TAPANI MATALA-AHO

On recurrences for some hypergeometric type polynomials


<http://www.numdam.org/item?id=AST_1991__198-199-200___237_0>
ON RECURRENCES FOR SOME HYPERGEOMETRIC TYPE POLYNOMIALS

by

TAPANI MATALA-AHO

1. Introduction

A well-known result for Legendre type polynomials $P_n = 2F_1\left(\begin{smallmatrix} -n-n \\ 1 \end{smallmatrix} \bigg| \frac{t}{1} \right)$ is the recurrence relation $nP_n - (2n - 1)(1 + t)P_{n-1} + (n - 1)(1 - t)^2P_{n-2} = 0$. When $t=-1$ this implies the known sum formula $\sum_{k=0}^{2m} (\begin{smallmatrix} 2m \\ k \end{smallmatrix})^2(-1)^k = (-1)^m(\begin{smallmatrix} 2m \\ m \end{smallmatrix}) \quad (m = 0, 1, 2, \ldots)$. 

There are numerous extensions of this recurrence for special values of $t$. ASKEY and WILSON [2] achieved three term recurrences for the sums $\sum_{k=0}^{n} \binom{n}{k}(\begin{smallmatrix} n+k+d \\ k+d \end{smallmatrix})\binom{n+k+b+c}{k+e} \binom{n+k+c+f}{k+f}$ $(a + d = b + c)$ by contiguous relations for hypergeometric series. PERLSTADT [6] obtained recurrences for sums $\sum_{k=0}^{n} \binom{n}{k}^r$ $(r = 2, \ldots, 6)$ by the method of telescoping series.

In the following we want to apply so called 'SISTER CELINE’S method' to the following kind of hypergeometric polynomials - say $3F_2\left(\begin{smallmatrix} -n,A,B \\ 1,1 \end{smallmatrix} \bigg| \frac{t}{1} \right)$, where $(A, B) = (-n, -n), (-n, 1/2)$ or $(-n, n + 1)$ and $4F_3\left(\begin{smallmatrix} -n,A,B,C \\ 1,1,1 \end{smallmatrix} \bigg| \frac{t}{1} \right)$, where $(A, B, C) = (-n, -n, -n)$ or $(-n, n + 1, n + 1)$. For the hypergeometric notation we refer to RAINVILLE [8]. These polynomials satisfy four or five order recurrences and with special values of the parameter $t$ we shall obtain some three term recurrences like the APÉRY recurrences [1] considered in detail by van de CORPUT [7]. Also we shall get DIXON’S sum formula $\sum_{k=0}^{2m} (\begin{smallmatrix} 2m \\ k \end{smallmatrix})^3(-1)^k = (-1)^m(\begin{smallmatrix} 2m \\ m \end{smallmatrix})^3(\begin{smallmatrix} m \\ m \end{smallmatrix}) \quad (m = 0, 1, 2, \ldots)$ (MACMAHON [5]). More references can be found from ASKEY and WILSON [2] and PERLSTADT [6].

2. Sister Celine’s method

From the remarks of BAILEY [3] it is clear that $3F_2$-polynomials satisfy at most four term recurrences and $4F_3$-polynomials satisfy at most five term recurrences. To obtain the recurrence

$$a_0(n)p_n + (a_1(n) + a_2(n)t)p_{n-1} + (a_3(n) + a_4(n)t + a_5(n)t^2)p_{n-2} + \ldots = 0$$

for the polynomial $p_n = p_n(t)$ of degree $n$ we shall use SISTER CELINE’S method, see FASENMYER [4] or RAINVILLE [8]. Note that some of the coefficient polynomials of $p_n, p_{n-1}, \ldots$
MATALA-AHO T.

may be zero e.g. in (4), (13) and (14). In these cases we shall rise the index \( \eta \) so that in all cases our results have the form

\[
A(n, t) p_n + B(n, t) p_{n-1} + C(n, t)p_{n-2} + ... = 0.
\]

Let us set \( p_n(t) = \sum_{k=0}^{n} \epsilon(k, n)t^k \). In the case of the above mentioned polynomials it is easy to write polynomials \( p_{n-1}, tp_{n-1}, p_{n-2}, tp_{n-2}, t^2p_{n-2}, ... \) in the form

\[
\sum_{k=0}^{n} r_i(k, n)\epsilon(k, n)t^k \quad (i = 1, 2, ...)
\]

(see the proof of Theorem 1), where the \( r_i(k, n) \)'s are certain rational expressions of \( k \) and \( n \).

Now from (1) shall we obtain the identity

\[
a_0(n) + a_1(n) r_1(k, n) + a_2(n) r_2(k, n) + a_3(n) r_3(k, n) + ... = 0.
\]

Thus it is straightforward to get enough equations to solve the unknowns \( a_i(n) \) (i = 0, 1, ...). Let the number of \( a_i \)'s be \( m \). In all our cases it sufficed to set \( k = 1, ..., m \) to achieve a solvable system of equations. In some cases due the symmetry it was possible to reduce significantly the number of equations (see Theorem 1).

3. Theorems

In the following theorem we shall state recurrences for

\[
J_n(t) = \binom{n, n, n}{1, 1} \quad \text{and} \quad H_n(t) = \binom{n, n, n}{1, 1}.
\]

Now due the relations \( J_n(1/t) = (-1/t)^nJ_n(t) \) and \( H_n(1/t) = (1/t)^nH_n(t) \) there are only six unknowns \( a_i(n) \) in the first case and 14 in the second case.

THEOREM 1. The recurrences for \( J_n \) and \( H_n \) are given by

\[
(3n - 5)n^2J_n - (9n^3 - 24n^2 + 17n - 4)(1 - t)J_{n-1}
\]

\[
+ (3n - 4)(3n^2 - 7n + 3 + (21n^2 - 49n + 24)t + (3n^2 - 7n + 3)t^2)J_{n-2}
\]

\[
- (3n - 2)(n - 2)^2(1 - t)^3J_{n-3} = 0
\]

and

\[
(b_0(n) + b_1(n)t + b_0(n)t^2)e_n + (b_2(n) + b_3(n)t + b_3(n)t^2 + b_2(n)t^3)H_{n-1} +
\]

\[
(b_4(n) + b_5(n)t + b_6(n)t^2 + b_5(n)t^3 + b_4(n)t^4)H_{n-2} +
\]

\[
(b_7(n) + b_8(n)t + b_9(n)t^2 + b_8(n)t^3 + b_7(n)t^4 + b_7(n)t^5)H_{n-3} +
\]

238
\[ (b_{10}(n) + b_{11}(n)t + b_{12}(n)t^2 + b_{13}(n)t^3 + b_{12}(n)t^4 + b_{11}(n)t^5 + b_{10}(n)t^6)H_{n-4} = 0, \]

where

\[ b_0(n) = (n - 1)n^3(64n^4 - 544n^3 + 1697n^2 - 2303n + 1151), \]
\[ b_1(n) = (n - 1)n^3(272n^4 - 2312n^3 + 7231n^2 - 9859n + 4958), \]
\[ b_2(n) = -(n - 1)(256n^7 - 2560n^6 + 10212n^5 - 20844n^4 + 23317n^3 - 14341n^2 + 4655n - 630), \]
\[ b_3(n) = -(n - 1)(1344n^7 - 13440n^6 + 53688n^5 - 109816n^4 + 123098n^3 - 75804n^2 + 24615n - 3330), \]
\[ b_4(n) = 384n^8 - 4800n^7 + 25318n^6 - 73416n^5 + 127613n^4 - 135624n^3 + 85700n^2 - 29385n + 4230, \]
\[ b_5(n) = -6304n^8 + 78800n^7 - 418938n^6 + 1237876n^5 - 2243833n^4 + 2489364n^3 - 1694065n^2 + 640890n - 103320, \]
\[ b_6(n) = -10(3296n^8 - 41200n^7 + 218996n^6 - 646196n^5 + 1157126n^4 - 1286706n^3 + 867069n^2 - 323775n + 51462), \]
\[ b_7(n) = -(n - 2)(256n^7 - 3072n^6 + 15140n^5 - 39448n^4 + 58199n^3 - 48180n^2 + 20600n - 3555), \]
\[ b_8(n) = -(n - 2)(9024n^7 - 108288n^6 + 536320n^5 - 1412352n^4 + 2119776n^3 - 1798290n^2 + 793925n - 142335), \]
\[ b_9(n) = -5(n - 2)(8384n^7 - 100608n^6 + 498988n^5 - 1318024n^4 + 1987917n^3 - 1698342n^2 + 756763n - 137142), \]
\[ b_{10}(n) = (n - 2)(n - 3)^3(64n^4 - 288n^3 + 449n^2 - 285n + 65), \]
\[ b_{11}(n) = (n - 1)(n - 2)(n - 3)^3(16n^3 - 56n^2 + 75n - 30), \]
\[ b_{12}(n) = -5(n - 2)(n - 3)^3(128n^4 - 576n^3 + 913n^2 - 597n + 141), \]
\[ b_{13}(n) = 10(n - 2)(n - 3)^3(112n^4 - 504n^3 + 797n^2 - 519n + 122), \]

respectively.

**PROOF:** We shall shortly describe the proof of the recurrence (4). For technical reasons we shall lower the index \( n \) to \( n - 2 \) in the formula (4). When we denote

\[ H_n = \sum_{k=0}^{n} \epsilon(k, n)t^k, \]

where \( \epsilon(k, n) = (-n)^k/(k!)^4 \), then

\[ t^iH_{n-j} = \sum_{k=i}^{n+i-j} \left( \frac{(k-i+1)(-n+k-j+1)}{(-n)_j} \right)^4 \epsilon(k, n)t^k \quad (i \leq j) \]
so that \( r_3 = \frac{(k - n)(k - n + 1)/n(n - 1)}{k(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)} \), \( r_4 = \frac{(k - n)/n(n - 1)}{k(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)} \), ... Hence the left hand side of (4) can be combined into one sum

\[
\sum_{k=0}^{n} \frac{N(k, n)}{(n(n-1)(n-2)(n-3)(n-4)(n-5))} \epsilon(k, n)t^k,
\]

where

\[
N(k, n) = \frac{(n-2)^4(n-3)^4(n-4)^4(n-5)^4(((k-n)^4(k-n+1)^4+(k-1)^4k^4b_0(n-2)+(k-n)^4k^4b_1(n-2))+...+(k-n)^4(k-n+1)^4(k-n+2)^4(k-2)^4(k-1)^4k^4b_{13}(n-2) = 0}
\]

identically on \( k \) and \( n \). Counting this could have been quite tedious without the assistance of some symbolic mathematical program system like Macsyma, Musimp or Reduce.

When \( t = 1 \) we get from (3) \( J_n = \frac{-3(3n-4)(3n-2)}{n^2} J_{n-2} \). So we have DIXON's result

\[
\sum_{k=0}^{2m} \binom{2m}{k} (-1)^k = (-1)^m \left( \sum_{k=0}^{m} \binom{3m}{m} \right) \quad (m = 0, 1, 2, ...) \quad \text{(MACMAHON [5])}.
\]

In the following corollary we shall state three term recurrences for \( j_n = \sum_{k=0}^{n} \binom{n}{k}^3 \), \( h_n = \sum_{k=0}^{n} \binom{n}{k}^4 \) and \( g_n = \sum_{k=0}^{n} \binom{n}{k}^4(-1)^k \). The recurrences for numbers \( j_n \) and \( h_n \) are results of FRANEL (PERLSTADT [6]), while the recurrence (7) for \( g_n \) is perhaps new.

**Corollary 1.** The recurrences for \( j_n, h_n \) and \( g_n \) are given by

(5) \[ n^2j_n - (7n^2 - 7n + 2)j_{n-1} - 8(n-1)^2j_{n-2} = 0, \]

(6) \[ n^3h_n - 2(2n-1)(3n^2 - 3n + 1)h_{n-1} - 4(n-1)(4n-3)(4n-5)h_{n-2} = 0 \]

and

(7) \[ (n-1)(12n^2 - 63n + 83)n^3g_n + 4(408n^6 - 3774n^5 + 13760n^4 - 25203n^3 + 24465n^2 - 11970n + 2340)g_{n-2} + 16(n-2)(12n^2 - 15n + 5)(n-3)^3g_{n-4} = 0 \]

respectively.

**Proof:** We shall prove the first formula, the proof of second formula goes similarly. The third formula is an immediate consequence of (4) with \( t = -1 \).
Let us define the operator $E$ by $E r_n = r_{n+1}$. As a consequence of formula (3) with $t = -1$ we get

$$
(3n - 5)n^2 j_n - 2(9n^3 - 24n^2 + 17n - 4)j_{n-1} -
(3n - 4)(15n^2 - 35n + 18)j_{n-2} - 8(3n - 2)(n - 2)^2 j_{n-3} = 0.
$$

Then (8) is equivalent to

$$
P(n)j_{n-3} = ((3n - 5)n^2 E^3 - 2(9n^3 - 24n^2 + 17n - 4)E^2 -
(3n - 4)(15n^2 - 35n + 18)E - 8(3n - 2)(n - 2)^2)j_{n-3} = 0,
$$

where the operator $P(n)$ factors in the following way

$$
P(n) = ((3n - 5)E + 3n - 2)((n - 1)^2 E^2 - (7(n - 1)(n - 2) + 2)E - 8(n - 2)^2).
$$

Let us denote

$$
s_{n-3} = ((n - 1)^2 E^2 - (7(n - 1)(n - 2) + 2)E - 8(n - 2)^2)j_{n-3}.
$$

Thus (9) is equivalent to

$$
((3n - 5)E + 3n - 2)s_{n-3} = 0
$$

i.e. $(3n - 5)s_{n-2} = -(3n - 2)s_{n-3}$. From (10) one gets $s_0 = 2^2 j_2 - (7 \cdot 2 \cdot 1 + 2)j_1 - 8j_0 = 0,$

so $s_k = 0$ $(k = 0, 1, 2, \ldots)$ and thus (10) implies (5).

In the second case one sets $t = 1$ in (4) and gets $P(n)h_{n-4} = 0$, where the operator $P(n)$ factors in the following way

$$
P(n) = ((n - 1)(20n^3 - 115n^2 + 215n - 132)E + 2(2n - 5)(20n^3 - 55n^2 + 45n - 12))

((n - 1)^3 E^2 - 2(2n - 3)(3n^2 - 9n + 7)E - 4(n - 2)(4n - 7)(4n - 9)).
$$

Analogously to the first case one obtains the recurrence (6).

By similar method like in Theorem 1 we can achieve recurrences for

$$
F_n(t) = _3F_2 \left( \begin{array} {c} -n, -n, 1/2 \\ 1, 1 \end{array} \right| t \right), \quad B_n(t) = _3F_2 \left( \begin{array} {c} -n, -n, n + 1 \\ 1, 1 \end{array} \right| t \right)
$$

and

$$
A_n(t) = _4F_3 \left( \begin{array} {c} -n, -n, n + 1, n + 1 \\ 1, 1, 1 \end{array} \right| t \right).
$$

THEOREM 2. The recurrences for $F_n, B_n$ and $A_n$ are given by

$$
4(4n - 7)n^2 F_n - 2(6(4n^3 - 11n^2 + 8n - 2) + (16n^3 - 44n^2 + 34n - 9)t)F_{n-1} +
$$
\[ (4(12n^3 - 45n^2 + 52n - 18) + 2(2n - 3)t + (4n - 3)(2n - 3)^2t^2)F_{n-2} \]
\[ - 4(4n - 3)(n - 2)(1 - t)^2F_{n-3} = 0 \]

and

\[ \begin{align*}
(b_0(n) + b_1(n)t)B_n & + (b_2(n) + b_3(n)t + b_4(n)t^2)B_{n-1} + \\
(b_5(n) + b_6(n)t + b_7(n)t^2 + b_8(n)t^3)B_{n-2} & + (b_9(n) + b_{10}(n)t + b_{11}(n)t^2)B_{n-3} = 0,
\end{align*} \]

where

\[ \begin{align*}
b_0(n) &= 3(9n - 14)n^2, & b_6(n) &= -4(159n^3 - 567n^2 + 626n - 200), \\
b_1(n) &= -4(8n - 13)n^2, & b_7(n) &= 4(67n - 40)(2n - 3)^2, \\
b_2(n) &= -3(27n^3 - 69n^2 + 47n - 10), & b_8(n) &= -16(8n - 5)(2n - 3)^2, \\
b_3(n) &= -10(12n^3 - 30n^2 + 23n - 6), & b_9(n) &= -3(9n - 5)(n - 2)^2, \\
b_4(n) &= 8(32n^3 - 84n^2 + 62n - 15), & b_{10}(n) &= (59n - 35)(n - 2)^2, \\
b_5(n) &= 3(3n - 5)(9n^2 - 17n + 6), & b_{11}(n) &= -4(8n - 5)(n - 2)^2,
\end{align*} \]

and

\[ \begin{align*}
(c_0(n) + c_1(n)t)A_n & + (c_2(n) + c_3(n)t + c_4(n)t^2)A_{n-1} + \\
(c_5(n) + c_6(n)t + c_7(n)t^2 + c_8(n)t^3)A_{n-2} & + (c_9(n) + c_{10}(n)t + c_{11}(n)t^2)A_{n-3} + \\
(c_{12}(n) + c_{13}(n)t)A_{n-4} &= 0,
\end{align*} \]

where

\[ \begin{align*}
c_0(n) &= (n - 1)(2n - 3)(2n - 5)^2n^3, \\
c_1(n) &= -(n - 1)(2n - 5)(4n - 7)(4n - 9)n^3, \\
c_2(n) &= -(n - 1)(2n - 1)(2n - 5)(8n^4 - 40n^3 + 62n^2 - 33n + 6), \\
c_3(n) &= -(n - 1)(2n - 1)(2n - 5)(4n - 3)(8n^3 - 34n^2 + 41n - 18), \\
c_4(n) &= 4(n - 1)(2n - 1)(2n - 5)(4n - 9)(16n^3 - 44n^2 + 34n - 9), \\
c_5(n) &= (2n - 3)(24n^6 - 216n^5 + 754n^4 - 1284n^3 + 1101n^2 - 441n + 66), \\
c_6(n) &= -(2n - 3)(864n^6 - 7776n^5 + 27514n^4 - 48444n^3 + 44516n^2 - 19485n + 3270), \\
c_7(n) &= 4(2n - 1)(2n - 3)^2(2n - 5)(64n^2 - 192n + 99), \\
c_8(n) &= -16(2n - 1)(2n - 3)^3(2n - 5)(4n - 3)(4n - 9), \\
c_9(n) &= -(n - 2)(2n - 1)(2n - 5)(8n^4 - 56n^3 + 134n^2 - 123n + 33), \\
c_{10}(n) &= -(n - 2)(2n - 1)(2n - 5)(4n - 9)(8n^3 - 38n^2 + 53n - 15), \\
c_{11}(n) &= 4(n - 2)(2n - 1)(2n - 5)(4n - 3)(16n^3 - 100n^2 + 202n - 129), \\
c_{12}(n) &= (n - 2)(n - 3)^3(2n - 1)^2(2n - 3), \\
c_{13}(n) &= -(n - 2)(n - 3)^3(2n - 1)(4n - 3)(4n - 5),
\end{align*} \]
HYPERGEOMETRIC TYPE POLYNOMIALS

respectively.

In the following corollary we shall state three term recurrences for $f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$, $e_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{1}{k}$, $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ and $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. The recurrence for the numbers $f_n$ is proved in Stienstra and Beukers [9], the recurrences for the Apéry numbers $b_n$ and $a_n$ can be found in Apéry [1] and Poorten Van de V. [7].

COROLLARY 2. The recurrences for $f_n$, $e_n$, $b_n$ and $a_n$ are given by

(15) $n^2 f_n - (10n^2 - 10n + 3) f_{n-1} + 9(n-1)^2 f_{n-2} = 0,$

(16) $4n^2 e_n - 2(10n^2 - 10n + 3) e_{n-1} + (4n-3)(4n-5)e_{n-2} = 0,$

(17) $n^2 b_n - (11n^2 - 11n + 3) b_{n-1} - (n-1)^2 b_{n-2} = 0$

and

(18) $n^3 a_n - (2n-1)(17n^2 - 17n + 5) a_{n-1} + (n-1)^3 a_{n-2} = 0$

respectively.

For example let $a_n = A_n(1) = _4 F_3 \left( \begin{array}{c} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{array} \right) \left| 1 \right)$ then the formula (14) gives $P(n)a_{n-4} = 0$, where

\[ P(n) = ((n-2)(2n-5)E^2 - (2n-1)(2n-5)E + (n-1)(2n-1)) 
\]
\[ ((n-2)^3 E^3 - (2n-5)(17(n-2)(n-3)+5)E + (n-3)^3). \]

Again like in the proof of Corollary 1 we see that $a_n$ satisfies (18).

REFERENCES


Tapani Matala-aho
University of Oulu
Department of Mathematics
90570 Oulu
Finland