let us write the equation $d_\mathfrak{k}\nu' = 0$ component by component. We obtain, in particular, the equation $r_0\nu'_p + d\nu'_{p-1} = 0$. Applying P, we get $Pr_0\nu'_p + Pd\nu'_{p-1} = 0$. As $Pd = 0$, this implies that $Pr_0\nu'_p = r_0P\nu'_p = r_0\nu'_p = 0$. Thus $\nu'_p \in \mathcal{H}^p(D) \otimes Z_K^{-\infty}(M)$. The preceding lemma allows us to find an element $\xi \in Z_K(D) \otimes Z_K^{-\infty}(M)$ such that $\nu' - \xi \in \sum_{j<p} B^j$. Thus the map $m^{-\infty}$ is surjective.

Let us prove that $\hat{m}^{-\infty}$ is an isomorphism. It suffices to show that $\hat{m}^{-\infty}$ is injective: Recall from Proposition 5 that $H_K(D)$ is free over $H_K(point)$. Furthermore we can find a basis P^p for the $H_K(point)$-module $H_K(D)$ and a basis P_q for the $H_K(point)$-module $H_K(D)_t$ such that $\int_M P^p P_q = \delta_p^q$. We can write an element $\alpha \in H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M)$ uniquely as $\alpha = \sum_a P^a \otimes \nu_a$. Let $\nu \in H_K^{-\infty}(D \times M)$. For any fixed a, consider the map $\nu \mapsto \int_D P_a(X)\nu(X)$. This is a well defined map from $H_K^{-\infty}(D \times M) \to H_K^{-\infty}(M)$. We have

$$\int_D P_a(X)(\hat{m}^{-\infty}\alpha)(X) = \nu_a(X).$$

Thus if $\hat{m}^{-\infty}(\alpha) = 0$, $\nu_a(X) = 0$, for all a, and hence $\hat{m}^{-\infty}$ is injective ∎

Applying Theorem 61 to the case where M is a point, we obtain the following

Corollary 64 *Let K be a compact Lie group. Let D be a compact K-manifold . Assume that $H_K(D)$ surjects on $H_K(point)$. Then*

$$H_K^{-\infty}(D) \cong C^{-\infty}(\mathfrak{k})^K \otimes_{S(\mathfrak{k}')^K} H_K(D).$$

Recall from [14] that if K is a compact connected Lie group acting on a compact symplectic manifold D in an Hamiltonian way, then $H_K(D)$ is free over $H_K(point)$. In particular, the following well-known lemma gives some examples of compact K-manifolds D such that $H_K(D)$ is free over $H_K(point) = S(\mathfrak{k}')^K$.

Lemma 65 *Let K be a compact connected Lie group and let L be a closed subgroup of K. Then $H_K(K/L)$ is free over $H_K(point)$ if and only if K and L have the same rank.*

Proof: Let $D = K/L$. Recall that the equivariant cohomology $H_K(D)$ is isomorphic to $H_L(point) = S(\mathfrak{l}')^L$ by the map $r_{\mathfrak{l}}$, induced from the inclusion of the base point $e \in K/L$. The restriction of polynomial functions on $\mathfrak{k}$ to polynomial functions on $\mathfrak{l}$ gives a homomorphism from $S(\mathfrak{k}')^K$ to $S(\mathfrak{l}')^L$. If $H_K(K/L)$ is free over $H_K(point)$, then in particular the homomorphism $S(\mathfrak{k}')^K \to S(\mathfrak{l}')^L$ is injective. This implies that K and L have the same rank. Conversely, suppose now that L is a closed subgroup of K with equal rank. Let T be a maximal torus of L (and hence of K. Let $W = N_K(T)/T, W_L = N_L(T)/T$ be the respective Weyl groups of (K,T) and (L,T). By Chevalley's theorem, $S(\mathfrak{l}')^L$ is isomorphic to $S(\mathfrak{t}')^{W_L}$, while the ground ring $S(\mathfrak{k}')^K$ is isomorphic to $S(\mathfrak{t}')^W$. Further $S(\mathfrak{t}')^{W_L}$ is a free module over $H_K(point) \cong S(\mathfrak{t}')^W$. ∎
The case where $D = K/T$ is a particularly important example. Considering this case we will see that we obtain as a consequence of Theorem 61 the following:

Proposition 66 *Let K be a compact connected Lie group with maximal torus T and Weyl group W. Let $C^{-\infty}(\mathfrak{t})^\epsilon$ be the space of W-anti-invariant generalized functions on $\mathfrak{t}$. Then the map from*

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \to C^{-\infty}(\mathfrak{t})$$

given by $P \otimes f \mapsto Pf$, for $P \in S(\mathfrak{t}'), f \in C^{-\infty}(\mathfrak{t})^\epsilon$, is an isomorphism.

Proof: Let us consider $D = K/T$ and $M = point$ in Theorem 61. We thus obtain an isomorphism

$$\hat{m}^{-\infty} : H_K(K/T) \otimes_{H_K(point)} C^{-\infty}(\mathfrak{k})^K \to H_K^{-\infty}(K/T). \tag{20}$$

We have $H_K(point) \cong S(\mathfrak{t}')^W$. Recall the isomorphisms

$$w_{K/T} : S(\mathfrak{t}') \to H_K(K/T) \tag{21}$$

where $w_{K/T}$ is the Chern-Weil homomorphism (see Definition 54 of section 5), the isomorphism F_o proved in Lemma 60

$$F_o : C^{-\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{t})^\epsilon, \tag{22}$$

and the isomorphism (cf. Theorem 46)

$$Ind_{K/T,o} : C^{-\infty}(\mathfrak{t}) \to H_K^{-\infty}(K/T). \tag{23}$$

Using the above identifications, the isomorphism (20) gives an isomorphism, again denoted by

$$\hat{m}^{-\infty} : S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \to C^{-\infty}(\mathfrak{t}). \tag{24}$$

By Proposition 59 of section 5, we have for $a \in S(\mathfrak{t}'), f \in C^{-\infty}(\mathfrak{k})^K$

$$w_{K/T}(a)f = Ind_{G/K,o}(aF_of).$$

Hence we obtain our proposition. ∎

7 Equivariant cohomology and subgroups

Let K be a compact connected Lie group and let M be a K-manifold. If L is a compact subgroup of K of equal rank, then $H_K(K/L)$ is free over $H_K(point)$. In this section, we will use Theorem 61 to compare $H_K^{-\infty}(M)$ and $H_L^{-\infty}(M)$ (and also $H_K(M)$ and $H_L(M)$).

Let $D = K/L$. The space $H_K(D)$ is isomorphic to $H_L(point) = S(\mathfrak{l}')^L$ by the Chern-Weil isomorphism:

$$w_D : H_L(point) = S(\mathfrak{l}')^L \to H_K(D). \tag{25}$$

Consider the natural restriction map $r_L : H_K(M) \to H_L(M)$ of a K-equivariant form on M to a L-equivariant form on M.

Let us recall the following proposition (see [16]; page 38). We include a proof for completeness.

Proposition 67 *Let K be a compact connected Lie group and let M be a K-manifold. Let L be a compact subgroup of K of equal rank. Then the map*

$$I_M : H_L(point) \otimes_{H_K(point)} H_K(M) \to H_L(M)$$

given by

$$P \otimes \omega \mapsto P(r_L\omega)$$

for $P \in H_L(point)$ and $\omega \in H_K(M)$ is an isomorphism.

In particular, $H_L(M)$ is generated by $r_L H_K(M)$ over $H_L(point) = S(\mathfrak{l}')^L$.

Proof: Consider the manifold $\mathcal{M} = K \times_L M$. By Theorem (51), the restriction map induced from the inclusion $i_{L,M}$ of the fiber M at $e \in D$ in the fibered space $\pi : \mathcal{M} \to D$ induces an isomorphism

$$i^*_{L,M} : H_K(\mathcal{M}) \to H_L(M) \tag{26}$$

Consider the map $\mu : [k,m] \mapsto km$ from $K \times_L M$ to M. The map $t = (\pi \times \mu)$ given by $t([k,m]) = (kL, k \cdot m)$ is a K-equivariant isomorphism from $\mathcal{M}$ to $D \times M$, where K acts on $D \times M$ as the diagonal action. Thus we have an isomorphism

$$t^* : H_K(D \times M) \to H_K(\mathcal{M}). \tag{27}$$

As follows from Theorem 61 and Lemma 65, the multiplication map $\hat{m}$ is an isomorphism:

$$\hat{m} : H_K(D) \otimes_{H_K(point)} H_K(M) \to H_K(D \times M). \tag{28}$$

Using the isomorphisms (25) -(28), we obtain an isomorphism

$$I : H_L(point) \otimes_{H_K(point)} H_K(M) \to H_L(M). \tag{29}$$

It remains to show that I is equal to I_M. For this, we have to compute for $P \in S(\mathfrak{l}')^L$ and $\omega \in H_K(M)$

$$\begin{aligned} i^*_{L,M}(\pi^*(w_D P) \wedge \mu^*\omega) &= i^*_{L,M}\pi^*(w_D P) \wedge i^*_{L,M}\mu^*\omega \\ &= P \wedge r_L\omega. \end{aligned}$$

This proves the proposition. ∎

In particular, consider $M = K/U$ for a closed subgroup U of K. Then the equivariant cohomology space $H_K(M)$ is equal to $H_U(point)$ via the Chern-Weil homomorphism w_M and we obtain :

Proposition 68 *Let K be a compact connected Lie group. Let L be a closed subgroup of K of same rank. Let U be a closed subgroup of K. Then the map $P \otimes A \mapsto P(r_L(w_M A))$ for $P \in S(\mathfrak{l}')^L$, $A \in S(\mathfrak{u}')^U$ determines an isomorphism:*

$$S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} S(\mathfrak{u}')^U \cong H_L(K/U) \tag{30}$$

Remark that by our hypothesis, $H_K(K/L) = S(\mathfrak{l}')^L$ is free over $S(\mathfrak{k}')^K$ with rank equal to $\dim H(K/L)$. Thus as a vector space

$$H_L(K/U) \cong H(K/L) \otimes_{\mathbb{R}} S(\mathfrak{u}')^U.$$

Corollary 69 *If U also has the same rank as K, then $H_L(K/U)$ is a free (finitely generated) $H_L(point)$-module.*

Proof: As $S(\mathfrak{u}')^U$ is free over $S(\mathfrak{k}')^K$, the $S(\mathfrak{l}')^L$ -module $S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} S(\mathfrak{u}')^U$ is free with rank equal to $\dim H(K/U)$. ∎

We turn now to the determination of the equivariant cohomology $H_L^{-\infty}(M)$ in the case where M is a K-manifold. We denote by $\chi_{K/L}$ the character of L with values in ± 1 given by $\gamma \to \det_{\mathfrak{k}/\mathfrak{l}} \gamma$. Recall the definition of $H_{L,\chi_{K/L}}^{-\infty}(M)$ from Definition 49 of section 5.

Theorem 70 *Let K be a compact connected Lie group and let M be a K-manifold. Let L be a compact subgroup of K of equal rank. Choose an orientation o on $\mathfrak{k}/\mathfrak{l}$. Then, there is a natural isomorphism of $H_L(point)$-modules:*

$$I_{M,o}^{-\infty} : H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_{L,\chi_{K/L}}^{-\infty}(M).$$

In particular, for $M = K/U$, we obtain

$$H_{L,\chi_{K/L}}^{-\infty}(M) \cong S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} C^{-\infty}(\mathfrak{u})^{\chi_{K/U}}.$$

Proof: The proof is almost the same as the proof of Proposition 67. Using the same notation, we use the chain of isomorphisms:

$$Ind_{K/L,o} : H_{L,\chi_{K/L}}^{-\infty}(M) \to H_K^{-\infty}(\mathcal{M}). \tag{31}$$

The isomorphism t of $\mathcal{M}$ with $D \times M$ induces an isomorphism

$$t^* : H_K^{-\infty}(D \times M) \to H_K^{-\infty}(\mathcal{M}). \tag{32}$$

As follows from Theorem 61 and Lemma 65, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D \times M). \tag{33}$$

Using the isomorphisms (25), (31)-(33), we obtain an isomorphism

$$(34) \qquad I_{M,o}^{-\infty} : H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_{L,\chi_{K/L}}^{-\infty}(M).$$

The description of $I_{M,o}^{-\infty}$ is as follows: for $P \in H_L(point) = S(\mathfrak{l}')^L$ and $\omega \in H_K^{-\infty}(M)$, the element $\alpha = I_{M,o}^{-\infty}(P \otimes \omega)$ is the unique element in $H_{L,\chi_{K/L}}^{-\infty}(M)$ such that $Ind_{K/L,o}\alpha = \pi^*(w_D P) \wedge \mu^*\omega$. It follows from Lemma 38 of section 4 that

$$\pi^*(w_D Q) Ind_{K/L,o}\alpha = Ind_{K/L,o} Q\alpha$$

for any $Q \in H_L(point)$. Hence $I_{M,o}^{-\infty}$ is an isomorphism of $H_L(point)$-modules. The last statement in Theorem 70 follows easily from Theorem 46. ∎

The isomorphism $I_M^{-\infty}$ is not so easy to determine explicitly as the isomorphism I_M. We determine it as much as we can.

Consider $H_K^{-\infty}(M)$ as a $H_K(M)$-module and $H_{L,\chi_{K/L}}^{-\infty}(M)$ as a $H_L(M)$-module.

Lemma 71 *If $\alpha \in H_K(M)$ and $\beta \in H_K^{-\infty}(M)$, then*

$$I_{M,o}^{-\infty}(1 \otimes \alpha\beta) = I_M(1 \otimes \alpha) I_{M,o}^{-\infty}(1 \otimes \beta).$$

Proof: If $I_M^{-\infty}(1 \otimes \beta) = \omega$, we have $\mu^*\beta = Ind_{K/L,o}\omega$. Then $\mu^*\alpha \wedge \mu^*\beta = Ind_{K/L,o}(r_L\alpha)\omega$ by Lemma 38 of section 4. This proves the lemma. ∎

We assume that K/L is orientable so that $\chi_{K/L} = 1$. Recall the map $F_o : C^{-\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{l})^L$ given in Definition 57 of section 5. Then

Lemma 72 *Let $f \in C^{-\infty}(\mathfrak{k})^K$, $\alpha \in H_K(M)$, then $I_{M,o}^{-\infty}(1 \otimes f\alpha) = (F_o f) r_L \alpha$.*

Proof: This follows immediately from Proposition 59 of section 5. ∎

Thus we know $I_M^{-\infty}$ on the subspace $S(\mathfrak{l}')^L \otimes_{H_K(point)} C^{-\infty}(\mathfrak{k})^K H_K(M)$ of $S(\mathfrak{l}')^L \otimes_{H_K(point)} H_K^{-\infty}(M)$.

If M is compact and $H_K(M)$ is a free module over $H_K(point)$, then $H_K^{-\infty}(M)$ is equal to $C^{-\infty}(\mathfrak{k})^K \otimes_{H_K(point)} H_K(M)$ (cf. Corollary 64 of section 6). Thus in this case, the isomorphism $I_{M,o}^{-\infty}$ is entirely determined by the knowledge of F.

8 Reduction to the maximal torus

Let K be a compact connected Lie group and let T be its maximal torus. Let $W = N(T)/T$ be the Weyl group.

Let M be a K-manifold. It is well known (cf. [16]; chapter 3, section 1, Proposition 1) that the natural restriction map $\mathcal{A}_K(M) \to \mathcal{A}_T(M)$ induces an isomorphism between $H_K(M)$ and $H_T(M)^W$. In this section, we prove a similar statement for the generalized K-cohomology $H_K^{-\infty}(M)$.

We first need to define a map from $H_T^{-\infty}(M)$ to $H_K^{-\infty}(M)$: Choose a non-zero K-invariant form ν' on K/T of maximal exterior degree. In particular, ν' determines an Euclidean measure $|d\nu'|$ on $\mathfrak{k}/\mathfrak{t}$ and an orientation o. For $f \in C^{-\infty}(\mathfrak{t}, \mathcal{A}(M))$, define $A_o(f) \in C^{-\infty}(\mathfrak{k}, \mathcal{A}(K \times M))$ by: If Φ is a test function on $\mathfrak{k}$

$$(A_o(f), \Phi dX)_{(k,m)} := \nu' \wedge (k \cdot (\int_{\mathfrak{t}} f(Y)\Phi(k \cdot Y)dY))_m$$

where dY is the Euclidean measure on $\mathfrak{t}$ which is quotient of dX by $|d\nu'|$ on $\mathfrak{k}/\mathfrak{t}$. In particular A_o depends only on the orientation o on K/T associated to ν'.

Lemma 73 *Let $D = K/T$. Consider $D \times M$ as a K-manifold under the diagonal action. Then the map A_o defines a cochain map from $(\mathcal{A}_T^{-\infty}(M), d_{\mathfrak{t}})$ to $(\mathcal{A}_K^{-\infty}(D \times M), d_{\mathfrak{k}})$.*

Proof: It is not difficult to check that if f is T -invariant, then $A_o(f)$ is in $\mathcal{A}_K^{-\infty}(D \times M)$. Now, as ν' is of maximal degree, we see that $d_{D\times M}(A_o(f)) = A_o(d_M f)$. It is also easy to prove that $\iota_{\mathfrak{k}} A_o(f) = A_o(\iota_{\mathfrak{t}} f)$. ∎

Consider the projection $\pi : D \times M \to M$ with fiber D. We denote the map $\pi_* : \mathcal{A}_K^{-\infty}(D \times M) \to \mathcal{A}_K^{-\infty}(M)$ by $\int_D$ (cf. Formula 8 of section 2).

Define $B(f) \in \mathcal{A}_K^{-\infty}(M)$ by

$$B(f) = \int_{D,o} A_o(f),$$

where the orientation on D is the orientation o. In particular, we see that B does not depend on o. If we denote by dk/dt the positive density on K/T associated to ν', we have for $f \in \mathcal{A}_T^{-\infty}(M)$ and Φ a test function on $\mathfrak{k}$,

$$\int_{\mathfrak{k}} B(f)(X)\Phi(X)dX = \int_{K/T} k \cdot (\int_{\mathfrak{t}} f(Y)\Phi(k \cdot Y)dY)dk/dt.$$

The map B is a cochain map from $(\mathcal{A}_T^{-\infty}(M), d_{\mathfrak{t}})$ to $(\mathcal{A}_K^{-\infty}(M), d_{\mathfrak{k}})$. The Weyl group W canonically acts on $\mathcal{A}_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$.

Theorem 74 *Let K be a compact connected Lie group and let T be its maximal torus. Let W be the Weyl group of K.*

The restriction of the cochain map B to $\mathcal{A}_T^{-\infty}(M)^W$ induces an isomorphism in cohomology

$$b : H_T^{-\infty}(M)^W \to H_K^{-\infty}(M).$$

Proof: Again, this theorem is an easy consequence of Theorem 61 of section 6. As in the proof of Theorem 70, we consider $\mathcal{M} = K \times_T M$ and we use the isomorphism (cf. Theorem 52 of section 5):

(35) $$Ind_{K/T,o} : H_T^{-\infty}(M) \to H_K^{-\infty}(\mathcal{M}).$$

Composing this with the isomorphism (see Formula 32 of section 7):

(36) $$t^* : H_K^{-\infty}(D \times M) \to H_K^{-\infty}(\mathcal{M}),$$

we obtain an isomorphism

(37) $$(t^*)^{-1} \circ \mathrm{ind}_{K/T,o} : H_T^{-\infty}(M) \to H_K^{-\infty}(D \times M).$$

It is not difficult to see that $(t^*)^{-1} Ind_{K/T,o} f = A_o f$, for $f \in H_T^{-\infty}(M)$,

Let ϵ be the character of W given by $\epsilon(w) = det_{\mathfrak{t}}(w)$. Let r be the action of W on $K/T \times M$ given by $w \cdot (kT, m) = (kw^{-1}T, m)$. This action commutes with the diagonal action of K and hence induces an action still denoted by r on $H_K^{-\infty}(D \times M)$. Under the isomorphism (37), the natural action of W on $H_T^{-\infty}(M)$ becomes the action $r \otimes \epsilon$.

The K-equivariant cohomology $H_K(D) \cong S(\mathfrak{t}')$ of D is free over $H_K(point) \cong S(\mathfrak{t}')^W$. Hence, as follows from Theorem 61, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

(38) $$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D \times M).$$

For the action of the group W on $H_K(D)$, induced by the action of W by right translation on $D = K/T$, the subspace $H_K(D)^\epsilon$ of $H_K(D)$ is a free $H_K(point)$ -module of rank one, in fact $H_K(D)^\epsilon = H_K(point) w_D(\chi)$, where $w_D(\chi) \in H_K(D)^\epsilon$ is the image under the Chern-Weil homomorphism w_D of the W-anti-invariant polynomial function

$$\chi(Y) = (2\pi)^{-\dim(D)/2} |W|^{-1} \det{}^{1/2}_{\mathfrak{k}/\mathfrak{t},o}(Y), Y \in \mathfrak{t}.$$

We have $\int_D w_D(\chi)(X) = 1$, for all $X \in \mathfrak{k}$. The space $H_K(D)^\epsilon$ is isomorphic to $H_K(point)$ under $\alpha \mapsto \int_D \alpha(X)$. Thus, by (38), the map $\alpha \mapsto \int_D \alpha$ induces an isomorphism (depending on the choice of an orientation on D)

$$H_K^{-\infty}(D \times M)^\epsilon \cong H_K^{-\infty}(M).$$

and hence, by (37),

$$H_T^{-\infty}(M)^W \cong H_K^{-\infty}(M).$$

The above isomorphism is given by the restriction to $H_T^{-\infty}(M)^W$ of the map $B = \int_{D,o} A_o$. Thus we obtain the formula of the theorem. ∎

In particular, when $M = point$, the isomorphism given by Theorem 74 is the well known isomorphism $b : C^{-\infty}(\mathfrak{t})^W \to C^{-\infty}(\mathfrak{k})^K$ given by:

$$(b(f), \Phi dX) = vol(K/T, dk/dt)\left(\int_{\mathfrak{t}} f(Y)\Phi(Y)dY\right)$$

if Φ is a K-invariant test function on $\mathfrak{k}$.

9 The case of a free action

Let G be a Lie group. Let P be a right G-manifold (i.e. G acts on P from the right). (Of course any left G-manifold can be thought of as a right G-manifold under $x \cdot g := g^{-1} \cdot x$, for $x \in P$ and $g \in G$.)

Definition 75 *Let P be a right G-manifold. We will say that the* action of G on P is principal *(or that G acts* principally *on P) if the orbit space P/G is a smooth manifold such that $P \to P/G$ is a smooth principal G-bundle.*

If G is compact, then a right action is principal if and only if the action is free.

Let P be a right G-manifold. If G acts principally on P, it is known [11] that the G- equivariant de Rham cohomology of P is isomorphic to the de Rham cohomology of the quotient space P/G. In this section, we prove similarly that the space $H_G^{-\infty}(P)$ is isomorphic to $H(P/G)$. We also consider the following more general situation:

(S):*Let G be a Lie group and let N be a closed normal subgroup of G. Let P be a right G-manifold. Assume that the subgroup N acts principally on P.*

We ask the question: Under what hypothesis are $H_G^{-\infty}(P)$ and $H_{G/N}^{-\infty}(P/N)$ isomorphic. In this section, we prove this affirmatively when G is compact connected. On the other hand when $N = G$, we need no compactness hypothesis on G to prove the isomorphism $H_G^{-\infty}(P) \cong H(P/G)$ and the proof for this case is comparatively easy. The reader only interested in the case where $N = G$ can go directly to the proof of Theorem 89.

An important example of this situation (S) is the following:

Example 76 *Let U and K be two Lie groups and let $G := U \times K$ be the direct product. Let L be a $U \times K$- manifold. For convenience, we assume that U acts on the left and K on the right. Assume the right action of K on L is principal. Let M be a K-manifold and let $P := L \times M$. Define the action of an element $g = (u,k) \in G = U \times K$ on P by :$(x,m) \cdot (u,k) = (u^{-1}xk, k^{-1}m)$, for $x \in L$, $m \in M$. Then the action of the (normal) subgroup K of G is principal on P. The quotient manifold $(L \times M)/K$ is the left U-space $\mathcal{M} = L \times_K M$ fibered over L/K with fiber M.*

Consider the quotient map $q : P \to P/N$ under the situation (S). Recall Definition 4 of a G-equivariant fibration with G-oriented fibers. Later in the section, we will need to impose the following conditions (77) and (78)

Condition 77 *There exists a G-orientation o for the fibers of q.*

This condition (77) is satisfied, for example, when $\det_{\mathfrak{n}} g > 0$, for all $g \in G$. In particular this is satisfied if G is connected.

Condition 78 *There exists a G-invariant connection ω for the principal N-bundle $q : P \to P/N$.*

This condition (78) is always satisfied when G is compact.

It is proved in [13] that the canonical map $q^* : H^{\infty}_{G/N}(P/N) \to H^{\infty}_G(P)$ is an isomorphism, when the condition (78) is satisfied. Furthermore, an explicit formula for the inverse of q^* is given in terms of the equivariant curvature of ω. The reader should however be warned that the natural map q^* is sometimes equal to zero when applied to the equivariant cohomology with generalized coefficients.

Whenever the conditions (77) and (78) are satisfied, we will construct a natural map (cf. Proposition 82)

$$m_o : H^{-\infty}_{G/N}(P/N) \to H^{-\infty}_G(P)$$

and will show that m_o is an isomorphism, if either $N = G$ or G is compact.

We begin by constructing a natural element $\gamma_o \in H^{-\infty}_G(P)$ (assuming the validity of conditions (77) and (78)):

Let $B := P/N$ be the space of N-orbits. Consider the projection $q : P \to B$. The vertical tangent bundle V is a G-equivariant real vector bundle over P. By assumption, the bundle V is a G-orientable vector bundle. As the group N acts principally, the bundle V is a trivial bundle over P canonically isomorphic to $P \times \mathfrak{n}$. The isomorphism is obtained by sending $(x, X) \in P \times \mathfrak{n}$ to the vertical tangent vector $(X_P)_x$. The action of an element $g \in G$ on $V = P \times \mathfrak{n}$ is given by $(x, Y) \cdot g = (xg, g^{-1} \cdot Y)$ for $x \in P, Y \in \mathfrak{n}$. (Observe that if $\det_{\mathfrak{n}}(g) > 0$, for all $g \in G$, then any choice of orientation of $\mathfrak{n}$ gives rise to a G-orientation of the vector bundle V, i.e., in this case the condition (77) is satisfied.)

Let us choose a G-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$. Using ω, we obtain a G-invariant decomposition

$$TP = V \oplus H$$

of the tangent bundle as sum of vertical and horizontal subbundles.

Similarly, using ω, we have an isomorphism

$$U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n} \tag{39}$$

given by $U(x, X) = (x, Y, Q)$, where $Q \in \mathfrak{g}/\mathfrak{n}$ is the projection of $X \in \mathfrak{g}$ and where $Y = \omega_x(X_P) \in \mathfrak{n}$.

Consider the dual bundle $V' = P \times \mathfrak{n}'$ to the vertical tangent bundle V. The projection $TP \to V$, given by the connection ω, determines a G-invariant injection s_ω of V' in the cotangent bundle $T'P$. Consider the canonical 1-form

α on the manifold $T'P$. Let $\alpha_\omega := s_\omega^*\alpha \in \mathcal{A}^1(V')$. It is a G-invariant differential form on V'. The form

$$\beta_\omega := e^{id_{\mathfrak{g}}\alpha_\omega} \in \mathcal{A}_G^\infty(V') \otimes_{\mathbb{R}} \mathbb{C}$$

is a closed G-equivariant differential form. Consider the projection $p : V' \to P$. We will prove below that if Φ is a test function on $\mathfrak{g}$, then $\int_{\mathfrak{g}} \beta_\omega(X)\Phi(X)dX$ is a differential form on $V' = P \times \mathfrak{n}'$ rapidly decreasing in the direction $\mathfrak{n}'$. Thus, as the vector bundle V' is G-oriented, we may define $p_*\beta_\omega$ as an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P)) \otimes_{\mathbb{R}} \mathbb{C}$ by setting:

$$\int_{\mathfrak{g}} (p_*\beta_\omega)(X)\Phi(X)dX = p_*(\int_{\mathfrak{g}} \beta_\omega(X)\Phi(X)dX)$$

(the map p_* depends on the choice of o).

A representative of the element γ_o will be defined as the integral of β_ω over the fibers of p, normalized in order that $\gamma_o(X)$ is a differential form on P with real coefficients.

Proposition 79 *Let $p : V' \to P$ be the projection as above. Let us choose a G-orientation o on the vector bundle V'. Let $n = \dim \mathfrak{n}$. Let $c_n = 1$, if n is even, and $c_n = -i$ if n is odd. Define*

$$\gamma_{\omega,o} := c_n(2\pi)^{-n}p_*(e^{id_{\mathfrak{g}}\alpha_\omega}).$$

Then $\gamma_{\omega,o}$ is an element of $\mathcal{A}_G^{-\infty}(P)$ and is $d_{\mathfrak{g}}$-closed.

The cohomology class of $\gamma_{\omega,o}$ in $H_G^{-\infty}(P)$ is independent of the choice of the G-invariant connection ω. It depends only on the G-orientation o. We denote it by γ_o.

Proof: Writing $\beta_\omega = e^{id_{\mathfrak{g}}\alpha_\omega}$, we compute $p_*\beta_\omega$.

Let E^j be a basis of $\mathfrak{n}$ with dual basis E_j. We write an element of $\mathfrak{n}'$ as $y = \sum_j y^j E_j$. Let

$$\omega = \sum_j \omega_j E^j \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$$

be the connection form. By definition $(\omega_k, E_P^j) = \delta_k^j$ and ω of course by definition vanishes on the horizontal vectors. Under the identification $V' \cong P \times \mathfrak{n}'$, the 1-form

$$\alpha_\omega = \sum_j y^j\omega_j.$$

Define, as in ([3], chapter 7), the moment $\mu \in \mathfrak{g}' \otimes C^\infty(P) \otimes \mathfrak{n}$ of the connection ω by setting, for any $X \in \mathfrak{g}$

$$\mu(X) = -\omega(X_P).$$

Thus $\mu(X)_x$ is an element of $\mathfrak{n}$ and $\mu(X+Y)_x = \mu(X)_x - Y$ for all $Y \in \mathfrak{n}$ and $x \in P$. Let us compute

$$d_X\alpha_\omega := (d_{\mathfrak{g}}\alpha_\omega)(X).$$

We obtain

$$d_X\alpha_\omega = (y, \mu(X)) + \sum_j (dy^j\omega_j + y^j d\omega_j).$$

We have $e^{id_X\alpha_\omega} = e^{i(y,\mu(X))}A(y, dy, \omega, d\omega)$, where A is a polynomial expression in $y^j, dy^j, \omega_j, d\omega_j$. If Φ is a test function on $\mathfrak{g}$, the integral

$$\int_{\mathfrak{g}} e^{i(y,\mu(X))}\Phi(X)dX$$

is a function on V' rapidly decreasing over the fiber $\mathfrak{n}'$. This can be seen as follows: Consider the isomorphism (39) of $P \times \mathfrak{g}$ with $P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$.

For $x \in P$, let

$$\mathfrak{q}_x := \{X \in \mathfrak{g}; \omega_x(X_P) = 0\}.$$

Thus $\mathfrak{q}_x$ is isomorphic to $\mathfrak{g}/\mathfrak{n}$ under the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ and

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_x. \tag{40}$$

We fix $x \in P$, and write $\mathfrak{q}_x = \mathfrak{q}$. Let $X \in \mathfrak{g}$. Using the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$, we write $X = Y + Q$, with $Y \in \mathfrak{n}, Q \in \mathfrak{q}$. We have $(y, \mu(X)) = -(y, Y)$. Writing $\Phi(X)dX = \Phi(Y, Q)dQdY$, we get

$$\int_{\mathfrak{g}} e^{i(y,\mu(X))}\Phi(X)dX = \int_{\mathfrak{n}} e^{-i(y,Y)}\Psi(Y)dY,$$

where $\Psi(Y) = \int_{\mathfrak{q}} \Phi(Y, Q)dQ$. Clearly $\Psi(Y)$ is a C^∞-function with compact support on $\mathfrak{n}$. As Fourier transform of test functions are rapidly decreasing, it follows that

$$\int_{\mathfrak{g}} e^{id_X\alpha_\omega}\Phi(X)dX = A(y, dy, \omega, d\omega)\int_{\mathfrak{n}} e^{-i(y,Y)}\Psi(Y)dY$$

is a form on V' rapidly decreasing over the fiber $\mathfrak{n}'$ of the projection $V' \to P$.

This proves that $p_*\beta_\omega$ exists, as an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P)) \otimes_{\mathbb{R}} \mathbb{C}$. It is clearly G-invariant, so that $p_*\beta_\omega \in \mathcal{A}_G^{-\infty}(P) \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore $p_*\beta_\omega$ is $d_{\mathfrak{g}}$-closed:

$$(d_{\mathfrak{g}}p_*\beta_\omega, \Phi dX) = p_*(d_{\mathfrak{g}}\beta_\omega, \Phi dX) = 0.$$

If ω_t is a one-parameter smooth family of G-invariant connections, we denote $\alpha_{\omega_t}, \beta_{\omega_t}$ by α_t, β_t respectively. We have

$$\frac{d}{dt}\beta_t = \frac{d}{dt}(e^{id_{\mathfrak{g}}\alpha_t}) = id_{\mathfrak{g}}((\frac{d}{dt}\alpha_t) \wedge \beta_t).$$

The integral of $(\frac{d}{dt}\alpha_t \wedge \beta_t)$ over the fiber $\mathfrak{n}'$ exists in the sense of generalized functions and

$$\frac{d}{dt}p_*\beta_t = id_{\mathfrak{g}}p_*(\frac{d}{dt}\alpha_t \wedge \beta_t). \tag{41}$$

Thus the cohomology class of $p_*\beta_t$ is independent of the choice of the connection ω_t.

We now compute explicitly the element $\gamma_{\omega,o}$ defined in Proposition 79 and show, in particular, that $\gamma_{\omega,o}(X)$ is a differential form with real coefficients.

Let us fix $x \in P$. The orientation o on the vector bundle V gives rise to an orientation o_x on $\mathfrak{n}$ (which may depend on the connected component of $x \in P$). Let E^j be an ordered basis of $\mathfrak{n}$. We will say that this order is o-compatible, if this basis is of orientation o_x. The exterior product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ of the components ω_i of the connection ω is a vertical form on P of maximum dimension.

Let $\nu' \in \Lambda^n\mathfrak{n}'$ be such that $(\nu', E^1 \wedge \cdots \wedge E^n) = 1$. The element ν' also determines an Euclidean measure dY on $\mathfrak{n}$ and a δ- function $|\nu'|^{-1}\delta_{\mathfrak{n}}(Y) \in C^{-\infty}(\mathfrak{n})$ (cf Section 2, Formula 10).

We can also write

$$|\nu'|^{-1}\delta_{\mathfrak{n}}(Y) = (2\pi)^{-\dim\mathfrak{n}} \int_{\mathfrak{n}'} e^{i(y,Y)}dy,$$

where dy is the measure on $\mathfrak{n}'$ dual to the Euclidean measure dY on $\mathfrak{n}$. Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ be the curvature of the connection ω. Thus $\Omega \in (\mathcal{A}^2(P)\otimes\mathfrak{n})^G$. We write

$$\Omega = \sum_j \Omega_j E^j.$$

Define the equivariant curvature of ω (as in [3], chap 7) by

$$\Omega(X) = \mu(X) + \Omega.$$

To simplify notation, we will use ν' to identify generalized functions and distributions and write $\delta_{\mathfrak{n}}$ instead of $|\nu'|^{-1}\delta_{\mathfrak{n}}$.

Let us show that the generalized function $\delta_{\mathfrak{n}}(\Omega(X)) \in C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ is well defined. We describe, at each point $x \in P$, $\delta_{\mathfrak{n}}(\Omega(X))_x$ as a generalized function on $\mathfrak{g}$ with values in the vector space $\Lambda T_x'P$. In the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_x$ given by formula 40, we write $X = Y + Q$. Then $\Omega(Y+Q)_x = -Y + \Omega_x$ and we define $\delta_{\mathfrak{n}}(\Omega(X))_x$ by its Taylor expansion:

$$\delta_{\mathfrak{n}}(\Omega(X))_x = \delta_{\mathfrak{n}}(-Y) + \sum_j \Omega_j(\partial_{E^j}\delta_{\mathfrak{n}})(-Y) + \cdots$$

Proposition 80 *Let $x \in P$ and let E^j be an ordered basis of $\mathfrak{n}$, with an order compatible with the orientation o_x. Then*

$$\gamma_{\omega,o}(X)_x = |\nu'|^{-1}\delta_{\mathfrak{n}}(\Omega(X))_x \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x.$$

Proof: The highest degree component of $e^{id\alpha_\omega} = e^{i(dy,\omega)+i(y,d\omega)}$ in dy_j's is equal to

$$c_n^{-1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y,d\omega)}.$$

The curvature Ω is equal to $d\omega$ modulo terms in ω_j. Thus

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y,d\omega)} = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \wedge e^{i(y,\Omega)}$$

and

$$\begin{aligned} p_*\beta_\omega(X)_x &= c_n^{-1}\left(\int_{\mathfrak{n}'} e^{i(y,\mu(X)+\Omega)} dy\right) \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\ &= c_n^{-1}(2\pi)^n |\nu'|^{-1}\delta_{\mathfrak{n}}(\Omega(X))_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x. \end{aligned}$$

This proves the proposition. ∎
In particular, we see that $\gamma_{\omega,o}$ indeed belongs to $\mathcal{A}_G^{-\infty}(P)$ and we obtain Proposition 79. ∎

Remark 81 *It is easy to check directly from the formula of the above proposition that $\gamma_{\omega,o}$ is $d_{\mathfrak{g}}$ closed.*

Fix $x \in P$ and consider the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ given by (40). The space $\mathfrak{q}$ is isomorphic to $\mathfrak{g}/\mathfrak{n}$. The generalized function $\gamma_{\omega,o}(X)_x$ iz constant in the direction $\mathfrak{q}$: $\gamma_{\omega,o}(Y+Q)_x = \gamma_{\omega,o}(Y)_x$. Thus, if $f(Q)$ is a generalized function on $\mathfrak{g}/\mathfrak{n}$, we can multiply $\gamma_{\omega,o}(Y)_x$ by $f(Q)$ and we obtain a generalized function on $\mathfrak{g}$ with values in $\Lambda T_x' P$.

Define the map

$$m_{\omega,o} : \mathcal{A}_{G/N}^{-\infty}(P/N) \to \mathcal{A}_G^{-\infty}(P)$$

by setting

$$m_{\omega,o}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega,o},$$

for $\alpha \in \mathcal{A}_{G/N}^{-\infty}(P/N)$.

The above discussion shows that $m_{\omega,o}$ is well defined. It is a cochain map of differential complexes

$$m_{\omega,o} : (\mathcal{A}_{G/N}^{-\infty}(P/N), d_{\mathfrak{g}/\mathfrak{n}}) \to (\mathcal{A}_G^{-\infty}(P), d_{\mathfrak{g}}).$$

Proposition 82 *The induced map in cohomology*

$$H^{-\infty}_{G/N}(P/N) \to H^{-\infty}_G(P),$$

from the cochain map

$$m_{\omega,o}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega,o},$$

does not depend upon the choice of the G-invariant connection ω on the principal N-bundle $q : P \to P/N$. We will denote it by m_o.

Proof: This follows easily from Formula 41 on the variation of $p_*\beta_\omega$. ∎

We describe now the properties of $m_{\omega,o}$ in relation to the $\mathcal{A}^\infty_G(P)$-module structure on $\mathcal{A}^{-\infty}_G(P)$.

If $\alpha \in \mathcal{A}^\infty_G(P)$, we can define $\beta \in \mathcal{A}^\infty_G(P)$, by setting

$$\beta(X) := \alpha(X + \Omega(X)).$$

Explicitly, for $x \in P$, $Y \in \mathfrak{n}$, $Q \in \mathfrak{q}$, $X = Y + Q$, then $X + \Omega(X) = Y + Q + \Omega_x - Y = Q + \Omega_x$, thus $\beta(X)_x = \alpha(Q + \Omega)_x$ is defined by its Taylor expansion

$$\alpha(\Omega + Q)_x = \alpha(Q)_x + \sum_j \Omega_j(\partial_{E^j}\alpha)(Q)_x + \cdots$$

Thus $\beta(X + Y)_x = \beta(X)_x$, for $X \in \mathfrak{g}, Y \in \mathfrak{n}$. Hence $\beta \in C^\infty(\mathfrak{g}/\mathfrak{n}, \mathcal{A}(P))$.

Let $\Gamma \subset \mathcal{A}(P)$ be the subspace of horizontal forms. The group G acts on Γ. The connection ω defines a horizontal projector $h : \mathcal{A}(P) \to \Gamma$ which commutes with the action of G. Define, for $\alpha \in \mathcal{A}^\infty_G(P)$,

$$W_\omega(\alpha)(X) := h(\alpha(X + \Omega(X))).$$

As $W_\omega\alpha = h(\beta)$, $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma)$. The G-invariance implies that $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma)^G$. As N is a normal subgroup of G, N acts trivially on $\mathfrak{g}/\mathfrak{n}$, in particular $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma^N)$. The space Γ^N is the space of forms on $B := P/N$, and we think of $W_\omega(\alpha)$ as an element of $C^\infty(\mathfrak{g}/\mathfrak{n}, \mathcal{A}(B))$. The G-invariance implies that $W_\omega(\alpha) \in \mathcal{A}^\infty_{G/N}(P/N)$. Thus we have obtained a map

$$W_\omega : \mathcal{A}^\infty_G(P) \to \mathcal{A}^\infty_{G/N}(P/N).$$

Remark 83 . *The map W_ω is a generalization of the Chern-Weil map: If $N = G$, let $\phi \in C^\infty(\mathfrak{g})^G$ and consider $\phi(X)1 \in \mathcal{A}^\infty_G(P)$. Then $W_\omega(\phi(X)1) = \phi(\Omega)$ is the characteristic form on P/G associated to ϕ by the classical Chern-Weil homomorphism.*

Proposition 84 *If $\alpha \in \mathcal{A}_G^\infty(P)$, then*

$$\alpha \wedge \gamma_{\omega,o} = q^*(W_\omega \alpha) \wedge \gamma_{\omega,o} = m_{\omega,o}(W_\omega(\alpha)).$$

In particular, for $\beta \in \mathcal{A}_{G/N}^{-\infty}(P/N)$,

$$\alpha \wedge m_{\omega,o}(\beta) = m_{\omega,o}(W_\omega \alpha \wedge \beta).$$

Proof: The proof follows easily from our formula given in Proposition 80 for $\gamma_{\omega,o}$. We have, for $x \in P$, $X = Y + Q$, $Y \in \mathfrak{n}, Q \in \mathfrak{q}$

$$\begin{aligned}
\alpha(X)_x \wedge \gamma_{\omega,o}(X)_x &= \alpha(Y+Q)_x \delta_{\mathfrak{n}}(\Omega - Y)_x (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\
&= \alpha(\Omega + Q)_x \delta_{\mathfrak{n}}(\Omega - Y)_x (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\
&= \alpha(X + \Omega(X))_x \wedge \gamma_{\omega,o}(X)_x.
\end{aligned}$$

As $\gamma_{\omega,o}$ is already of top degree in the vertical directions, we see that

$$\alpha(X + \Omega(X))_x \wedge \gamma_{\omega,o}(X)_x = h(\alpha(X + \Omega(X)))_x \wedge \gamma_{\omega,o}(X)_x$$

which is the formula we want. ∎

The following proposition is proved in [13]. We give another proof, which is an easy application of Proposition 84.

Proposition 85 *Let G be a Lie group and let $N \subset G$ be a closed normal subgroup of G. Let P be a right G-manifold such that the action of N is principal. Assume that the principal N-bundle $q : P \to P/N$ admits a G-invariant connection ω. Then the map*

$$W_\omega : \mathcal{A}_G^\infty(P) \to \mathcal{A}_{G/N}^\infty(P/N)$$

defined above is a cochain map:

$$W_\omega d_{\mathfrak{g}} = d_{\mathfrak{g}/\mathfrak{n}} W_\omega.$$

Furthermore, if $\beta \in \mathcal{A}_{G/N}^\infty(P/N)$ then

$$W_\omega q^* \beta = \beta.$$

Proof: The last equation follows from the definition of W_ω, as $\alpha(Y+Q)_x = (q^*\beta)(Q)$ is independent of the variable $Y \in \mathfrak{n}$, thus $\alpha(\Omega + Q)_x = \alpha_x(Q)$ and is horizontal.

Let us prove that W_ω commutes with differentials. Let $\alpha \in \mathcal{A}_G^\infty(P)$ and write γ instead of $\gamma_{\omega,o}$ and W instead of W_ω. Let us compute $d_{\mathfrak{g}}(\alpha \wedge \gamma)$:

As γ is $d_{\mathfrak{g}}$-closed, we obtain from Proposition 84,

$$d_{\mathfrak{g}}(\alpha \wedge \gamma) = d_{\mathfrak{g}}\alpha \wedge \gamma = q^*(W(d_{\mathfrak{g}}\alpha)) \wedge \gamma.$$

We also have

$$\begin{aligned} d_{\mathfrak{g}}(\alpha \wedge \gamma) &= d_{\mathfrak{g}}(q^*W(\alpha) \wedge \gamma) \\ &= d_{\mathfrak{g}}(q^*W(\alpha)) \wedge \gamma \\ &= q^*(d_{\mathfrak{g}/\mathfrak{n}}W(\alpha)) \wedge \gamma. \end{aligned}$$

Thus

$$m_{\omega,o}(W(d_{\mathfrak{g}}\alpha)) = m_{\omega,o}(d_{\mathfrak{g}/\mathfrak{n}}W(\alpha)).$$

But the map $m_{\omega,o}$ is easily seen to be injective and hence we obtain

$$W(d_{\mathfrak{g}}\alpha) = d_{\mathfrak{g}/\mathfrak{n}}W(\alpha).$$

∎

We also have ([13])

Theorem 86 *Let the notation and assumptions be as in the above proposition (85). Then the cochain map*

$$q^* : \mathcal{A}^{\infty}_{G/N}(P/N) \to \mathcal{A}^{\infty}_{G}(P)$$

induces an isomorphism in cohomology.

As $W_\omega q^* = I$, where I is the identity operator, the map W_ω provides an explicit inverse for q^* in cohomology.

Let us consider example 76 for the following special case: The manifold L is equal to the Lie group U and K is a closed subgroup of U. The manifold $L = U$ is a $U \times K$ manifold, where the action of $(u, k) \in U \times K$ on $x \in L$ is given by $x \cdot (u, k) = u^{-1}xk$. Let $G = U \times K$ and $P = U \times M$, for a K-manifold M. The action of both of the (normal) subgroups U and K of G on P are principal.

Consider first the action of U. The space P/U is our K-manifold M we started with. Consider the canonical Maurer-Cartan connection $\omega_U \in (\mathcal{A}^1(U) \otimes \mathfrak{u})^U$ defined by $\omega_U(X_U) = X$ for every $X \in \mathfrak{u}$. Here X_U is the vector field associated to the action of U on L by left translation, i.e $(X_U)_x$ is the tangent vector to the curve $\exp(-\epsilon X)x$. We have $\omega_U(Y_L)_x = -x \cdot Y$, for $x \in L$, $Y \in \mathfrak{k}$, as K acts by right translations on L. The connection ω_U extends trivially to a $(U \times K)$-invariant connection $\omega_U \otimes 1$ for the principal U-bundle $q_U : U \times M \to M$.

As before, trivialize the vertical bundle $V_U \cong P \times \mathfrak{u}$ (for q_U) by the map defined by sending the vertical tangent vector $(X_{U\times M})_{x,m}$ to (x, m, X), for $x \in$

$U, m \in M, X \in \mathfrak{u}$. Choose an orientation $o_{\mathfrak{u}}$ on $\mathfrak{u}$. We obtain a U-invariant orientation o_U on V_U, by setting $(o_U)_{x,m} = sign(\det_{\mathfrak{u}} x) o_{\mathfrak{u}}$. This orientation is $U \times K$ invariant, if and only if $\det_{\mathfrak{u}} k > 0$, for every $k \in K$. Thus, under this condition, we get a map

$$m_{o_U} : H_K^{-\infty}(M) \to H_{U \times K}^{-\infty}(U \times M).$$

Consider now the action of K. The quotient space $U \times M$ by the action of K is the induced space

$$\mathcal{M} = U \times_K M$$

with left action of U, that we considered in section 5. Assume there exists a U-invariant connection $\omega \in (\mathcal{A}^1(U)^U \otimes \mathfrak{k})^K$ for the principal K-bundle $U \to U/K$. Then the form $\omega \otimes 1$ on $U \times M$ is a U-invariant connection form for the principal K-bundle $q_K : U \times M \to U \times_K M$. Assume that $\det_{\mathfrak{k}} k > 0$ for every $k \in K$. Choose an orientation $o_{\mathfrak{k}}$ on $\mathfrak{k}$, then the fibration q_K has a unique $U \times K$-invariant orientation o_K given by $o_{e,m} = o_{\mathfrak{k}}$ for each point $m \in M$, where e is the identity of U. Thus, under these conditions, there exists a map

$$m_{o_K} : H_U^{-\infty}(U \times_K M) \to H_{U \times K}^{-\infty}(U \times M).$$

Given orientations $o_{\mathfrak{u}}, o_{\mathfrak{k}}$ of $\mathfrak{u}, \mathfrak{k}$ respectively, they determine an orientation o on $\mathfrak{u}/\mathfrak{k}$. We have $\det_{\mathfrak{u}/\mathfrak{k}} k > 0$ for all $k \in K$, as both numbers $\det_{\mathfrak{u}} k$ and $\det_{\mathfrak{k}} k$ are > 0, by assumption. Recall the map

$$Ind_{U/K,o} : H_K^{-\infty}(M) \to H_U^{-\infty}(U \times_K M)$$

from Section 5, Proposition 50

Lemma 87 *Let U be a Lie group and let K be a closed subgroup of U, such that the principal K-bundle $U \to U/K$ admits a U-invariant connection. Assume $\det_{\mathfrak{k}} k > 0$, $\det_{\mathfrak{u}} k > 0$ for all $k \in K$. Let $o_{\mathfrak{u}}, o_{\mathfrak{k}}, o$ be compatible orientations on $\mathfrak{u}, \mathfrak{k}, \mathfrak{u}/\mathfrak{k}$, then*

$$m_{o_U} = m_{o_K} Ind_{U/K,o}.$$

Proof: First, we explicitly compute the map m_{o_U}. Consider the canonical connection ω_U. Its curvature is 0. The equivariant curvature of ω_U is given, for the identity element e of U, $m \in M$, $X \in \mathfrak{u}, Y \in \mathfrak{k}$, by

$$\Omega_U(X, Y)_{(e,m)} = -X + Y.$$

Let $\ell = \dim U$. Let $\nu'_U \in \Lambda^\ell \mathfrak{u}'$ be a positive element (with respect to the orientation $o_{\mathfrak{u}}$). Let dx be the unique left U-invariant form on U, such that

$(dx)_e = \nu'$. Then ν'_U determines a δ-function $|\nu'_U|^{-1}\delta_{\mathfrak{u}}$ on $\mathfrak{u}$, and for $X \in \mathfrak{u}$, $Y \in \mathfrak{k}$,

$$\gamma_{\omega_U,o_U}(X,Y)_{(e,m)} = |\nu'_U|^{-1}\delta_{\mathfrak{u}}(Y-X)(dx)_e.$$

Thus, for $\alpha \in \mathcal{A}_K^{-\infty}(M)$,

$$(m_{\omega_U,o_U}\alpha)(X,Y)_{(e,m)} = |\nu'_U|^{-1}\delta_{\mathfrak{u}}(Y-X)(\alpha(Y)\wedge dx)_{(e,m)}.$$

More explicitly, for Φ_1 a test function on $\mathfrak{u}$, Φ_2 a test function on $\mathfrak{k}$,

$$(\int_{\mathfrak{u}\times\mathfrak{k}}(m_{\omega_U,o_U}\alpha)(X,Y)\Phi_1(X)\Phi_2(Y)dXdY)_{(e,m)} =$$
$$((\int_{\mathfrak{k}}\alpha(Y)\Phi_1(Y)\Phi_2(Y)dY)\wedge dx)_{(e,m)}$$

where dX is the Euclidean density on $\mathfrak{u}$ determined by ν'.

Now let us analyse the action of K on $U\times M$. Let $k = \dim K$, $n = \dim(U/K)$. Let E^i be an oriented basis of $\mathfrak{k}$ with dual basis $E_i \in \mathfrak{k}'$. Let $\nu'_K = E_1\wedge\cdots\wedge E_k$. The connection ω determines a K-invariant decomposition $\mathfrak{u} = \mathfrak{k}\oplus\mathfrak{r}$. Thus E_i can be thought of as an element of $\mathfrak{u}'$ vanishing on $\mathfrak{r}$. If $\omega = \sum_i \omega_i E^i$, the form ω_i is the unique left U-invariant 1-form on U such that $(\omega_i)_e = E_i$. Let $pr_{\mathfrak{k}}$ (resp. $\mathrm{pr}_{\mathfrak{r}}$) be the projection from $\mathfrak{u}$ to $\mathfrak{k}$ (resp. $\mathfrak{r}$) determined by ω.

Consider the connection $\tilde{\omega} := \omega\otimes 1$ for the principal K-bundle $q_K : U\times M \to U\times_K M$. Let $\Omega \in \mathcal{A}^2(U)\otimes\mathfrak{k}$ be the curvature of ω. The equivariant curvature of $\tilde{\omega}$ at the point $(e,m)\in U\times M$ is given, for $X\in\mathfrak{u}, Y\in\mathfrak{k}$ by

$$\Omega(X,Y)_{(e,m)} = (pr_{\mathfrak{k}}X - Y) + \Omega_e.$$

Thus the element $\gamma_{\tilde{\omega},o_K}$ is given by

$$(\gamma_{\tilde{\omega},o_K})(X,Y)_{(e,m)} = |\nu'_K|^{-1}\delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X - Y)+\Omega_e)(\omega_1\wedge\ldots\wedge\omega_k)_e.$$

Let $\mu' \in \Lambda^n\mathfrak{r}'$ be such that $\mu'\wedge\nu'_K = \nu'_U$. Let dr be the unique left U-invariant n-form on U such that $(dr)_e = \mu'$. The element μ' determines also a δ-function $|\mu'|^{-1}\delta_{\mathfrak{r}}$ on the vector space $\mathfrak{r}$. Let $\alpha\in\mathcal{A}_K^{-\infty}(M)$. By definition, for $X\in\mathfrak{u}$, $(Ind_{U/K,o}\alpha)(X)_{(e,m)}$ is the projection (on the horizontal elements for the diagonal K-action) of $|\mu'|^{-1}\delta_{\mathfrak{r}}(pr_{\mathfrak{r}}X)(\alpha(pr_{\mathfrak{k}}X)\wedge dr)_{(e,m)}$. As $\gamma_{\tilde{\omega},o_K}$ is already of top vertical dimension in the direction K, we have

$$((m_{\tilde{\omega},o_K}Ind_{U/K,o}\alpha)(X,Y))_{(e,m)} =$$
$$|\mu'|^{-1}\delta_{\mathfrak{r}}(pr_{\mathfrak{r}}X)(\alpha(\mathrm{pr}_{\mathfrak{k}}X)\wedge dr\wedge|\nu'_K|^{-1}\delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X-Y)+\Omega)\wedge\omega_1\wedge\ldots\wedge\omega_k)_{(e,m)}.$$

Now $dr \wedge \omega_1 \ldots \wedge \omega_k$ is the form dx of top degree on U and Ω is a form on U, so that $dx \wedge \delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X - Y) + \Omega) = dx \wedge \delta_{\mathfrak{k}}(pr_{\mathfrak{k}}X - Y)$. Thus, if Φ_1 is a test function on $\mathfrak{u}$ and Φ_2 a test function on $\mathfrak{k}$, we obtain

$$(\int_{\mathfrak{u}\times\mathfrak{k}}(m_{\tilde{\omega},o_K} Ind_{U/K,o}\alpha)(X,Y)\Phi_1(X)\Phi_2(Y)dXdY)_{e,m} =$$

$$((\int_{\mathfrak{k}}\alpha(Y)\Phi_1(Y)\Phi_2(Y)dY) \wedge dx)_{(e,m)}.$$

Comparing with the preceding calculation, we obtain the equality, for $X \in \mathfrak{u}, Y \in \mathfrak{k}, m \in M$,

$$(m_{\omega_U,o_U}\alpha)(X,Y)_{(e,m)} = (m_{\tilde{\omega},o_K} Ind_{U/K,o}\alpha)(X,Y)_{(e,m)}.$$

By U-invariance we obtain the equality at each point $(x,m) \in U \times M$ and the lemma is proved. ∎

Proposition 88 *Let U be a Lie group. Let K be a compact subgroup of U such that* $\det_{\mathfrak{u}} k = 1$, $\det_{\mathfrak{k}} k = 1$ *for all $k \in K$. Then for any K-manifold M, the maps*

$$m_{o_U} : H_K^{-\infty}(M) \to H_{U\times K}^{-\infty}(U \times M)$$

and

$$m_{o_K} : H_U^{-\infty}(U \times_K M) \to H_{U\times K}^{-\infty}(U \times M),$$

(defined earlier) are both isomorphisms.

Proof: Let $G = U \times K$. The space $P = U \times M$ is also the induced space $G \times_{\Delta(K)} M$, where K is embedded in $G = U \times K$ by the diagonal map Δ. It is easy to see from the explicit calculation above that the map m_{o_U} coincides with the map $Ind_{G/K,o}$. As K is compact, Theorem 52 of section 5 implies that m_{o_U} is an isomorphism. As $Ind_{U/K,o}$ is also an isomorphism, Lemma 87 gives us the proposition. ∎

Let us return to the general situation (S) of a right G-manifold P, with principal action of a normal subgroup N of G, satisfying Conditions (77) and (78). Then the map $m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P)$ is defined. Although it would be desirable to know that m_o is always an isomorphism, we are able to prove it only under additional hypotheses.

First consider the case where $G = N$. Thus $H_{G/N}^{-\infty}(P/N)$ is simply equal to $H(P/G) = H(B)$. As $P \to B = P/G$ is a principal bundle with group G, we can find a (G-invariant) connection ω for the bundle $P \to B$. We assume that this fibration has G-oriented-fibers. Thus we can construct a canonical element (up to the G-orientation o) $\gamma_o \in H_G^{-\infty}(P)$ and the map m_o.

Let Ω be the curvature of the connection ω for $P \to P/G$. If $\phi(X) \in C^\infty(\mathfrak{g})^G$, then $\phi(\Omega) \in H(B)$ and is independent of the choice of ω. We thus define a structure of $C^\infty(\mathfrak{g})^G$-module on $H(B)$ via the Chern -Weil homomorphism: $\phi \cdot \beta = \phi(\Omega) \wedge \beta$, for $\phi \in C^\infty(\mathfrak{g})^G$ and $\beta \in H(B)$.

Theorem 89 *Let G be a Lie group acting principally (from the right) on a manifold P. Assume further that the quotient map $q : P \to P/G$ has G-oriented fibers under an orientation o. Then the map*

$$m_o : H(P/G) \to H_G^{-\infty}(P)$$

given by $m_o(\alpha) = q^*\alpha \wedge \gamma_o$ is an isomorphism of $C^\infty(\mathfrak{g})^G$-modules.

Proof: The fact that m_o is a morphism of $C^\infty(\mathfrak{g})^G$-modules, follows readily from Proposition 84.

It remains to see that m_o is an isomorphism of vector spaces:

If $P = G \times B$ is the direct product of G and B, with the action of an element $g_0 \in G$ given by $(g, m) \cdot g_0 = (g_0^{-1}g, m)$, for $g \in G, m \in B$, then by Proposition 88 (for $U = G$, $K = e$, $M = B$), the equivariant cohomology $H_G^{-\infty}(P)$ is isomorphic with $H(B)$ under the map m_o. Thus our theorem is true when the fibration $P \to B$ is trivial. (Remark: a trivial principal G-bundle is usually trivialized as $G \times B$, where the action of G is on the right $(g, m) \cdot g_0 = (gg_0, m)$. We can use the isomorphism $(g, m) \to (g^{-1}, m)$ to change this usual trivialization to the trivialization used above.)

Let us now return to the general situation . Choose a (G-invariant) connection form ω for the principal G-bundle q. Consider the element $\gamma_o \in H_G^{-\infty}(P)$ given in Proposition 79 (with respect to the given G-orientation o). Let U be an open subset of B. Denote by γ_U the restriction of γ_o to $q^{-1}(U)$. We denote by $m_U : H(U) \to H_G^{-\infty}(q^{-1}(U))$ the map m_o restricted to $q^{-1}(U)$: $m_U(\alpha) = q^*\alpha \wedge \gamma_U$.

Lemma 90 *Let U and V be two open subsets of B. Assume that the maps m_U, m_V, $m_{U\cap V}$ are isomorphisms, then $m_{U\cup V}$ is an isomorphism.*

Proof: This lemma is proved by a standard Mayer-Vietoris argument: Both the sequences

$$0 \to \mathcal{A}(U \cup V)) \to \mathcal{A}(U) \oplus \mathcal{A}(V) \to \mathcal{A}(U \cap V) \to 0,$$

$$\begin{aligned} 0 \to \mathcal{A}_G^{-\infty}(q^{-1}(U \cup V)) \to \mathcal{A}_G^{-\infty}(q^{-1}(U)) \oplus \mathcal{A}_G^{-\infty}(q^{-1}(V)) \\ \to \mathcal{A}_G^{-\infty}(q^{-1}(U \cap V)) \to 0 \end{aligned}$$

are exact. The surjectivity of the last map can be seen as follows: Choose a partition of unity f_U, f_V for $U \cup V$ subordinate to U, V. Then the functions $q^* f_U, q^* f_V$ are G-invariant functions. If $\beta \in \mathcal{A}_G^{-\infty}(q^{-1}(U \cap V))$, then β is the image of $(-(q^* f_V)\beta, (q^* f_U)\beta) \in \mathcal{A}_G^{-\infty}(q^{-1}(U)) \oplus \mathcal{A}_G^{-\infty}(q^{-1}(V)))$.

Thus the above Mayer-Vietoris sequences induce the long exact sequences in cohomology. The lemma follows from the five lemma. ∎

Now Theorem 89 follows by recalling that there is a finite open cover U_i of the base B such that the bundle $q^{-1}(U_i) \to U_i$ is trivial. ∎

We now return to the general situation (S) . Assume now that G is compact. Then the condition (78) on the existence of a G-invariant connection for the map $q : P \to P/N$ is always satisfied. We prove the following theorem

Theorem 91 *Let G be a compact Lie group and let P be a right G-manifold. G. Let N be a closed normal subgroup of G acting freely on P. We assume furthermore that the fibers of $q : P \to P/N$ admit a G-orientation o. Then the map*

$$m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P)$$

is an isomorphism.

Proof: We choose a G-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$ and use notation of the proof of Proposition 79. Let $\Gamma \subset \mathcal{A}(P)$ be the space of horizontal forms. The action of G on $\mathcal{A}(P)$ preserves the subspace Γ. Let us consider the algebra homomorphism $C : \Lambda\mathfrak{n}' \to \mathcal{A}(P)$, determined by sending $E_j \in \mathfrak{n}'$ to ω_j. We still denote by

$$C : \Gamma \otimes \Lambda\mathfrak{n}' \to \mathcal{A}(P)$$

the map given by $C(\alpha \otimes \xi) = \alpha \wedge C(\xi)$, for $\alpha \in \Gamma, \xi \in \Lambda\mathfrak{n}'$. The map C is an isomorphism. Furthermore C commutes with the action of G.

Recall the isomorphism (11) $U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$. Let us explicitly write U, using coordinates. Let $n = \dim \mathfrak{n}$. Choose a basis Q^a, $n < a \leq \dim \mathfrak{g}$, of $\mathfrak{g}/\mathfrak{n}$. We choose a basis G^j of $\mathfrak{g}$ such that the first n-vectors are the vectors E^j and the last ones are representatives of Q^a. Let $Q = \sum_{i>n} x_i Q^i$ be an element of $\mathfrak{g}/\mathfrak{n}$. Define, for $p \in P$,

$$k_j(p, Q) = (\omega_p(\sum_{i>n} x_i G^i)_P, E_j).$$

Then $Q \mapsto k_j(p, Q)$ is a linear function in Q varying smoothly in p. Let $X = \sum_i x_i G^i$. In these coordinates $x = (x_i)$, we have

$$U(p, X) = (p, Y(x), Q(x))$$

with
$$Q(x) = \sum_{i>n} x_i Q^i, Y(x) = \sum_{j=1}^{n} (x_j + k_j(p, Q(x)) E^j.$$

We denote by U^* the isomorphism
$$U^* : C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P)) \to C^{-\infty}(\mathfrak{g}, C^\infty(P))$$
given by $(U^*s)(X,p) = s(U(p,X))$, for $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$.

Formula above shows that U^*s is indeed smooth in p and generalized in X.

Let
$$A = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda \mathfrak{n}'.$$

With the help of U and C, we can define an isomorphism
$$T : A \to C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$$
by the following formula: For $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$, $\alpha \in \Gamma \otimes \Lambda \mathfrak{n}'$,
$$T(s\alpha) = (U^*s)C(\alpha).$$

The group G acts on A by the action induced by the adjoint representation of G on $\mathfrak{n}, \mathfrak{g}/\mathfrak{n}$, and its natural action on Γ. We denote by A_G the space of G-invariants in A. Then T commutes with the action of G and induces an isomorphism still denoted by T between A_G and $\mathcal{A}_G^{-\infty}(P)$.

Consider the $\mathbb{Z}_+$-gradation on A
$$A^* = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda^* \mathfrak{n}'.$$

We still denote by $d_\mathfrak{g}$ the operator on the space A obtained from the operator $d_\mathfrak{g}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ under the isomorphism T.

We write $Y \in \mathfrak{n}$ as $Y = \sum y_j E^j$. Let $\iota(E^j)$ be the contraction on $\Lambda \mathfrak{n}'$ by the vector E^j. Let $j_\mathfrak{n}$ be the operator of degree -1 on A given by
$$j_\mathfrak{n} = \sum_j y_j \iota(E^j).$$

The components Ω_j of the curvature Ω are horizontal forms. Thus exterior multiplication by Ω_j is an operator on Γ. We can consider the operator f acting on $C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda \mathfrak{n}'$ given by
$$f = \sum_j \Omega_j \otimes \iota(E^j).$$

The operator f is homogeneous of degree -1.

Let us write, using the $\mathbb{Z}_+$-grading of A, the operator $d_\mathfrak{g}$ on A as a sum of homogeneous operators d_i of degree i.

Lemma 92 *We have*

$$d_{\mathfrak{g}} = d_{-1} + d_0 + d_1$$

with $d_{-1} = -j_{\mathfrak{n}} + f$.

Proof: Let Q^a be a basis of $\mathfrak{g}/\mathfrak{n}$. Let $p \in P$. At the point $p \in P$, consider the decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_p$$

given in (40). We write $Q_p^a \in \mathfrak{q}_p$ for the unique element of $\mathfrak{q}_p$ above $Q^a \in \mathfrak{g}/\mathfrak{n}$. We have $\omega_p(Q_P^a) = 0$. The contraction by Q_P^a produces an operator ι^a on the space of horizontal forms Γ. Let $Q = \sum_a q_a Q^a \in \mathfrak{g}/\mathfrak{n}$. The coordinate function q_a acts on $C^{-\infty}(\mathfrak{g}/\mathfrak{n})$ by multiplication. The operator $\iota_{\mathfrak{g}/\mathfrak{n}} = \sum q_a \iota^a$ is an operator of degree 0 on $C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda\mathfrak{n}'$.

It is easy to see that the operator $\iota = \iota_{\mathfrak{g}}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ becomes the operator $\iota_{\mathfrak{g}/\mathfrak{n}} \oplus j_{\mathfrak{n}}$ on A under the isomorphism T.

Let us now analyse the differential d_P under the isomorphism T. If $I = \{1 \leq i_1 < i_2 \ldots < i_k \leq n\}$ is an ordered multiindex, we write $\omega_I = \omega_{i_1} \wedge \ldots \wedge \omega_{i_k}$. Let $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P)), \alpha \in \Gamma$. We compute $d_P((U^*s)\alpha \wedge \omega_I)$. As d_P is a derivation, we analyse the exterior differential of each term of this product.

As $d\omega + \frac{1}{2}[\omega, \omega] = \Omega$, where Ω is horizontal, we see that the differential $d\omega_j$ of the component ω_j of the connection ω is the sum of an element of $C(\Lambda^2\mathfrak{n}')$ and of $\Omega_j \in \Gamma$.

The differential d_P does not necessarily keep the space Γ of horizontal forms stable, but $d_P(\alpha) \in \Gamma \oplus \Gamma \otimes \mathfrak{n}'$, for $\alpha \in \Gamma$.

Finally, for $s(Y, Q, p) \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$, we have, with $Y = \sum_j y_j E^j$,

$$T^{-1} d_P T s = d_P s + \sum_j \partial_{y_j} s(Y, Q, p) d_P k_j(p, Q).$$

Combining all these observations, we see that d_P becomes a sum of the homogeneous operators $f_{-1} + f_0 + f_1$ under the isomorphism T. Furthermore the term f_{-1} is the operator f. Hence we obtain the lemma. ∎

We now prove Theorem 91 by an induction argument similar to the argument of the proof of Theorem 39. Actually we will make use of the bigrading

$$A^{k,q} = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma^q) \otimes \Lambda^k\mathfrak{n}',$$

where Γ^q refers to the $\mathbb{Z}^+$-grading on Γ given by the exterior degree.

Each of the spaces $A^{k,q}$ is stable by G, since N is a normal subgroup of G. As the group G is compact, the proof of Proposition 22 and Remark 23 (of section 3) implies that the homology groups of the operator $j_{\mathfrak{n}} : A_G^{*,q} \to A_G^{*-1,q}$ are equal to zero, except in maximal degree $n = \dim \mathfrak{n}$.

Let $\alpha \in \mathcal{A}_G^{-\infty}(P)$ be such that $d_{\mathfrak{g}}\alpha = 0$. We first show that α is homologous to an element β divisible by $\gamma_{\omega,o}$. We work with A_G and write again α for the element $T^{-1}(\alpha) \in A_G$. Let $\alpha = \sum_{k \geq k_0} \alpha_k$ with $\alpha_k \in A_G^k$. From the degree consideration (cf. Lemma 92), we see that $(j_{\mathfrak{n}} - f)\alpha_{k_0} = 0$.

Write now $\alpha_{k_0} = \sum_{q \geq q_0} \alpha_{(k_0,q)}$ with $\alpha_{(k_0,q)} \in A^{k_0,q}$. The operator f sends $A^{k_0,q}$ to $A^{k_0-1,q+2}$. Thus, we see again from degree considerations in q that $j_{\mathfrak{n}}(\alpha_{k_0,q_0}) = 0$. So if $k_0 < n$, there exists $\beta \in A_G^{k_0+1,q_0}$ such that $\alpha_{k_0,q_0} = j_{\mathfrak{n}}\beta$. The element $\alpha + d_{\mathfrak{g}}\beta$ is homologous to α and its term of degree k_0 is in $\sum_{q > q_0} A^{k_0,q}$. (Of course $\alpha + d_{\mathfrak{g}}\beta$ has no term of degree strictly less than k_0.) By successive approximations, we thus see that we can construct a representative of α in A_G^n. Now, let $\alpha \in A_G^n$ be such that $d_{\mathfrak{g}}\alpha = 0$. In particular, $(j_{\mathfrak{n}} - f)(\alpha) = 0$. We can write at the point $p \in P$ $\alpha(Y,Q)_p = \lambda(Y,Q,p)(\omega_1 \wedge \ldots \wedge \omega_n)_p$ where $\lambda(Y,Q,p) \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}) \otimes \Lambda H_p^*$, where H_p is the space of horizontal vectors. Let us write $Y = \sum_j y_j E^j$. For every $j, 1 \leq j \leq n$, the equation $(j_{\mathfrak{n}} - f)\alpha = 0$ implies

$$(y_j - \Omega_j)\lambda(Y,Q,p) = 0.$$

It is not difficult to see (using for example the translation $\lambda(Y,Q,p) \mapsto \lambda(Y+\Omega,Q,p))$ that $\lambda(Y,Q,p) = \delta_{\mathfrak{n}}(-Y+\Omega)\beta(Q,p)$ where $\beta(Q,p)$ is a generalized function on $\mathfrak{g}/\mathfrak{n}$ with values in ΛH_p^*. This way, we construct an element $\beta \in C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma)$. As $\alpha \in A_G$ is G-invariant, $\beta \in (C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma))^G$. As N is normal, the group N acts trivially on $\mathfrak{g}/\mathfrak{n}$. Thus, we see that $\beta \in (C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma^N))^{G/N} = \mathcal{A}_{G/N}^{-\infty}(P/N)$ and $\alpha = q^*\beta \wedge \gamma_{\omega,o}$.

The equation $d_{\mathfrak{g}}\alpha = 0$ and the injectivity of the map $m_{\omega,o}$ at the cochain level (cf. Proof of Proposition 85) implies that $d_{\mathfrak{g}/\mathfrak{n}}\beta = 0$. Thus β is closed and the map m_o is surjective. The injectivity is proved by a similar argument. ∎

Let K and U be compact subgroups of a Lie group L. Then L can be thought of as a $U \times K$-manifold under $x \cdot (u,k) = u^{-1}xk$, for $x \in L$, $u \in U$ and $k \in K$. If M is a K-manifold, we consider the $U \times K$ manifold $P = L \times M$, with twisted action as in Example 76.

Thus, specializing Theorem 91 to this example, we obtain

Proposition 93 *Let K, U be compact subgroups of a Lie group L. Assume that there exists a $U \times K$-invariant orientation o_K for the principal K-bundle $q_K : L \times M \to L \times_K M$. Then the map*

$$m_{o_K} : H_U^{-\infty}(L \times_K M) \to H_{U\times K}^{-\infty}(L \times M)$$

is an isomorphism. In particular, when $M =$ point,

$$H_U^{-\infty}(L/K) \cong H_{U\times K}^{-\infty}(L).$$

In the case of a free action, we have seen that $H_G^{-\infty}(M)$ is isomorphic to $H_G(M)$ under the multiplication by γ_o. However it may happen that the natural inclusion $H_G(M) \to H_G^{-\infty}(M)$ is identically 0. This is for example the case for the action of G on itself, at least when G is compact: The element $1 \in H_G(G) = \mathbb{R}$ has integral zero over G, while the integral of $\gamma_o \in H_G^{-\infty}(G)$ is equal to $vol(G, dg)\delta_{\mathfrak{g}}(X)$, as follows from the explicit formula for γ_o given above.

Assume that M is compact and oriented. Thus $\int_M$ defines a map from $H_G^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$. It is clear from the formula, given in Proposition 80 for the generator γ_o that if $\alpha = \beta \wedge \gamma_o$ with $\beta \in H_G(M)$, then $\int_M \alpha$ is a derivative $P(\partial)\delta_{\mathfrak{g}}$ of the $\delta_{\mathfrak{g}}$- function on $\mathfrak{g}$. Moreover, the order of the derivative is less or equal that of $\dim(B)/2$. We will determine explicitly this map in a special case.

Let K be a compact connected semi-simple Lie group and let T be its maximal torus. Let W be the Weyl group of (K,T). Let $S(\mathfrak{t})^W$ be the subalgebra of W-invariants in $S(\mathfrak{t})$. Let I be the ideal in $S(\mathfrak{t})$ generated by all the invariants of positive degree. Similarly, let $S(\mathfrak{t}')^W$ be the subalgebra of W-invariants in $S(\mathfrak{t}')^W$ and let J be the ideal in $S(\mathfrak{t}')$ generated by all the invariants of positive degree. Let $\delta_{\mathfrak{t}}$ be the δ function on $\mathfrak{t}$ determined by the Euclidean measure on $\mathfrak{t}$ associated to the Killing form.

If $f \in C^{-\infty}(\mathfrak{t})$ and $Q \in S(\mathfrak{t})$, then the derivative $Q(\partial)f$ of f by the constant coefficient differential operator $Q(\partial)$ is well defined.

Similarly, if $P \in S(\mathfrak{t})$ is a polynomial function on $\mathfrak{t}'$ and $Q \in S(\mathfrak{t}')$, we can define $Q(\delta)P$. An element $P \in S(\mathfrak{t})$ is called *harmonic*, if $Q(\delta)P = 0$, for all $Q \in J$. We denote by $\mathcal{H}$ the set of harmonic elements of $S(\mathfrak{t})$.

Lemma 94 *Let*

$$\mathcal{J} := \{f \in C^{-\infty}(\mathfrak{t}), Pf = 0, \text{for all}\, P \in J\}$$

be the set of generalized functions on $\mathfrak{t}$ annihilated by all the W-invariant functions P without constant terms under multiplications. Then $\mathcal{J}$ is equal to

$$\mathcal{J} = \{Q(\partial)\delta_{\mathfrak{t}}; Q \in \mathcal{H}\}.$$

Proof: Choose a W-invariant norm $|x|$ on $\mathfrak{t}$. If $f \in \mathcal{J}$, then f is annihilated by the invariant polynomial function $|x|^2$. Hence f is supported at the origin and there exists a $Q \in S(\mathfrak{t})$ such that $f = Q(\partial) \cdot \delta_{\mathfrak{t}}$. The equation $J \cdot Q(\partial) \cdot \delta_{\mathfrak{t}} = 0$ implies, by Fourier transform, that Q is harmonic. ∎

Let $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{r}$ be the T-invariant decomposition of $\mathfrak{k}$, and let $n = \dim \mathfrak{r}$. Choose compatible orientations $o_{\mathfrak{k}}, o_{\mathfrak{r}}, o_{\mathfrak{t}}$ on $\mathfrak{k}, \mathfrak{r}, \mathfrak{t}$. Let κ', μ', ν' be the forms of maximal degree on $\mathfrak{k}, \mathfrak{r}, \mathfrak{t}$ respectively, associated to the Killing form $(,)$ and our choice of orientations . We denote also by κ' the left K-invariant form on K

coinciding with κ' at the identity e of K. Similarly, we extend ν' (resp. μ') as a left K-invariant $\dim\mathfrak{t}$-form (resp. $\dim\mathfrak{r}$-form) on K coinciding with ν' (resp μ') at e.

Let $\lambda \in \mathfrak{t}'$. The bilinear form on $\mathfrak{r}$ given by $B_\lambda(X,Y) = (\lambda, [X,Y])$ is an element of $\Lambda^2\mathfrak{r}'$. Let $\Delta = \{\alpha \in i\mathfrak{t}'\}$ be the set of roots of $(\mathfrak{k}, \mathfrak{t})$. Choose an order on Δ compatible with the orientation $o_\mathfrak{r}$, as in ([12], page 40). Let U be the polynomial function $U(\lambda) = \prod_{\alpha>0}(\lambda, i\alpha)$. Then $U \in \mathcal{H}$ and the map $P \in S(\mathfrak{t}') \mapsto P(\partial)U$ induces an isomorphism from $S(\mathfrak{t}')/J$ to $\mathcal{H}$. Furthermore it is easy to see that $B_\lambda^{n/2} = ((n/2)!)U(\lambda)\mu'$.

Consider the free action of T on $M = K$ by $k \cdot t = kt$. The space $H_T(K)$ is isomorphic to $H(K/T)$. The Chern-Weil map $W : S(\mathfrak{t}') \to H(K/T)$ is surjective, with kernel J. Thus we identify $H_T(K)$ with the $S(\mathfrak{t}')$-module $S(\mathfrak{t}')/J$.

Proposition 95 *The map*

$$\int_K : H_T^{-\infty}(K) \to C^{-\infty}(\mathfrak{t})$$

is an isomorphism from $H_T^{-\infty}(K)$ to $\mathcal{J}$. Furthermore, we have

$$\int_K \gamma_o = (-1)^{n/2} vol(K) U(\partial) \cdot \delta_\mathfrak{t},$$

where $n = \dim K$.

Proof: Consider the curvature Ω of $K \to K/T$, determined by the T-invariant decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{r}$. It is an element of $\Lambda^2\mathfrak{r}' \otimes \mathfrak{t}$. Let us compute $\exp\Omega$ in the algebra $\Lambda\mathfrak{r}' \otimes S(\mathfrak{t})$. The component of $\exp\Omega$ of exterior degree n is given by the formula $(\exp\Omega)_{[n]} = \mu' \otimes U$. The term of exterior degree n of $\delta_\mathfrak{t}(\Omega - X)$ is thus equal to $(-1)^{n/2}\mu' \otimes U(\partial)\delta_\mathfrak{t}$. Formula for $\gamma_o \in \mathcal{A}_T^{-\infty}(K)$ given in Proposition 80 shows that the term of maximal exterior degree of γ_o is

$$(\gamma_o)_{[\dim K]}(X) = (-1)^{n/2}(U(\partial) \cdot \delta_\mathfrak{t})(X)\kappa'.$$

Integrating over K, we obtain the formula for $\int_K \gamma_o$ given in the proposition.

As $H_T(K)$ is generated by 1 over $S(\mathfrak{t}')$, we obtain the equality $H_T^{-\infty}(K) = (S(\mathfrak{t}')/J)\gamma_o$. Furthermore, as seen by Fourier transform, the map $P \mapsto P(U(\partial) \cdot \delta_\mathfrak{t})$ induces an isomorphism from $S(\mathfrak{t}')/J$ to $\mathcal{J}$ and we obtain our proposition. ∎

Corollary 96 *The natural map $H_T(K) \to H_T^{-\infty}(K)$ is identically* 0.

Proof: Elements of $H_T(K) = H(K/T)$ come from the base, thus have integral zero on K. ∎

Remark 97 *If a torus acts on a compact oriented manifold M without fixed points, every (equivariant) cohomology class in $H_T(M)$ is of integral equal to zero, as follows from the localization formula (see [3], chap 7). The preceding example (i.e T acting on K by right translations) gives a striking case of an action of T without fixed points, where any non-zero equivariant cohomology class with generalized coefficients has a non-zero integral.*

10 A spectral sequence for T-equivariant cohomology

Let K be a compact connected Lie group and M a K- manifold. Let T be a maximal torus of K. In section 8, we have seen that the K-equivariant cohomology $H_K^{-\infty}(M)$ of M can be computed in terms of the T-equivariant cohomology of M. In this section, we will establish a spectral sequence relating the $S(\mathfrak{t}')$- modules $H_T(M)$ and $H_T^{-\infty}(M)$.

Let T be an abelian Lie group (not necessarily compact) and let M be a T-manifold Let $\mathfrak{t}$ be the Lie algebra of T. Then, as T is abelian,

$$\mathcal{A}_T(M) = S(\mathfrak{t}') \otimes \mathcal{A}(M)^T.$$

Similarly

$$\mathcal{A}_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T).s$$

We can then consider $\mathcal{A}_T^{-\infty}(M)$ as obtained from the space $\mathcal{A}_T(M)$ by "extension" of coefficients.

Let us consider the space:

$$\Omega = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'$$

$\mathbb{Z}$-graded by its exterior degree with respect to $\Lambda\mathfrak{t}'$, i.e.

$$\Omega^p = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}') \otimes \Lambda^p\mathfrak{t}'.$$

Let E^i be a basis of $\mathfrak{t}$ with dual basis E_i of $\mathfrak{t}'$. An element $X \in \mathfrak{t}$ is written as $X = \sum_i x_i E^i$. We can consider an element of

$$V = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}')$$

as a form $\alpha(X, Y) \in \mathcal{A}(M)^T$ depending in a generalized way on the first variable $X \in \mathfrak{t}$ and in a polynomial way on the second variable $Y \in \mathfrak{t}$.

We consider on V the $\mathbb{Z}/2$-grading given by the parity of an element in $\mathcal{A}(M)^T$. Consider the $S(\mathfrak{t}')$-module structure on the space V defined by $(E_i \cdot$

$\alpha)(X,Y) = (x_i - y_i)\alpha(X,Y)$, i.e. $E_i(\theta \otimes P) = x_i\theta \otimes P - \theta \otimes y_i P$, for $\theta \in C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$ and $P \in S(\mathfrak{t}')$.

Let j be the Koszul differential of degree -1 on $\Omega = V \otimes \Lambda\mathfrak{t}'$, got from the $S(\mathfrak{t}')$-module V (cf. Formula 12 of section 3), i.e.

$$j = \sum_i (x_i - y_i) \otimes \iota_\Lambda(E^i).$$

(As usual, in extending $\iota_\Lambda(E^i)$ to the tensor product of the two superspaces V and $\Lambda\mathfrak{t}'$, we respect the sign rules (2) and (3) of section 1).

If $\alpha(X,Y) = \theta(X) \otimes P(Y) \in V$, the restriction $\alpha(X,X) = \theta(X)P(X)$ of α to the diagonal is well defined. Thus, for any $\beta \in \Omega$, the restriction $\beta(X,X)$ of β to the diagonal is an element of $C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes \Lambda\mathfrak{t}'$. Let us denote by $r(\beta)$ the component of exterior degree zero of $\beta(X,X)$. Thus the map r is a map from Ω to $\mathcal{A}_T^{-\infty}(M)$.

We can also write

$$\Omega = C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) \otimes \Lambda\mathfrak{t}',$$

where by definition

$$C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) = \sum_{p \geq 0} C^{-\infty}(\mathfrak{t}, \mathcal{A}_T^p(M))$$

and $\mathcal{A}_T^*(M)$ refers to the $\mathbb{Z}_+$-grading of $\mathcal{A}_T(M)$ defined in section 2.

We extend pointwise the differential $d_\mathfrak{t}$ of $\mathcal{A}_T(M)$ to $C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M))$ by defining $(\overline{d_\mathfrak{t}} f)(X) = d_\mathfrak{t}(f(X))$. Consider the operator

$$d_0 = \overline{d_\mathfrak{t}} \otimes I$$

of degree 0 (with respect to the $\mathbb{Z}$-grading of $\Lambda\mathfrak{t}'$) on Ω.

The operators j and d_0 satisfy $j^2 = 0$, $d_0^2 = 0$, $jd_0 + d_0 j = 0$.

Consider the $\mathbb{Z}/2$ grading on Ω given by the parity of forms on $\mathcal{A}(M)^T$ together with the $\mathbb{Z}$-grading of $\Lambda\mathfrak{t}'$. Then d_0 and j are odd operators. Define the operator

$$D = j + d_0.$$

The operator D is an odd operator on Ω of square equal to 0.

Let $H(\Omega, D)$ be the cohomology space of D. It is a $\mathbb{Z}/2$-graded space.

Proposition 98 *The map $r : \Omega \to \mathcal{A}_T^{-\infty}(M)$ satisfies $rD = d_\mathfrak{t} r$. Moreover r induces an isomorphism in cohomology. Thus the cohomology of the complex (Ω, D) is isomorphic to $H_T^{-\infty}(M)$.*

Proof: Since $rj = 0$, and $r(\theta(X) \otimes y_i \iota(E^i_M)P(Y)) = x_i \iota(E^i_M)(\theta(X)P(X))$ if $\theta(X) \in C^{-\infty}(\mathfrak{t})$ and $P(Y) \in \mathcal{A}_T(M)$, the first assertion is immediate,. Let $n = \dim \mathfrak{t}$. As V is a tensor product of the free module $S(\mathfrak{t}')$ by $C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$, the space V is a free $S(\mathfrak{t}')$-module (cf. Corollary 16 of section 3). Thus by Proposition 14 the Koszul complex

$$0 \to \Omega^n \xrightarrow{j} \quad \cdots \xrightarrow{j} \Omega^1 \xrightarrow{j} \Omega^0 \xrightarrow{r} \mathcal{A}_T^{-\infty}(M) \to 0$$

is exact at all the levels Ω^i, for all $i > 0$. Exactness at Ω^0 is easy to check.

Let Ω' be the exact complex for j defined by $\Omega'^i = \Omega^i$ if $i > 0$ and $\Omega'^0 = Kerr$. Choose any homotopy h of Ω' of degree 1 i.e. $hj + jh = I_{\Omega'}$. Consider $N = hd_0 + d_0 h$. Then N is an operator of degree 1 on Ω'. We have $hD + Dh = I + N$ on Ω', and N is a nilpotent operator commuting with D. Let us prove that r is surjective in cohomology: Let $\theta \in \mathcal{A}_T^{-\infty}(M)$ be such that $d_{\mathfrak{t}}\theta = 0$. We lift θ as a form in two variables $\Theta(X, Y) = \theta(X)$ constant in Y, i.e $\Theta = \theta \otimes 1$. Then $r(D\Theta) = 0$ i.e $D\Theta = d_0\Theta \in \Omega'$. Thus $(I + N)D\Theta = (hD + Dh)D\Theta = DhD\Theta$ i.e. $D(\Theta - (I+N)^{-1}hD\Theta) = 0$. The element $w(\Theta) := \Theta - (I+N)^{-1}hD\Theta$ still satisfies $r(w(\Theta)) = \theta$, and is a cocycle for D.

Similarly, we prove that r is injective: Let $\alpha \in \Omega$ be such that $D\alpha = 0$ and $r(\alpha) = d_{\mathfrak{t}}\theta$. Then $\alpha' := \alpha - D\Theta$ satisfies $D\alpha' = 0$ and $r\alpha' = 0$. Then $\alpha' = D(I+N)^{-1}h\alpha'$ is a boundary. This proves the proposition. ∎

We give below a more explicit way to construct a representative in $H(\Omega, D)$ of an element in $H_T^{-\infty}(M)$.

If $Y \in \mathfrak{t}$, we define as in section 4 the tensor product contraction $\iota_t(Y) = \iota(Y_M) + \iota_\Lambda(Y)$ on Ω. The horizontal space Ω_{hor} is then defined as

$$\Omega_{hor} = \{\alpha \in \Omega, \iota_t(Y)\alpha = 0 \,\text{for all}\, \mathrm{Y} \in \mathfrak{t}\}.$$

The space Ω_{hor} is stable by D. There is a canonical projection map (see Definition 28 of section 4) from Ω to Ω_{hor} given by

$$h = \prod_i (I - \epsilon_i \iota_t(E^i))$$

where ϵ_i denotes the multiplication by E_i on $\Lambda\mathfrak{t}'$.

We denote by $w : \mathcal{A}_T^{-\infty}(M) \to \Omega$ the map $w(\theta) = h(\Theta)$, where Θ is the lift of θ constant in Y.

We have

$$w(\theta) = \Theta + (-1)^{|\theta|} \sum_i \iota(E^i_M)\Theta \otimes E_i - \sum_{i<j} \iota(E^i_M)\iota(E^j_M)\Theta \otimes E_i \wedge E_j + \cdots$$

Lemma 99 *The map w satisfies: $wd_{\mathfrak{t}}\theta = Dw\theta$. Further w induces an isomorphism in cohomology, inverse to the map in cohomology induced by r.*

Proof: Both terms of this equation belong to Ω_{hor}. Thus, to prove that they are equal, we need only to compute their terms of zeroth exterior degree. The element $wd_{\mathfrak{t}}\theta$ has zero-exterior degree term equal to the lift of $(d_{\mathfrak{t}}\theta)(X) = d_M\theta(X) - \sum_i x_i\iota(E^i_M)\theta(X)$, constant in Y.

The element $Dw\theta$ has zero-exterior degree term

$$d_M\theta(X) - \sum_i y_i\iota(E^i_M)\theta(X) + \sum_{i,j}(y_i - x_i)\iota_\Lambda(E^i)(\epsilon_j(\iota(E^j_M)\theta(X)))$$

which is equal to

$$d_M\theta(X) - \sum_i x_i\iota(E^i_M)\theta(X).$$

It is clear that $rw = 1$. But since the map r induces an isomorphism in cohomology, we get that w also induces isomorphism in cohomology inverse to that of r. ∎

The complex (Ω, D) admits an increasing filtration $\mathcal{F} = \{\mathcal{F}_x\}_{0\leq p\leq \dim \mathfrak{t}}$ by the exterior degree in $\Lambda\mathfrak{t}'$ i.e . $\mathcal{F}_p = \otimes_{k\leq p}\Omega^k$. This canonically gives rise to a convergent homology spectral sequence E^r converging to $H(\Omega, D)$.

Lemma 100 *Assume that T is compact* abelian *and M is a paracompact T-manifold, such that $H^*_T(M)$ is finite dimensional in each degree. Then*

$$E^1_p = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda^p\mathfrak{t}'.$$

Proof: By definition

$$\begin{aligned}
E^1_p &= H(\mathcal{F}_p/\mathcal{F}_{p-1}, D) \\
&= H(\mathcal{F}_p/\mathcal{F}_{p-1}, d_0) \qquad \text{since}\, j(\mathcal{F}_p) \subset \mathcal{F}_{p-1}, \\
&\cong H(C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) \otimes \Lambda^p\mathfrak{t}', d_0) \\
&\cong H(C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)), \overline{d_{\mathfrak{t}}}) \otimes \Lambda^p\mathfrak{t}'.
\end{aligned}$$

It is easy to see that $Ker(\overline{d}_{\mathfrak{t}}) = C^{-\infty}(\mathfrak{t}, Z_T(M))$ and moreover $Im(\overline{d}_{\mathfrak{t}}) \subset C^{-\infty}(\mathfrak{t}, B_T(M))$. Further, by Theorem 117 (of the Appendix), we get a continuous splitting of the map

$$\mathcal{A}^{n-1}_T(M) \xrightarrow{d_{\mathfrak{t}}} B^n_T(M),$$

and hence $Im(\overline{d}_{\mathfrak{t}}) = C^{-\infty}(\mathfrak{t}, B_T(M))$. Also $H^n_T(M)$ being finite dimensional, the projection $Z^n_T(M) \to H^n_T(M)$ admits a continuous splitting. From this we easily conclude that

$$H(C^{-\infty}(\mathfrak{t}, \mathcal{A}^*_T(M)), \overline{d}_{\mathfrak{t}}) \cong C^{-\infty}(\mathfrak{t}, H^*_T(M)).$$

This proves the lemma. ∎

Remark 101 *If T is compact and $H^*(M)$ is finite dimensional in each degree then so is $H_T^*(M)$. This follows from the Serre spectral sequence for the fibration $M \to E(T) \times_T M \to B(T)$. In particular, for compact M, $H_T^*(M)$ is finite dimensional in each degree.*

The differential $d^1 : E_p^1 \to E_{p-1}^1$ of degree -1 induced by $D = d_0 + j$ on $E^1 = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda\mathfrak{t}'$ is the Koszul differential j associated to the canonical $S(\mathfrak{t}')$-module stuctures on $C^{-\infty}(\mathfrak{t})$ and $H_T(M)$. Hence, combining Proposition 98, Lemma 100 and Lemma 17 we obtain the main result of this section.

Theorem 102 *Let T be a compact abelian Lie group and let M be a manifold such that $H_T^*(M)$ is finite dimensional in each degree. Then the cohomology group $H_T^{-\infty}(M)$ has an increasing $\mathbb{Z}_+$-filtration H_p, and a convergent homology spectral sequence with*

$$E_p^2 = Tor_p^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$$

and

$$E_p^\infty = H_p/H_{p-1},$$

where $C^{-\infty}(\mathfrak{t})$ and $H_T(M)$ have their canonical $S(\mathfrak{t}')$-module structures.

This spectral sequence is functorial with respect to the T-equivariant smooth maps. Further the total $\mathbb{Z}/2$-grading given by the standard $\mathbb{Z}_+$ degree on $H_T^*(M)$ together with the p index in Tor is compatible with the $\mathbb{Z}/2$-grading of $H_T^{-\infty}(M)$.

We obtain a number of corollaries:

Corollary 103 *Let M, N be T-manifolds such that $H_T^*(M)$ is finite dimensional in each degree, with a T- equivariant smooth map $f : M \to N$. Assume that the induced map $f^* : H_T^*(N) \to H_T^*(M)$ is an isomorphism in T- equivariant cohomology . Then the induced map*

$$f^* : H_T^{-\infty}(N) \to H_T^{-\infty}(M)$$

is also an isomorphism.

Proof: It follows immediately from the above spectral sequence. ∎

The following corollary was obtained in section 6 for compact T- manifolds as a consequence of Theorem 61 (cf. Corollary 64).

Corollary 104 *For any T-manifold M such that $H_T^*(M)$ is a projective finitely generated $S(\mathfrak{t}')$- module, the canonical map*

$$\beta_{T,M} : C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} H_T(M) \to H_T^{-\infty}(M)$$

is an isomorphism.

Proof: Since $H_T(M)$ is $S(\mathfrak{t}')$-projective, the spectral sequence of Theorem 102 has $E_p^2 = 0$, unless $p = 0$. In particular the spectral sequence degenerates at the E^2-term itself. Also

$$E_0^2 = Tor_0^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \approx C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} H_T(M).$$

This proves the corollary. ∎

Let T be an abelian Lie group and let M be a T-manifold. The T-equivariant de Rham complex with generalized coefficients admits a graded subcomplex obtained by forming the algebraic tensor product

$$\tilde{\mathcal{A}}_T^{-\infty}(M) := C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T.$$

This subcomplex is stable by the action of $S(\mathfrak{t}')$. We denote the cohomology of this subcomplex by $\tilde{H}_T^{-\infty}(M)$. We have the following comparison:

Proposition 105 *Let T be a compact abelian Lie group and let M be a T-manifold such that $H_T^*(M)$ is finite dimensional in each degree. Then the canonical map $\tilde{H}_T^{-\infty}(M) \to H_T^{-\infty}(M)$, induced from the inclusion $\tilde{\mathcal{A}}_T^{-\infty}(M) \to \mathcal{A}_T^{-\infty}(M)$, is an isomorphism.*

Proof: Recall the definition of the complex (Ω, D) and define a subcomplex

$$\tilde{\Omega} := C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T \otimes S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'.$$

The cochain map $r : \Omega \to C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$ restricts to a cochain map (denoted by) $\tilde{r} : \tilde{\Omega} \to C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T$. By the same proof as that of Proposition 98, we can easily see that $\tilde{r}$ induces isomorphism in cohomology.

Thus the augmented complex

$$\tilde{\Omega} \xrightarrow{\tilde{r}} \tilde{\mathcal{A}}_T^{-\infty}(M)$$

maps by the natural inclusion i into the augmented complex

$$\Omega \xrightarrow{r} \mathcal{A}_T^{-\infty}(M).$$

The filtration $\{\mathcal{F}_p\}$ of Ω gives rise to the filtration $\{\tilde{\mathcal{F}}_p := \mathcal{F}_p \cap \tilde{\Omega}\}$ of $\tilde{\Omega}$. In particular, we get the induced map $\tilde{E}_p^r \to E_p^r$, where $\tilde{E}_p^r$ is the spectral sequence corresponding to the filtration $\{\tilde{\mathcal{F}}_p\}$. We have $\tilde{E}_p^1 = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda^p\mathfrak{t}'$. In particular, $\tilde{E}_p^1 \to E_p^1$ is an isomorphism, and hence the inclusion $\tilde{\Omega} \to \Omega$ induces isomorphism in cohomology. But then the map $i : \tilde{\mathcal{A}}_T^{-\infty}(M) \to \mathcal{A}_T^{-\infty}(M)$ also induces an isomorphism in cohomology. ∎

The spectral sequence obtained in Theorem 102 may sometimes be used to determine the torsion groups $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$. For example, if K is a compact connected Lie group with maximal torus T and if M is a K-manifold such that $Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$ is equal to zero except for $i = i_0$ (for some i_0), then from the degenerate spectral sequence of Theorem 102 and Theorem 70,

$$Tor_{i_0}^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{k}')^K} H_K^{-\infty}(M).$$

As we next show, this hypothesis is valid when M is homogeneous under K. Let U be a closed subgroup of K. Let us choose a maximal torus T_U of U and let T be a maximal torus of K containing T_U.

Proposition 106 *Let $M = K/U$, where K is a compact connected Lie group and U a closed subgroup. Let $\chi : U \to \pm 1$ be the character $\chi(u) := \det_{\mathfrak{k}/\mathfrak{u}} u$. Then the group*

$$Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) = 0 \qquad \text{for}\, i \neq d := \dim(T/T_U)$$

and

$$Tor_d^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{k}')^K} C^{-\infty}(\mathfrak{u})^{\chi}.$$

Proof: Let $\mathfrak{t}_U$ be the Lie algebra of T_U. Let $W_U \subset GL(\mathfrak{t}_U)$ be the Weyl group of the pair (U, T_U) (i.e.$W_U = N_U(T_U)/T_U$). If $P \in S(\mathfrak{t}')^W$ is a W-invariant function, its restriction to $\mathfrak{t}_U$ is W_U invariant (to see this, use Chevalley's theorem to conclude that P is the restriction to $\mathfrak{t}$ of a K-invariant polynomial on $\mathfrak{k}$) . From Proposition 68 of section 7, we have $H_T(M) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}'_U)^{W_U}$. Thus

$$Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}'_U)^{W_U}).$$

Consider $N := S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}'_U)$ as a $(S(\mathfrak{t}'), W_U)$-module by the action of $S(\mathfrak{t}')$ on the left and the action of W_U on the right factor. Then $N^{W_U} = S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}'_U)^{W_U})$. The space $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)$ thus carries a canonical structure of $(S(\mathfrak{t}'), W_U)$-module and moreover by the standard averaging process

$$Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)^{W_U}.$$

Thus, to prove the vanishing part, we prove that $Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N) = 0$ except for $i = d$.

Let $\mathfrak{t}_1 = \mathfrak{t}/\mathfrak{t}_U$, so that $\mathfrak{t}'_1 \subset \mathfrak{t}'$. Consider the partial Koszul complex $j_1 = j_{\mathfrak{t}_1} : S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'_1 \to S(t'_U)$. This gives a $S(\mathfrak{t}')$-free resolution of $S(\mathfrak{t}'_U)$. As $S(\mathfrak{t}')$ is free over $S(\mathfrak{t}')^W$, the complex $S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'_1)$ with differential $I \otimes j_1$ gives a $S(\mathfrak{t}')$-free resolution of $S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(t'_U)$. Thus $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)$ is the homology of the complex

$$C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} (S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'_1)) \cong C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'_1)$$

i.e. of the complex
(42)

$$0 \to C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda^d \mathfrak{t}_1') \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda^{d-1} \mathfrak{t}_1')$$
$$\cdots \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \mathfrak{t}_1') \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \to 0.$$

Consider the isomorphism obtained in Proposition 66 of section 6

$$C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \cong C^{-\infty}(\mathfrak{t})$$

induced from the multiplication map. Hence, the map $P_1 \otimes F \otimes P_2 \mapsto P_1 \otimes FP_2$ gives an isomorphism of

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}')$$

with

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t}).$$

Thus we obtain an isomorphism of the complex (42) with the complex

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (C^{-\infty}(\mathfrak{t}) \otimes \Lambda \mathfrak{t}_1')$$

under the differential $I \otimes j_1^{-\infty}$. By Proposition 22 of section 3, the homology of the complex $(C^{-\infty}(\mathfrak{t}) \otimes \Lambda \mathfrak{t}_1', j_1^{-\infty})$ is non-zero only in degree d. As $S(\mathfrak{t}')$ is free over $S(\mathfrak{t}')^W$, we obtain the vanishing part of the proposition. Furthermore, by the remark just before this proposition and Theorem 46 of section 5, we obtain the assertion regarding Tor_d. ∎

11 Localization formula

Let T be a torus, i.e. a compact connected abelian Lie group, acting on a compact oriented manifold M. Let $\alpha \in H_T^{-\infty}(M)$. The integral $\Theta(X) := \int_M \alpha(X)$ of α is a generalized function on $\mathfrak{t}$. When $\alpha \in H_T^\infty(M)$, the localization formula (see [3], chapter 7) gives $\Theta(X)$ in terms of the restriction of α to the fixed submanifold M^T of M. As shown by Proposition 95 of section 9 (where M^T is empty but the map $\int_M$ is not zero), it is not possible to determine $\int_M \alpha$ in terms of $\alpha|M^T$ in the generalized case. The main reason for the difference between the $C^{\pm\infty}$-cases is that the space $H_T^{-\infty}(point) = C^{-\infty}(\mathfrak{t})$ is not torsion free over $S(\mathfrak{t}')$. Indeed, for $\alpha \in H_T^{-\infty}(M)$, we will find a non-zero polynomial $P \in S(\mathfrak{t}')$ and determine $P(X) \int_M \alpha(X)$ in terms of $\alpha|M^T$, as in the C^∞-case.

The localization formula, we are going to give in the generalized case, involves choosing a T-equivariant embedding of M in a real representation space

V of T. This is always possible, see ([9], Chap 6, Theorem 4.1). Let V_0 be the subspace of T-fixed vectors and let

$$V = V_0 \oplus V_1$$

be the T-invariant decomposition. Thus $\det_{V_1}(X)$ is a non zero polynomial on $\mathfrak{t}$. The fixed submanifold $M_0 = M^T$ of M is given by $M \cap V_0$. The space V_1 is even dimensional. Let us choose an orientation on V_1. This orientation determines a polynomial square root of $\det_{V_1}(X)$. Using a T-invariant metric on V, we view the normal bundle $\mathcal{N}$ of M_0 in M as a T-equivariant subbundle of the trivial bundle $M_0 \times V_1$. The bundle $\mathcal{N}$ is T-orientable and is of even rank. Let us denote by $\mathcal{Q}$ the supplementary bundle:

$$M_0 \times V_1 = \mathcal{N} \oplus \mathcal{Q}.$$

The bundle $\mathcal{Q}$ is a T-equivariant bundle over M_0. We choose orientations of $V_1, \mathcal{N}, \mathcal{Q}$ in a compatible way. Let $u_{\mathcal{N}} \in H_{cpt,T}(\mathcal{N})$, $u_{\mathcal{Q}} \in H_{cpt,T}(\mathcal{Q})$ be the T-equivariant Thom classes (see Definition 10 section 2) of $\mathcal{N}$, $\mathcal{Q}$ respectively. Let $\chi(\mathcal{N}) \in H_T(M_0)$ (resp.$\chi(\mathcal{Q})$) be the equivariant Euler class of the bundle $\mathcal{N} \to M_0$ (resp. $\mathcal{Q}$). By definition (we differ here from the definition of [3], chapter 7), the restriction of $u_{\mathcal{N}}$ (resp. $u_{\mathcal{Q}}$) to M_0 via the zero section is equal to $\chi(\mathcal{N})$ (resp. $\chi(\mathcal{Q})$). We have the following equality in $H_T(M_0)$:

$$(43) \qquad (-2\pi)^{-\dim V_1/2} \det{}_{V_1}^{1/2}(X) \cong \chi(\mathcal{N})(X)\chi(\mathcal{Q})(X).$$

Let us fix an orientation of M and consider the compatible orientation of M_0. Following is the localization formula in generalized cohomology.

Theorem 107 *Let T be a torus acting on a compact oriented manifold M. For $\alpha \in H_T^{-\infty}(M)$, we have the equality*

$$(-2\pi)^{-\dim V_1/2} \det{}_{V_1}^{1/2}(X) \int_M \alpha(X) = \int_{M^T} \alpha(X)\chi(\mathcal{Q})(X)$$

as elements of $C^{-\infty}(\mathfrak{t})$.

Proof: The proof is obtained by imitating the proof in the C^∞-case given in [2]. Consider the Thom class $u_1(X) \in H_{cpt,T}(V_1)$ of the T-vector space V_1, thought of as a T-equivariant vector bundle $q : V_1 \to$ *point*. We have

$$(-2\pi)^{-\dim V_1/2} q^*(\det{}_{V_1}^{1/2}(X)) \sim u_1(X)$$

as elements of $H_T(V_1)$. Consider the map $p : M \to V_1$ induced by the projection of $V = V_0 \oplus V_1$ to V_1. Thus

$$(-2\pi)^{-\dim V_1/2} \det{}_{V_1}^{1/2}(X) \int_M \alpha(X) = \int_M \alpha(X) p^* u_1(X).$$

We can take a representative of u_1 (as a cohomology class in $H_{cpt,T}(V_1)$) supported in a sufficiently small neighborhood of 0 in V_1. Thus we may assume that p^*u_1 is compactly supported in a T-stable tubular neighborhood U of $M_0 = p^{-1}(0)$. Let π be the T-equivariant projection of $U \to M_0$. Let i be the inclusion of M_0 in M. We have seen in section 2, Proposition 8, that the restriction $\alpha|U$ of α to U is equivalent to $\pi^*i^*\alpha$ in $H_T^{-\infty}(U)$. As p^*u_1 is compactly supported in U, we obtain

$$\begin{aligned}\int_M \alpha(X)p^*u_1(X) &= \int_U \alpha(X)p^*u_1(X)\\ &= \int_U \pi^*i^*\alpha(X)p^*u_1(X)\\ &= \int_{M_0} \alpha(X)\pi_*p^*u_1(X).\end{aligned}$$

Let $\beta = p^*u_1 \in H_{cpt,T}(U)$. It remains to show that $\pi_*\beta(X) = \chi(\mathcal{Q})(X)$ in $H_T(M_0)$.

The restriction of $\beta(X)$ to M_0 is equal to $(-2\pi)^{-\dim V_1/2}\det{}_{V_1}^{1/2}(X)$. The tubular neighborhood U of M_0 in M is T- equivariantly diffeomorphic to the normal bundle $\mathcal{N} \to M_0$. Let $u_{\mathcal{N}}$ be the equivariant Thom class of $\mathcal{N}$. Then, it is well known (see the proof of Proposition 11 of section 2) that $\beta = (\pi^*\pi_*\beta)u_{\mathcal{N}}$ in $H_{cpt,T}(U)$. By restricting this equality to M_0, we obtain

$$(-2\pi)^{-\dim V_1/2}\det{}_{V_1}^{1/2}(X) \cong \pi_*\beta(X)\chi(\mathcal{N})(X)$$

in $H_T(M_0)$. As $\det_{V_1}(X)$ is a non zero polynomial, $\chi(\mathcal{N})(X)$ is invertible on the open set $\det_{V_1}(X) \neq 0$. By Formula (43), we obtain the equality $\pi_*\beta(X) = \chi(\mathcal{Q})(X)$. This proves the theorem. ∎

Let us illustrate the localization formula in the simple example of $M = P_1(\mathbb{C})$.

Let p_1 be the point at infinity of M. Then $U = M - \{p_1\}$ is isomorphic to $\mathbb{C}$. We consider the action of $T = \{e^{i\theta}\}$ on $P_1(\mathbb{C})$ given by $z \mapsto e^{i\theta}z$. This action has two fixed points $p_0 = 0$ and $p_1 = \infty$. We write still p_0, p_1 for the injections of p_0 and p_1 in M.

We write an element of $\mathfrak{t}$ as $X = \theta J$, with $\exp 2\pi J = 1$. Let us first describe the T-equivariant cohomology of M. It is a free $S(\mathfrak{t}')$-module with two generators α, β. We can normalize these two generators, by requiring

$$p_0^*(\alpha) = 1, \qquad \int_M \alpha = 0$$

while

$$p_0^*(\beta) = 0, \qquad \int_M \beta = 1.$$

Identifying $X = \theta J$ with θ, some specific representatives of α and β are

$$\alpha = 1$$

$$\beta(\theta) = (2\pi)^{-1}(\theta|z|^2(1+|z|^2)^{-1} + i(1+|z|^2)^{-2}dz \wedge d\bar{z}).$$

The restriction maps $p_0^*, p_1^* : H_T(M) \to S(\mathfrak{t}')$ satisfy

$$2\pi(p_1^* - p_0^*) = \theta \int_M .$$

Consider now $H_T^{-\infty}(M)$. Let $\delta(\theta)$ be the δ function at 0. The element

$$v(\theta) = \delta(\theta)\beta(\theta)$$

is in $H_T^{-\infty}(M)$.

As $\theta\delta(\theta) = 0$, we have

$$v(\theta) = (-2i\pi)^{-1}\delta(\theta)(1+|z|^2)^{-2}dz \wedge d\bar{z}.$$

Thus, the element v does not have component in zero exterior degree, in particular its restriction to $M^T = \{p_0\} \cup \{p_1\}$ is zero. The integral $\int_M v(\theta)$ is equal to $\delta(\theta)$ and is supported at 0. This is compatible with the localization theorem which asserts that $\theta \int_M v(\theta) = 0$.

Let $P = p_0^* \oplus p_1^*$ be the map:

$$P : H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t}) \oplus C^{-\infty}(\mathfrak{t}).$$

Thus v is in the kernel of P.

In fact, we have the exact sequence

$$0 \to \mathbb{C}v \to H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t}) \oplus C^{-\infty}(\mathfrak{t}) \to 0.$$

The exactness of this sequence can be seen as follows: As $H_T(M)$ is free over $S(\mathfrak{t}')$, we have

$$H_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t})\alpha + C^{-\infty}(\mathfrak{t})\beta.$$

Writing $\nu = f\alpha + g\beta$, we see that if $P\nu = 0$, then $f = 0$ and $\theta g = 0$. Thus ν is proportional to v. Let us see that P is surjective. The restriction maps $p_0^*, p_1^* : H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t})$ still satisfy

$$(2\pi)(p_1^* - p_0^*) = \theta \int_M .$$

Thus we have $p_0^*\nu = f$ and $(2\pi)p_1^*\nu = \theta g + (2\pi)f$. As it is always possible to divide by θ in the space $C^{-\infty}(\mathfrak{t})$, we see that $P = p_0^* \oplus p_1^*$ is surjective.

12 Appendix – A splitting for $d_{\mathfrak{g}}$

Recall from section 1 that, for a manifold M, $\mathcal{A}^*(M)$ denotes the $\mathbb{Z}_+$-graded space of smooth forms on M and $d = d_M$ denotes the exterior derivative. Let $B^*(M)$ (resp. $Z^*(M)$) denote the space of exact (resp. closed) forms. As in the earlier part of this article, we assume M to be paracompact and equip $\mathcal{A}(M)$ with the C^∞-topology and $B(M), Z(M)$ are given the subspace topology. We make the following

Definition 108 *The exterior derivative d is said to admit a* continuous splitting *on M, if there exists a continuous graded linear map $s : B^*(M) \to \mathcal{A}^{*-1}(M)$ (of degree -1) such that $d \circ s = I$.*

Observe that, by virtue of Hodge theorem, d admits a continuous splitting on any compact manifold M. In fact, we will prove in an elementary way the following

Theorem 109 *Let M be any (paracompact) manifold. Then d admits a continuous splitting on M.*

Proof: The proof of the theorem will be broken into the following lemmas.

Lemma 110 *For any manifold M, $Z(M)$ and $B(M)$ are closed subspaces of $\mathcal{A}(M)$.*

Proof: Being the kernel of d, $Z(M) \subset \mathcal{A}(M)$ is clearly a closed subspace. By Poincaré duality, we have

$$B(M) = \{\alpha \in Z(M); \int_M \alpha\gamma = 0\}$$

for all $\gamma \in Z_{cpt}(M)_t$. Thus $B(M)$ is a closed subspace of $\mathcal{A}(M)$. ∎

Lemma 111 *Let M be any contractible manifold. Then d admits a continuous splitting on M.*

Proof: Choose a C^∞-contraction $\phi : \mathbb{R} \times M \to M$, i.e. $\phi|_{\{0\}\times M} = I_M$ and $\phi|_{\{1\}\times M} = m_0$, for some fixed point $m_0 \in M$. Define the map $H : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ by

$$H(\omega) = \int_0^1 \phi^*\omega$$

for $\omega \in \mathcal{A}^*(M)$. Then H is a homotopy operator, i.e.

$$dH\omega + Hd\omega = \omega - \phi_1^*\omega$$

for $\omega \in \mathcal{A}^*(M)$, where $\phi_1 : M \to M$ is defined by $\phi_1 := \phi|_{\{1\}\times M}$. Now define $s : B^*(M) \to \mathcal{A}^{*-1}(M)$, by $s = H|_{B^*(M)}$. Then s gives a splitting for d. ∎

Lemma 112 *Let M be a manifold, and $W \subset U$ be two open subsets satisfying $\bar{W} \subset U$, where $\bar{W}$ is the closure of W. Let $[\omega] \in H(M)$ be a cohomology class such that $[\omega]_{|U} = 0$, as an element of $H(U)$. Then there exists a form $\tilde{\omega} \in Z(M)$ which satisfies:*

1. *$\tilde{\omega}|_W$ is identically zero, and*

2. *$[\tilde{\omega}] = [\omega]$ as elements of $H(M)$.*

Proof: Choose an open subset V of M such that $\bar{W} \subset V \subset \bar{V} \subset U$. Write $\omega|_U = d\theta$, for some $\theta \in \mathcal{A}^*(U)$. Choose a C^∞-function f on M such that $f \equiv 1$ on $\bar{W}$ and $f \equiv 0$ on $M \backslash V$. Then $f\theta$ is a smooth form on the whole of M. Now set $\tilde{\omega} = \omega - d(f\theta)$. Then $\tilde{\omega}$ satisfies the requirements of the lemma. ∎

Lemma 113 *Let U and V be two open subsets of a manifold M, such that the exterior derivative d admits a continuous splitting on U, V and $U \cap V$. Assume further that $H(U \cap V)$ is finite dimensional. Then d admits a continuous splitting on the union $W := U \cup V$.*

Proof: Choose a continuous splitting s_1 (resp. s_2) of d on U (resp. V), and define a splitting $s : B^*(U) \oplus B^*(V) \to \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V)$ by $s = s_1 \oplus s_2$ (i.e., $s(\omega_1 + \omega_2) := s_1(\omega_1) + s_2(\omega_2)$, for $\omega_1 \in B^*(U)$ and $\omega_2 \in B^*(V)$).

Consider the commutative diagram (where the upper horizontal sequence is exact):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^{*-1}(W) & \xrightarrow{\gamma_1} & \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V) & \xrightarrow{\gamma_2} & \mathcal{A}^{*-1}(U \cap V) & \longrightarrow & 0 \\
 & & {\scriptstyle d_W}\downarrow & & {\scriptstyle d_U}\downarrow{\scriptstyle d_V} & & {\scriptstyle d_{U\cap V}}\downarrow & & \\
0 & \longrightarrow & B^*(W) & \xrightarrow{\hat{\gamma}_1} & B^*(U) \oplus B^*(V) & \xrightarrow{\hat{\gamma}_2} & B^*(U \cap V) & &
\end{array}$$

where $\gamma_1, \gamma_2, \hat{\gamma}_1$ and $\hat{\gamma}_2$ are the canonical maps. From the commutativity of the above diagram, $\gamma_2 s \hat{\gamma}_1(\omega) \in Z^{*-1}(U \cap V)$, for any $\omega \in B^*(W)$; and moreover, from the definition of the coboundary map $\delta : H^{*-1}(U \cap V) \to H^*(W)$, we get $\delta[\gamma_2 s \hat{\gamma}_1(\omega)] = [\omega] = 0$. In particular, the cohomology class $[\gamma_2 s \hat{\gamma}_1(\omega)]$ lies in the image $F \subset H(U \cap V)$ of $H(U) \oplus H(V)$ under γ_2.

Now choose any linear map $\beta : F^{*-1} \to Z^{*-1}(U) \oplus Z^{*-1}(V)$ such that $[\gamma_2 \beta(x)] = x$, for all $x \in F^{*-1}$. Since F is finite dimensional ($H(U \cap V)$ being finite dimensional by assumption), any such β is automatically continuous. With the help of β, we define the continuous linear map

$$s_\beta : B^*(W) \longrightarrow \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V),$$

by $s_\beta(\omega) = s\hat{\gamma}_1(\omega) - \beta[\gamma_2 s\hat{\gamma}_1(\omega)]$, for $\omega \in B^*(W)$. It can be easily seen that $\gamma_2 s_\beta(\omega) \in B^{*-1}(U \cap V)$. Choose a partition of unity $\{f_U, f_V\}$ subordinate to the cover $\{U, V\}$ of W and choose a continuous splitting s_3 of d on the intersection $U \cap V$. Then $f_V \cdot (s_3\gamma_2 s_\beta(\omega)) \in \mathcal{A}^{*-2}(U)$ and $f_U \cdot (s_3\gamma_2 s_\beta(\omega)) \in \mathcal{A}^{*-2}(V)$, for any $\omega \in B^*(W)$. Finally, define the continuous map $\theta : B^*(W) \to \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V)$ by

$$\theta(\omega) = s_\beta(\omega) - \left(d_U(f_V \cdot (s_3\gamma_2 s_\beta(\omega))) \oplus d_V(-f_U \cdot (s_3\gamma_2 s_\beta(\omega)))\right) ,$$

for $\omega \in B^*(W)$.

It is easy to see that $\gamma_2 \circ \theta = 0$, in particular, the map θ lifts to a (continuous) map $\bar{\theta} : B^*(W) \to \mathcal{A}^{*-1}(W)$ and moreover the map $\bar{\theta}$ provides a continuous splitting for d_W, i.e., $d_W \circ \bar{\theta} = I$. This completes the proof of the lemma. ∎

Let us now prove a stabilization lemma for splittings.

Proposition 114 *Let $\{V_i\}_{i=1,2,\ldots}$ be an open and locally finite cover of M. Set, for any $k = 1, 2, \ldots$,*

$$U_k = \bigcup_{i=1}^{k} V_i .$$

Also set $U_0 = \emptyset$. Assume that d admits a continuous splitting on V_k and $U_k \cap V_{k+1}$ and assume that $H(V_k)$ and $H(U_k \cap V_{k+1})$ are finite dimensional for all $k \geq 1$. Then d_M admits a continuous splitting.

Proof: We proceed as in the above lemma with the pair $U = U_k$ and $V = V_{k+1}$, but with a special choice of the map β. Mayer-Vietoris long exact sequence implies that $H(U_k)$ is finite dimensional. Let $F_k \subset H(U_k \cap V_{k+1})$ be the image of $H(U_k) \oplus H(V_{k+1})$ under the map $\gamma_2 := \gamma_2^k$. For $i \leq k$, let $H(U_k)_i$ be the subspace of elements of $H(U_k)$ which restrict to 0 in $H(U_i)$. We have the following

Lemma 115 *For any i, there exists $k(i) \geq i$ such that for all $k \geq k(i)$,*

$$\gamma_2 \left(H(U_k)_i \oplus H(V_{k+1})\right) = F_k.$$

Proof: For any $k \geq i$, let $R_k \subset H(U_i)$ be the image of $H(U_k)$ under the natural restriction. As $H(U_i)$ is finite dimensional, the decreasing sequence of subspaces R_k of $H(U_i)$ is stationary. Thus there exists an index $k(i)$ such that for $k \geq k(i)$, $R_k = R_{k(i)}$. Let us show that for $k \geq k(i)$, $\gamma_2 \left(H(U_k)_i \oplus H(V_{k+1})\right) = F_k$. Indeed let $\alpha = \nu - \kappa \in F_k$ where ν (resp. κ) is the restriction to $U_k \cap V_{k+1}$ of an element still denoted by $\nu \in H(U_k)$ (resp. $\kappa \in H(V_{k+1})$). As $R_k = R_{k+1}$, we may write $\nu = \nu_0 + \nu'|_{U_k}$, with $\nu_0 \in H(U_k)_i$ and $\nu' \in H(U_{k+1})$. Thus $\alpha = \nu_0 - (\kappa - \nu'|_{V_{k+1}})$ is in $\gamma_2(H(U_k)_i + H(V_{k+1}))$. ∎

We continue with the proof of Proposition 114. We fix an open refinement $\{W_i\}$ of $\{V_i\}$, i.e., $\bar{W}_i \subset V_i$ and $\cup_i W_i = M$. Let us choose more carefully the map $\beta_k : F_k \to Z(U_k) \oplus Z(V_{k+1})$. Let $b(k)$ be the largest integer $i \geq 0$ such that $\gamma_2(H(U_k)_i \oplus H(V_{k+1})) = F_k$ and choose $\beta_k = \beta_k^0 \oplus \beta_k'$ valued in $Z(U_k)_{b(k)} \oplus Z(V_{k+1})$, where $Z(U_k)_i$ is the space of closed differential forms θ on U_k such that $[\theta]|_{U_i} = 0$ as an element of $H(U_i)$. Furthermore with the help of Lemma 112, we can assume that for any $\omega \in F_k$ the component $\beta_k^0(\omega)$ vanishes identically on the open subset $\bigcup_{i=1}^{b(k)} W_i$.

With this choice of β_k, we get a continuous splitting s_{k+1} of d on the manifold U_{k+1}. This completes the inductive procedure to construct a splitting s_k of d on the manifold U_k, for all values of k. Now define a map $s : B^*(M) \to \mathcal{A}^{*-1}(M)$ by

$$s(\omega) = \lim_{k \to \infty} s_k(\omega_{|U_k}), \qquad \text{for } \omega \in B(M)\,.$$

Observe that for any relatively compact open subset V of M, there exists a large enough k_0 (depending only upon V) such that $(s_k(\omega_{|U_k}))_{|V} = (s_{k_0}(\omega_{|U_{k_0}}))_{|V}$ for all $k \geq k_0$. In particular, the map s is well defined and continuous. It is clear that s provides a splitting of d on the whole of M. ∎

To prove Theorem 109, it is then sufficient to prove the existence of a covering of M satisfying the conditions of Proposition 114.

Lemma 116 *Consider a locally finite covering of M by geodesically convex open subsets with respect to a fixed Riemannian metric on M, then this covering satisfies the conditions of Proposition 114.*

Proof: For any open subset U of M, let $n(U)$ be the smallest number of geodesically convex open subsets of U required to cover U (if no such finite cover exists, we decree $n(U) = \infty$). We first prove by induction on $n(U)$ that d_U admits a continuous splitting on any open subset $U \subset M$ with $n(U) < \infty$: The case $n(U) = 1$ is taken care of by Lemma 111. Observe that for any two convex open subsets of M, their intersection is also convex, so the general case follows by induction on $n(U)$ and Lemma 113 (together with [8], Chapter 1, Proposition 5.3.1). Thus a locally finite open cover $M = \bigcup V_i$, $i \geq 1$ by geodesically convex open subsets of M satisfies the hypothesis of Proposition 114. ∎

This completes the proof of Theorem 109. ∎

Let us generalize Theorem 109 to the equivariant case.

Theorem 117 *Let G be a compact Lie group acting on a paracompact manifold M. Then, for all $p \in \mathbb{Z}_+$, the subspaces $Z_G^p(M)$ and $B_G^p(M)$ are closed subspaces of $\mathcal{A}_G^p(M)$. Furthermore the equivariant de Rham differential $d_{\mathfrak{g}} : \mathcal{A}_G^p(M) \to B_G^{p+1}(M)$, where $\mathfrak{g} := \mathrm{Lie}\ G$, admits a continuous splitting for all $p \geq 0$.*

Proof: We start with the following preliminary lemmas.

Lemma 118 *Let G be a compact group acting on a manifold M. If $H(M)$ is finite dimensional, then $H_G^p(M)$ is finite dimensional, for all $p \in \mathbb{Z}_+$.*

Proof: Consider the filtration of $\mathcal{A}_G(M)$ by the subspaces

$$\mathcal{A}_G(M)_k = \left(S(\mathfrak{g}') \otimes \sum_{i=0}^{k} \mathcal{A}^i(M)\right)^G$$

of $\mathcal{A}_G(M)$ consisting of equivariant differential forms of exterior degree less or equal to k. Consider the space $Z_G(M)_k$ of closed equivariant differential forms with exterior degree less or equal to k. It is easy to see (as in the proof of Proposition 5) that the map $\alpha \mapsto \alpha_{[k]}$ induces an injective map from $V_k = Z_G(M)_k/\left(Z_G(M)_{k-1} + (d_{\mathfrak{g}}\mathcal{A}_G(M)_{k-1})\right)$ to $(S(\mathfrak{g}') \otimes H^k(M))^G$. Clearly V_k surjects on $H_G(M)_k/H_G(M)_{k-1}$, where $H_G(M)_k$ is the subspace of $H_G(M)$ consisting of those cohomology classes with a representative in $Z_G(M)_k$. This, in particular gives that $H_G(M)_k$ is finite dimensional in each $\mathbb{Z}$-graded degree and for all k. This proves the lemma.

Lemma 119 *Let $f : V \to W$ be a continuous linear map between Fréchet spaces such that* $\operatorname{Im} f$ *is of finite codimension in W. Then* $\operatorname{Im} f$ *is a closed subspace of W.*

Proof: Let us take a vector space complement U of $\operatorname{Im} f$ in W, which is finite dimensional by assumption. Also let K be the kernel of f. Consider the direct sum $V \oplus U$ and define a continuous linear map $\hat{f} : V \oplus U \to W$ by $\hat{f}|_V = f$ and $\hat{f}|_U$ is the inclusion. It is a surjective linear map between Fréchet spaces, so it is an open map (by the open map theorem). In particular, the map $\hat{f}$ gives a (linear) homeomorphism $(V/K) \oplus U \to W$. But V/K is closed in the direct sum $(V/K) \oplus U$, and hence its image $\operatorname{Im} f$ is closed in W. ∎

Thus we obtain

Lemma 120 *If $H_G^p(M)$ is finite dimensional, the space $B_G^p(M) \subset \mathcal{A}_G^p(M)$ is a closed subspace, for all $p \in \mathbb{Z}_+$.*

Lemma 121 *Let $p : \mathcal{V} \to M$ be a G-equivariant real vector bundle over a G-manifold M. If $d_{\mathfrak{g}}$ has a continuous splitting on M, then $d_{\mathfrak{g}}$ admits a continuous splitting on $\mathcal{V}$.*

Proof: Consider the map $\phi(t, v) = tv$ from $\mathbb{R} \times \mathcal{V}$ to $\mathcal{V}$. This map commutes with the action of G. Keeping the same notation for the operator H as in

Lemma 32, and denoting by i the inclusion of M as the zero section of $\mathcal{V}$, we obtain

$$\omega - p^*i^*\omega = (d_{\mathfrak{g}}H + d_{\mathfrak{g}}H)\omega$$

for all $\omega \in \mathcal{A}_G(\mathcal{V})$. Thus if s is a continuous splitting for $d_{\mathfrak{g}}$ on M, then $s_{\mathcal{V}} = (H + p^*si^*)|_{B_G(\mathcal{V})}$ is a continuous splitting for $d_{\mathfrak{g}}$ on $\mathcal{V}$. ∎

Lemma 122 *Let N be a contractible manifold. Let $K \subset G$ be a closed subgroup of G. Consider the G-manifold $M = (G/K) \times N$, where G acts by left action on the first factor and trivially on the second factor. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the G-manifold M.*

Proof: Note first that if $N = point$, i.e., $M = G/K$ is homogeneous, then (see section 5) $\mathcal{A}_G(M) \cong (S(\mathfrak{g}')\otimes\Lambda(\mathfrak{g}/\mathfrak{k})')^K$ is finite-dimensional in each degree, thus $d_{\mathfrak{g}}$ admits a continuous splitting on M. Proceeding as in Lemma 121, we obtain a continuous splitting for the product $(G/K) \times N$. ∎

We will also use the following equivariant analogue of Lemma 113, which follows by the same proof (using a G-invariant partition of unity, which exists since G is compact).

Lemma 123 *Let G be a compact Lie group and let M be a G-manifold with G-stable open subsets U and V. Assume that $d_{\mathfrak{g}}$ admits a continuous splitting on U, V and $U \cap V$. Assume further that $H^p_G(U \cap V)$ is finite dimensional in each degree p. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the union $W := U \cup V$.*

Proposition 124 *Let G be a compact Lie group and let M be a G-manifold with a locally finite covering by G-stable open subsets V_i, $i \geq 1$. Let $U_k = \bigcup_{i=1}^k V_i$. Assume that $d_{\mathfrak{g}}$ admits a continuous splitting on V_k and $U_k \cap V_{k+1}$, for all $k \geq 1$. Assume further that $H^p_G(U_k \cap V_{k+1})$ and $H^p_G(V_k)$ are finite dimensional in each degree p. Then*

(1) $d_{\mathfrak{g}}$ admits a continuous splitting on M,

(2) $B^p_G(M)$ is a closed subspace of $\mathcal{A}^p_G(M)$.

Proof: Mayer-Vietoris long exact sequence implies that $H^p_G(U_i)$ is finite-dimensional. We construct a continuous splitting s_i for $d_{\mathfrak{g}}$ on U_i as in the proof of Proposition 114. Then the splittings s_i stabilize to give rise to a continuous splitting s on M. Let us prove the assertion (2). Let $\alpha \in \overline{B^*_G(M)}$. Since $B_G(U_i)$ is a closed subspace of $\mathcal{A}_G(U_i)$ by Lemma 120, $\alpha|_{U_i} = d_{\mathfrak{g}}s_i(\alpha|_{U_i})$. By construction of s we see that $s_i(\alpha|_{U_i}) \in \mathcal{A}^{*-1}_G(U_i)$ stabilizes in an element $\beta \in \mathcal{A}^{*-1}_G(M)$ such that $d_{\mathfrak{g}}\beta = \alpha$. ∎

Lemma 125 *Let X be a G-manifold with exactly one orbit type. Then $d_{\mathfrak{g}}$ admits a continuous splitting on X.*

Proof: Since G is compact, our assumption implies that the space of orbits $M := X/G$ is a manifold and that the quotient map $q : X \to X/G$ is a locally trivial fibration (see [9], Chapter 2, Theorem 5.8). Take a locally finite open covering of M by convex open subsets V_i (in particular V_i being contractible, q is trivial over V_i) and set $\tilde{V}_i := q^{-1}(V_i)$. Now take any $x_0 \in \tilde{V}_i$, then $\tilde{V}_i$ is G-diffeomorphic to $G \cdot x_0 \times V_i$, where G acts trivially on V_i and G acts canonically on the orbit $G \cdot x_0$. Thus $d_{\mathfrak{g}}$ admits a continuous splitting on $\tilde{V}_i$ by Lemma 122. Arguing as in the proof of Lemma 116, we see that the covering $X = \bigcup_i \tilde{V}_i$ satisfies the hypothesis of Proposition 124. Thus we obtain our lemma. ∎

Let M be a compact G-manifold with boundary $\delta(M)$. Observe that $\delta(M)$ is automatically G-stable. Let $M^o = M \backslash \delta(M)$. We call M^o the interior of M.

Lemma 126 *Let M be a compact G-manifold possibly with boundary. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the interior M^o of M.*

Proof: By [9], chapter 8, Theorem 3.13, any compact manifold has finitely many orbit types. Let us assume by induction on the number of orbit types $\eta(M^o)$ of M^o that the lemma is true for all compact manifolds M (possibly with boundary) with $\eta(M^o) \leq n$ and take a compact manifold M with $\eta(M^o) = n+1$. The case where $\eta(M^o) = 1$ is taken care of by Lemma 125. Let G_1 be an isotropy subgroup such that G_1 is not properly contained in any other isotropy subgroup and let N be the set of all the elements of M^o with isotropy group G_1. Then $N = (M^o)^{G_1}$ by the maximality property of G_1. Thus N is a closed submanifold of M^o. Let K be the normalizer of G_1 in G. The map $(g, x) \mapsto g \cdot x$ induces an isomorphism of $G \times_K N$ on its image F. Thus F is a G-invariant closed submanifold of M^o. Cover M^o as $M^o = U \cup V$ where U is a G-stable open tubular neighborhood of F and $V = M^o \backslash F$. Clearly $\eta(V) = n$ and $\eta(F) = 1$. Also U, $U \cap V$ and V are interiors of compact G-manifolds with boundary. Further the existence of a continuous splitting for the operator $d_{\mathfrak{g}}$ on F (guaranteed by Lemma 125) gives rise to a splitting of $d_{\mathfrak{g}}$ on U, by Lemma 121. By induction hypothesis, $d_{\mathfrak{g}}$ admits a continuous splitting on V and $U \cap V$. Further $U \cap V$ being the interior of a compact manifold with boundary, $H(U \cap V)$ is finite dimensional and so is $H^p_G(U \cap V)$ for every p. Now the lemma follows by applying Lemma 123 to the open cover $M^o = U \cup V$. ∎

Let us now prove Theorem 117. Choose a G-invariant Morse function

$$f : M \to [0, \infty)$$

for the G-manifold M in the sense of ([22], par. 4). We further assume that f is a proper map. Let $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be the complete list (possibly infinite) of critical values of f. Choose real numbers $\mu_i < \overline{\mu}_i$ such that $\lambda_i < \mu_i < \overline{\mu_i} < \lambda_{i+1}$. Define $V_i := f^{-1}(\mu_{i-1}, \overline{\mu}_i)$ (μ_0 is set as $-\infty$) and $U_k = \bigcup_{i=1}^k V_i$. Then $V_k, U_k, U_k \cap V_{k+1}$ are interiors of compact G-manifolds with boundaries and the open cover V_i of M satisfies the hypothesis of Proposition 124. This completes the proof of Theorem 117. ∎

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