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EXPONENTIAL SUMS AFTER BOMBIERI AND IWANIEC

by

M.N. HUXLEY

BOMBIERI and IWANIEC [BI1, BI2] obtained $\theta = 9/56$ for the Lindelöf exponent (the least θ for which the Riemann zeta function satisfies $\zeta(1/2+it) = O(t^{\theta+\epsilon})$ as $t \to \infty$.)

They remarked that their method might not be special to the Lindelöf problem ; in fact, as the saying goes, "they wrought [worked] better than they knew".

To show that one property is uniformly distributed with respect to another property, one forms exponential sums

$$S = \sum_{M}^{2M-1} e(f(m)) \ , \qquad (1)$$

where

$$e(x) = \exp 2\pi i x, \qquad f(m) = TF(m/M)$$

with $F(x)$ in the function class $C^n[1-\delta, 2+\delta]$ for some $\delta > 0$ and $n \geq 4$. The case $F(x) = \log x$ gives Dirichlet series. If $F(x)$ is a polynomial of degree d with rational coefficients, denominator q, and if $T = M^d$, then the sum S is approximately

$$MS_q/q \ ,$$

where S_q is a complete exponential sum with denominator q. One imposes conditions to prevent $F(x)$ from being well approximated by a polynomial for a long interval of values of m. A sufficient condition is that $F(x)$ be holomorphic on a neighbourhood of the segment $1 \leq x \leq 2$ of the real axis, and satisfies there

$$F'(x) = (1+o(1))x^{-s}$$

for some real $s > 0$. This condition is called the "virial" or "monomial" condition. It holds in many applications.

There are three useful ideas for treating exponential sums :

O. Subdivide the range for m,
A. Cauchy's inequality,
B. Poisson summation.

The name "Step A" is usually given to Weyl's differencing lemma, which may be analysed as subdivision, followed by Cauchy, followed by averaging. Van der Corput's method [see GK, I or K]consists of iterating these steps. The simplest form of Van der Corput's method, applying steps O, A, B (read from left to right) gives

$$S = O(M^{1/2}T^{1/6}) .$$

The method can be applied to exponential sums in several variables, and it becomes extremely complicated.

Bombieri and Iwaniec obtained

$$S = O(M^{1/2}T^{9/56+\epsilon})$$

by taking the steps in the order O, B, A. The method is arithmetic, and is essentially limited to one variable. Their subdivisions correspond to approximations to $f(m)$ by quadratic polynomials with rational coefficients. If the denominator q of the leading coefficient is small, the short interval is a "major arc", length N say, and the sum over the short interval is approximately

$$NS_q/q .$$

If q is large, the short interval is a "minor arc", and one expects the sum over the short interval to be small. This behaviour is seen in computer studies of exponential sums, notably those of DESHOUILLERS [D]. The Cauchy inequality is employed to show that the minor arc contribution is small in L_p norm (for some suitable p). In some ways the treatment resembles applying Hardy and Littlewood's Farey dissection to

$$\int_0^1 \sum_{n=1}^{N} e(f(n+\alpha M))d\alpha . \tag{2}$$

If all arcs are treated as major (steps O,B alone), one gets

$$S = O(M^{1/2}T^{1/6+\epsilon}).$$

This method is no worse than that of Van der Corput.

At the same time JUTILA [J1-8] has been considering sums

$$\sum_{M}^{2M-1} \tau(m) e(f(m)) , \tag{3}$$

where $\tau(m)$ is the divisor function or the Fourier coefficient of a modular form, beginning with steps O, B where O is subdivision according to the rational approximation to the first derivative, B is Voronoi or Wilton summation. In this context the numbers $\tau(m)\, e(-am/q)$ are the coefficients of the modular form twisted by the matrix $\begin{bmatrix} q & -a \\ 0 & q \end{bmatrix}$, and the Wilton summation formula is still available. These ideas could extend to any motivic L-series characterised by the three conditions :

D. An ordinary Dirichlet series with denominators n^{-s},
E. An Euler product,
F. Functional equations for the L-series and its twists.

One may fit Bombieri and Iwaniec's ideas into this frame by taking $\tau(m)$ to be the theta-function coefficients, 2 if m is a perfect square, 0 if not, and by considering $F(x)$ as a function of x^2. This change of variable explains why the derivatives do not correspond.

There are two successful applications of the Bombieri-Iwaniec method to sums with an extra variable. The Weyl step O, A replaces the sum S of (1) with double sums of the form

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(f(m+h) - f(m)). \tag{4}$$

This sum suggests the simpler sum

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(hf'(m)). \tag{5}$$

The sum (5) was estimated by IWANIEC and MOZZOCHI [IM] using the same method. The rational polynomial approximation to $hf'(x)$ is found by multiplying the approximation to $f'(x)$ by h, so h must not be too large. HEATH-BROWN and HUXLEY [HBH] estimated (4) - actually in the form

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(f(m+h) - f(m-h)). \tag{6}$$

This in turn gives estimates for

$$\int_{-U}^{U} |S(T_o + T)|^2 dT, \tag{7}$$

where $S(T)$ is the sum (1) considered as a function of T, if H goes up to T_o/U in size.

More general multiple exponential sums have not been treated, since one cannot find a good approximation by a rational polynomial.

The Iwaniec-Mozzochi sum (5) is connected with numerical integration. The prettiest case is the discrepancy for a circle (or more generally a smooth closed curve), the number of integer points minus the area. For a circle radius R, approximating the circle by a polygon whose sides lie along lattice lines $x =$ integer, $y =$ integer shows that the discrepancy is $O(R)$. Voronoi's method, applied by Sierpiński, approximates the circle by a polygon with rational gradients. Sierpiński obtained a discrepancy $O(R^{2/3})$ if the centre of the circle is at an integer point. The method can be modified [H2] to give $O(R^{2/3}(\log R)^{4/3})$ in general.

Exponential sums are introduced by way of the row-of-teeth function

$$\rho(t) = [t] - t + 1/2 \;\cong\; \sum_{h \neq 0} \frac{e(ht)}{2\pi i h}\,.$$

Thus

$$\sum_{m} \rho(\surd(R^2 - m^2))$$

can be expressed in terms of terms of the sums (5). The subdivision in step O corresponds to the sides of the Sierpiński polygon, with q as the denominator of the rational gradient a/q.

Minor arc contributions can be classified as follows.

E1. The "main term", estimated in L_p norm,
E2. Edge effects from ends of ranges of summation,
E3. Approximation errors in each summand in each Poisson summation.

The O, B, A sequence is dangerous because the errors of types (E2) and (E3) from each short sum in the subdivision must be added. For the sum S of (1) there is a finite Poisson summation modulo q, followed by a Poisson summation in m, giving an Airy integral. For the double sums (5) and (6)

Poisson summation in m is followed by Poisson summation in h. The second Poisson summation gives the Bessel function

$$Hi_{-1/2}(t) = \sqrt{\left[\frac{2}{\pi t}\right]}\, e^{-it}\ ,$$

with an (E3) error term because the Bessel function is given by an integral from 0 to ∞ , and one only integrates over a bounded range.

The main term on a minor arc is a sum over vectors $\underset{\sim}{x}$ of an exponential $e(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)$.

For the sum (5)

$$\underset{\sim}{x} = (k\ell, \ell, \ell\sqrt{k}, \ell/\sqrt{k})$$

summed over a range (depending on the minor arc) of the form

$$L_1 \le \ell \le L_2, \quad \max(K_1, c_1\ell^2) \le k \le \min(K_2, c_2\ell^2).$$

The exponent is really a power series in $1/\sqrt{k}$, but the further terms can be treated as type (E3) errors. The vector $\underset{\sim}{y}$ is $\underset{\sim}{y}(a/q)$ indexed by the gradient a/q of the Sierpiński polygon :

$$\underset{\sim}{y} = \left[\frac{\overline{a}}{q}, \frac{\overline{a}b}{q} - \frac{\nu}{q}, \frac{1}{\sqrt{(\mu q^3)}}, \frac{\kappa}{\sqrt{(\mu q^3)}}\right]\ ,$$

where a, b and q are integers, $\overline{a}$ being the inverse modulo q, and κ, μ and ν are real, all depending on the minor arc. The obvious technical difficulty, that the ranges for k and ℓ are not independent and vary with the minor arc, can be overcome. For this and other technical reasons the sum is squared, and the $\underset{\sim}{x}$ vectors are replaced by differences

$$\underset{\sim}{x}^{(k_1,k_2,\ell_1,\ell_2)} = (k_1\ell_1 - k_2\ell_2, \ldots).$$

The treatment of the sum (6) is analogous but more complicated. The Bessel integral is perturbed, with larger (E3) errors, and the third entry of the vector is

$$k\sqrt{\ell}(1 + \ell^2/48k^2).$$

The simpler sum (1) gives a very similar $\underset{\sim}{y}$ vector and an $\underset{\sim}{x}$ vector

$$\underset{\sim}{x}^{(h)} = (h^2, h, h^{3/2}, h^{1/2}).$$

One raises the sum to the r-th power and uses

$$\underset{\sim}{x}^{(h_1,\ldots,h_r)} = \underset{\sim}{x}(h_1) + \cdots + \underset{\sim}{x}(h_r).$$

Cauchy's inequality takes the form of the Large Sieve [B], generalised to be symmetric between the "integer" $\underset{\sim}{x}$ and the "rational" $\underset{\sim}{y}$ vectors.
Moreover vectors of each type may coincide with one another, so that instead of the number of vectors, one counts the number of coincident pairs. For the $\underset{\sim}{x}$ vectors, this is like the Hilbert-Kamke problem. BOMBIERI and IWANIEC [BI2] gave a bound for $r = 4$ using ingenious exponential sum arguments. WATT [W1] gave an elementary argument based on the fact that the variety

$$h_1^2 + \cdots + h_r^2 = h_{r+1}^2 + \cdots h_{2r}^2 \ ,$$
$$h_1 + \cdots + h_r = h_{r+1} + \cdots h_{2r}$$

is an affine cylinder. One wants to show that most coincidences are the trivial ones when h_{r+1} to h_{2r} are h_1 to h_r permuted. This is easy for $r = 3$, but false for $r \geq 7$. WATT [W3] has a weaker result for $r = 5$, combining elementary and exponential sum methods. HUXLEY and KOLESNIK [HK] have essentially settled the case $r = 5$. IWANIEC and MOZZOCHI [IM] treated the corresponding k_r, ℓ_r coincidence problem elementarily; see also [W2]. For the sum (6) Heath-Brown noticed that the perturbing factor $(1 + \ell^2/48k^2)$ can be neglected in a range of h in which the h^3 term in (6) cannot be neglected.

Bombieri and Iwaniec gave a bound for the number of coincident $\underset{\sim}{y}$ vectors in the case $F(x) = \log x$ only. HUXLEY and WATT [HW1] gave the same bound for general $F(x)$. KOLESNIK [GK] later found a simpler idea which leads to the same bound. All five authors only assume that the entries y_1 and y_3 coincide; this uses f'' (as a/q) and the residual term in $f^{(3)}$. The entries y_2 and y_4 involve f' also, and are harder to use. The coincidence of $\underset{\sim}{y}$ vectors may be regarded as resonance between different arcs of the curve. The possible resonances correspond to matrices in the modular group $SL(2, Z)$. One would like to show that most matrices of $SL(2, Z)$ give no resonance. Further progress may entail the use of "Kloostermania", the Fourier theory of $SL(2, Z)$.

The various sums (1) to (7) may be averaged over a family of related functions $f_i(x)$. This is important in many of JUTILA's applications [J3-8]. In some cases [HW1,W4] one uses the second and fourth entries of the minor arc vector $\underset{\sim}{y}$.

The latest results for the sum S of (1) are

$$S = O[M^{1/2}T^{89/560+\varepsilon}]$$

by WATT [W3], giving $\theta = 89/560$ in the Lindelöf problem, and a corresponding bound for sums with an exponential and a Dirichlet character [W4], and

$$S = O[M^{1/2}T^{11/70+\varepsilon}]$$

for M near $T^{1/2}$ by HUXLEY and KOLESNIK [HK]. The latter result gives only $\theta = 17/108$ in the Lindelöf problem ($11/70 = 0.1571\ldots < 17/108 = 0.1574\ldots < 89/560 = 0.1589\ldots < 9/56 = 0.1607\ldots$).

For the sum (5) IWANIEC and MOZZOCHI [IM] and HUXLEY [H3] get

$$O\left[HT^{1/4+\varepsilon}\left[\frac{HT}{M}\right]^{1/10}\right] ,$$

which becomes $O(M)$ for

$$H = O(MT^{-7/22-\varepsilon}) ,$$

and gives the discrepancy estimate $O(R^{7/11+\varepsilon})$ for the circle [IM] or a more general smooth closed curve [H3]. The same bound for the sum (6) in [HBH] estimates the integral (7) as $O(MU)$ for

$$U \geq T^{7/22+\varepsilon} .$$

It leads indirectly to

$$\int_0^T |\zeta(1/2+it)|^2 = T(\log T/2\pi + 2\gamma - 1) + O(T^{7/22+\varepsilon}) .$$

This implies $\theta = 7/44$ in the Lindelöf problem, but $7/44 = 0.1590\ldots$ is worse than WATT's 89/560 [W3].

JUTILA [J1-8] has many bounds for the sum (3). Two applications are

$$\sum_{\chi \bmod D} \int_T^{T+T^{2/3}} \left|L\left[\frac{1}{2}+it,\chi\right]\right|^4 dt = O(DT^{2/3}(DT)^\varepsilon)$$

where $\tau(n)$ is the divisor function and

$$\sum_{\chi \bmod D} \int_T^{T+T^{2/3}} \left|\varphi\left[\frac{1}{2}+it,\chi\right]\right|^2 dt = O(DT^{2/3}(DT)^\varepsilon)$$

where $\tau(n)$ is the Fourier coefficient of a modular form, and $\varphi(s,\chi)$ is its Hecke L-series, normalised to have critical line Re $s = 1/2$.

FOUVRY and IWANIEC [FI1] have also considered using steps B and A without subdivision, but in several variables, provided that the monomial condition holds in each variable. This idea should give new exponents in some classical problems.

Finally, in the spirit of the Journées, some problems. If

$$y = f(x), \quad M \leq x \leq 2M$$

is a smooth curve, then there are connections between

a) the exponential sum S of (1) over the interval M to $2M-1$,
b) the rounding error sum $\sum \rho(f(m))$ over the same interval,
c) the number of integer points within a distance δ of the curve,
d) the number of integer points on the curve.

BOMBIERI and PILA [BP] have an upper bound $O(T^{\epsilon})$ for problem (d). For (a) there is the classical Van der Corput iteration, whilst for (c) there is an analogous elementary iteration [H5], in which Step A is differencing, and Step B is interchanging the variables x and y. Is there an iteration for the rounding error (b) ?

What conditions ensure that the Diophantine approximations to f, f' and f'' at integer values of x are independent ? A quantitative result could allow one to count coincidences among minor arc $\underset{\sim}{y}$ vectors properly, or even to avoid putting moduli round the minor arc sums.

Are there any counterexamples of curves which are not rational algebraic curves of genus zero and of low degree, but for which the sums (a), (b) or (c) are unexpectedly large ?

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