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COHOMOLOGICAL p–NILPOTENCE CRITERIA FOR COMPACT LIE GROUPS

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Introduction

In [Q1] Quillen discussed cohomological criteria for p–nilpotence of finite groups. He proved that for odd primes p a finite group G is p–nilpotent if and only if the restriction map from the mod p cohomology $H^*(G;\mathbb{F}_p)$ to the mod p cohomology $H^*(G_p;\mathbb{F}_p)$ of a p–Sylow subgroup G_p is an F–isomorphism. Recall that a map $A \xrightarrow{\varphi} B$ of graded $\mathbb{F}_p$ algebras is called an F–isomorphism if and only if $a \in \mathrm{Kern}\varphi$ implies $a^n = 0$ for some n and for each $b \in B$ some power b^{p^n} is in the image of φ [Q2]. Furthermore Quillen sketched a proof of the following result which he attributed to Atiyah: If p is any prime and $H^i(G;\mathbb{F}_p) \to H^i(G_p;\mathbb{F}_p)$ is an isomorphism for all sufficiently large i, then G is p–nilpotent.

Quillen's main result in [Q2] can be interpreted as follows: For a compact Lie group G with classifying space BG the F–isomorphism type of $H^*(BG;\mathbb{F}_p)$ is determined by the sets $\mathrm{Rep}(V,G)$ of G–conjugacy classes of homomorphisms from elementary abelian p–groups V to G [HLS]. In particular, one can rephrase Quillen's p–nilpotence criterion in the following form: For an odd prime p a finite group G is p–nilpotent if and only if inclusion induces a bijection $\mathrm{Rep}(V,G_p) \xrightarrow{i} \mathrm{Rep}(V,G)$ for all elementary abelian p–groups V ([HLS; Prop. 4.2.3.]).

If G is a compact Lie group with maximal torus T, normalizer NT, Weyl group $W(G) = NT/T$, then G_p will denote the preimage of W_p in NT. In this case G_p will be called a p–Sylow normalizer and is known to be a good substitute for a p–Sylow subgroup.

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In this paper we give for odd primes a characterization of those compact Lie groups G for which $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ is a bijection for all V, or equivalently $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is an F–isomorphism (Theorem 2.1.). The possibility of such a characterization was already mentioned in [HLS, Sect. 4.2.5.]. It seems appropriate to call such groups p–nilpotent compact Lie groups. We will also generalize Atiyah's criterion to the compact Lie group case (Theorem 2.5.). Our interest in such characterizations comes from the importance of BG_p for the study of the (stable) homotopy type of BG.

The paper is organized as follows. In section 1 we give the precise definition of a p–nilpotent compact Lie group and discuss some properties of such groups. We do not intend a systematic group theoretical study of this concept but will rather concentrate on properties which are relevant for our cohomological characterizations. These characterizations are stated and proved in section 2.

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1. p–nilpotent compact Lie groups

1.1 DEFINITION. *A compact Lie group G is called p–nilpotent if and only if there is a finite normal subgroup N of order prime to p which together with G_p generates G.*

1.2 REMARKS.

(a) For finite groups this reduces to the classical definition of p–nilpotence. Then N consists of all elements of order prime to p and G/N is isomorphic to G_p, i.e. G is a semidirect product $N \rtimes G_p$. In this case N is also called the normal p complement of G_p in G.

(b) In the compact Lie group case G is in general not a semidirect product. For example, if $G = \langle S^1,\ x, y \mid [x, S^1] = [y, S^1] = x^3 = y^3 = 1,$ $[x, y] = \zeta$ with ζ a primitive 3rd root of unity in $S^1\rangle$ and $p \neq 3$, then

$G_p = S^1$ and the normal subgroup $N = \langle x, y\rangle$ shows that G is p–nilpotent. However, $N \cap G_p \neq \{1\}$ and hence $G \not\cong N \rtimes G_p$. It is also obvious that G is not a semidirect product $\widetilde{N} \rtimes G_p$ for some other $\widetilde{N} \triangleleft G$.

Our definition of p–nilpotence above will be justified by the results below, which together with this example show that it would not be adequate to require the existence of a finite normal p–complement in the compact Lie group case.

1.3 PROPOSITION. *Let G be a compact Lie group and p be any prime. Then the following statements are equivalent.*

(a) *G is p–nilpotent.*

(b) $\text{Rep}(Q, G_p) \xrightarrow{i} \text{Rep}(Q, G)$ *is a bijection for all p–groups Q.*

(c) *If Q is any finite p–subgroup of G, then $N_G(Q)/C_G(Q)$, the quotient of the normalizer of Q in G by the centralizer of Q in G, is a finite p–group.*

(d) *Each finite subgroup H of G is p–nilpotent.*

(e) *G is a finite extension of a torus, i.e. there exists an exact sequence $T \hookrightarrow G \twoheadrightarrow \pi$ with π finite, and G has a finite p-nilpotent subgroup H with $H/H \cap T = \pi$ and $T_p = \{t \in T \mid t^p = 1\} \subset H$.*

(f) *G is an extension of a torus by a finite p–nilpotent group π and the conjugation action of the normal p–complement ν of π_p in π is trivial on T.*

Proof. (a) $\Rightarrow$ (b): Onto is equivalent to saying that any p–subgroup Q of G is conjugate to a subgroup of G_p, i.e. that the Q–set G/G_p has a nonempty Q–fixed point set $(G/G_p)^Q$. This follows from $\chi((G/G_p)^Q) \equiv \chi(G/G_p) \not\equiv 0 \bmod p$ where χ denotes Euler characteristic (cf. [HLS; Prop. 4.2.1.]).

To show that i is $1-1$ consider the projection $G_p \overset{\pi}{\twoheadrightarrow} G_p/G_p \cap N \cong G/N$. It suffices to show that π induces an injection on $\text{Rep}(Q, ?)$. So let α_1, α_2 be two homomorphisms with $\pi\alpha_1 = g\pi\alpha_2 g^{-1}$ for some $g \in G_p$. By factoring out the kernel we may assume that $\pi\alpha_1$ is mono. Identify Q with its image in $G_p/G_p \cap N$. Then α_1 and $g\alpha_2 g^{-1}$ are sections of $\pi^{-1}(Q) \overset{\pi}{\twoheadrightarrow} Q$. Now $\text{Kern}\pi = G_p \cap N$ is a subgroup of T of order prime to p and hence

$H^1(Q, G_p \cap N) = 0$, i.e. α_1 and $g\alpha_2 g^{-1}$ are even conjugate by an element in $G_p \cap N$ and we are done.

(b) $\Rightarrow$ (c): For any group G the automorphism group $\mathrm{Aut}(Q)$ acts on $\mathrm{Rep}(Q, G)$. If Q is a subgroup of G, then $N_G(Q)/C_G(Q)$ identifies naturally with the isotropy subgroup of the inclusion $Q \hookrightarrow G$, considered as an element in the $\mathrm{Aut}(Q)$–set $\mathrm{Rep}(Q, G)$.

Now (b) implies that we can assume that Q is a subgroup of G_p and that it suffices to show that $N_{G_p}(Q)/C_{G_p}(Q)$ is a p–group. So suppose that $x \in N_{G_p}(Q)$ has order prime to p in $N_{G_p}(Q)/C_{G_p}(Q)$. As in [HLS, sect. 4.3.] we may assume that x itself has order prime to p, i.e. $x \in T$. Then one sees as in [HLS, Lemma 4.3.3.] that x acts trivially on the quotient of Q by its Frattini–subgroup $\phi(Q)$ and hence trivially on Q (cf. [H, Satz III 3.18.]). Therefore x is in $C_{G_p}(Q)$ and we are done.

(c) $\Rightarrow$ (d): If Q is a subgroup of H, then $N_H(Q)/C_H(Q)$ is a subgroup of $N_G(Q)/C_G(Q)$ and hence the Frobenius criterion [H, Satz IV, 5.8.] implies that H is p–nilpotent.

For the remaining implications we need a Lemma. For a natural number ℓ let T_ℓ denote $\{t \in T \mid t^\ell = 1\}$.

1.4 LEMMA. *Let G be an extension of a torus T by a finite group π of order $|\pi|$. Then there is a finite subgroup F of G with $F/F \cap T = \pi$ and $F \cap T = T_{|\pi|}$.*

Proof. Interpret the (class of the) extension $T \hookrightarrow G \twoheadrightarrow \pi$ as an element $[e] \in H^2(\pi; T)$ and use that $|\pi| \cdot [e] = 0$ together with the long exact cohomology sequence arising from the short exact sequence $T_{|\pi|} \hookrightarrow T \xrightarrow{\bullet|\pi|} T$ of π–modules.

□

We continue with the proof of Proposition 1.3.

(d) $\Rightarrow$ (e): Assume that G is not a finite torus extension. Then $G_{(1)}$, the connected component of 1, is not abelian and hence contains a compact connected nonabelian Lie group of rank 1, i.e. either $SO(3)$ or $SU(2)$. Now $SO(3)$ contains A_4, the alternating group on four letters, as symmetry group of a regular tetrahedron. As neither A_4 nor its twofold cover in $SU(2)$ are 2–nilpotent, we may assume that p is odd. Next consider $\widetilde{G} := NT \cap G_{(1)}$. This is a finite torus extension, so there is a finite subgroup $\widetilde{F}$ as in Lemma 1.4. Let $\widetilde{H}$ be the finite subgroup of G, generated by $\widetilde{F}$ and T_p (finite because T_p is normal). If $G_{(1)} \neq T$, then the Weyl group $W(G_{(1)})$ is nontrivial. Pick a reflection in $W(G_{(1)})$ and represent it by an element $r \in \widetilde{H}$. Then r defines a nontrivial element of order 2 in $N_{\widetilde{H}}(T_p)/C_{\widetilde{H}}(T_p)$ and hence $\widetilde{H}$ is not p–nilpotent.

We conclude that $G_{(1)}$ is a torus and G is a finite torus extension. Now let $F \subset G$ be as in 1.4. Then $H = \langle F, T_p \rangle$ is the finite group with the desired properties.

(e) $\Rightarrow$ (f): If N is the normal p complement of H_p in H, then $N/N \cap T$ is the normal p complement of π_p in π. Therefore it suffices to show that N commutes with T. Now N and T_p are both normal in H and have trivial intersection, hence they commute. Finally, a smooth automorphism of T which fixes T_p is clearly trivial, if p is odd, or has order at most 2, if $p = 2$. Hence N commutes with T and we are done.

(f) $\Rightarrow$ (a): Let G' be the preimage in G of the normal p complement ν. Then Lemma 1.4 gives a subgroup F' of G' with $F'/F' \cap T = \nu$ and $F' \cap T = T_{|\nu|}$, where $|\nu|$ is the order of ν. Clearly, F' is a finite group of order prime to p which together with G_p generates G. However, F' need not be normal.

Therefore consider the subgroup $N = \langle F', T_{|\nu|^2} \rangle \subset G$. This is still a finite group of order prime to p. We claim that N is normal. For this it suffices to show that $gF'g^{-1} \subset N$ for all $g \in G$. So let x be in F'. Then $gxg^{-1} = yt$ for some $y \in F'$, $t \in T$, since ν is normal in π. It suffices to show that $t^{|\nu|^2} = 1$. This follows because the order of elements in F' clearly divides $|\nu|^2$ and because y commutes with t by assumption.

This finishes the proof of 1.3.

□

2. Cohomological p–nilpotence criteria

Before we state our main result we recall that a subgroup V of G_p is said to be weakly closed in G_p with respect to G if $gVg^{-1} \subset G_p$, $g \in G$, implies $gVg^{-1} = V$.

2.1 THEOREM. *Let G be a compact Lie group and p be an odd prime. Then the following statements are equivalent.*

(a) *G is p–nilpotent.*

(b) $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ *is bijective for all elementary abelian p–groups V.*

(c) *Let V be any normal elementary abelian p–subgroup of G_p which contains T_p. Then V is weakly closed in G_p with respect to G and $N_G(V)/C_G(V)$ is a finite p–group.*

2.2 REMARKS.

(a) We recall that condition 2.1.(b) is equivalent to the map $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ being an F isomorphism. In fact, a transfer argument shows that this map is mono for all compact Lie groups G. If G is also p–nilpotent then the Leray–Serre spectral sequence of the fibration $B(N \cap G_p) \to BG_p \to B(G_p/G_p \cap N) = B(G/N)$ with mod p acyclic fibre shows that $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is also onto and hence a genuine isomorphism.

(b) In the finite case condition 2.1.(c) above gives just Quillen's group theoretical version of his p–nilpotence criterion ([Q1, Thm. 1.5.]). The proof of implication (c) $\Rightarrow$ (a) below is essentially a careful modification of the proof of Theorem 1.5. in [Q1].

(c) For $p = 2$ there are examples of compact Lie groups G which satisfy conditions 2.1.(b) and 2.1.(c) but which are not 2–nilpotent. $G = SU(2)$ is an example of a connected group and $G = Q_8 \rtimes \mathbb{Z}/3$, the semidirect product of the quaternion group with $\mathbb{Z}/3$ (cf. [Q1]), is an example of a finite group.

A cohomological criterion for p–nilpotence that works for all primes will be given below in Theorem 2.5.

<u>Proof of Theorem 2.1.</u>

<u>(a) $\Rightarrow$ (b):</u> This follows from Proposition 1.3.

<u>(b) $\Rightarrow$ (c):</u> Clearly, (b) implies that a normal elementary abelian p–subgroup V of G_p is weakly closed with respect to G. The proof of Proposition 1.3. ((b) $\Rightarrow$ (c)) shows that $N_G(V)/C_G(V)$ is a p–group.

(c) $\Rightarrow$ (a): If G is not a finite torus extension, then we see as in the proof of Proposition 1.3. ((d) $\Rightarrow$ (e)) that $N_G(T_p)/C_G(T_p)$ contains a nontrivial element of order 2 in contradiction to our assumptions.

Therefore G is a finite torus extension. Denote G/T by π and let F be a finite subgroup of G with $T \cap F = T_{|\pi|}$ and $F/F \cap T = \pi$ as in Lemma 1.4. By criterion (e) of Proposition 1.3. it suffices to show that the finite group $H = \langle F, T_p \rangle$ is p–nilpotent.

We pick a p–Sylow subgroup H_p of H which is contained in G_p.

2.3 LEMMA. *Let V be any abelian subgroup of H (resp. H_p) which contains T_p. Then V is normal in H (resp. H_p) if and only if V is normal in G (resp. G_p), provided p is odd.*

Proof. Suppose V is abelian and contains T_p. Then V commutes with T_p and hence with T (p is odd!). Therefore, if H normalizes V, then $\langle H, T\rangle = G$ normalizes V. Similarly with H_p and G_p. The converse is trivial.

□

We return to the proof of 2.1. ((c) $\Rightarrow$ (a))

Lemma 2.3 implies that any normal elementary abelian p–subgroup V of H_p containing T_p is weakly closed in H_p with respect to H. Furthermore, $N_H(V)/C_H(V)$ is a subgroup of $N_G(V)/C_G(V)$, in particular a p–group.

Therefore, the p–nilpotence of H is a consequence of the following slight generalization of Quillen's Theorem 1.5. in [Q1].

2.4 PROPOSITION. *Let p be an odd prime and G be a finite group with p–Sylow subgroup G_p. Let U be a normal elementary abelian p–subgroup of G and assume that each normal elementary abelian p–subgroup V of G_p containing U is weakly closed in G_p with respect to G and that $N_G(V)/C_G(V)$ is a p–group for such V. Then G is p–nilpotent.*

Proof of 2.4. The proof is almost the same as in [Q1]. For the convenience of the reader we repeat the main steps.

The hypothesis of 2.4. are inherited by all subgroups of G which contain G_p. Therefore we can do induction on the order of such subgroups.

Let V be a subgroup of G_p which contains U and is maximal with respect to being elementary abelian and normal in G_p. Then V is a maximal elementary abelian subgroup of G (cf. [Q1, Prop. 4.1.]) and hence $C_G(V)$ is p–nilpotent by [H, Satz IV, 5.5.]. Now there are two cases:

Case 1: V is normal in G. Then G is p–nilpotent because $C_G(V)$ is p–nilpotent and $G/C_G(V) = N_G(V)/C_G(V)$ is a p–group.

Case 2: V is not normal in G. Then let W be a maximal G–normal subgroup of V which contains U. Define subgroups V_1 of V and N of G by

$$V_1/W = V/W \cap Z(G_p/W) \quad (Z \text{ denotes the center})$$
$$N = N_G(V_1).$$

Then everything works precisely as in [Q1].

- N contains G_p and is properly contained in G, hence N is p–nilpotent by induction.
- V_1/W is a central subgroup of G_p/W which is weakly closed with respect to G/W. Therefore, Grün's Theorem implies $H^1(G/W) \xrightarrow{\cong} H^1(N/W)$ and the cohomology 5–term exact sequences of the group extensions $W \hookrightarrow G \twoheadrightarrow G/W$, $W \hookrightarrow N \twoheadrightarrow N/W$ yield $H^1(G) \xrightarrow{\cong} H^1(N)$.
- Finally, Tate's H^1–criterion [T] implies that G is p–nilpotent.

□ □

The following result generalizes Atiyah's p–nilpotence criterion and is valid for all primes.

2.5 THEOREM. *Let G be a compact Lie group and suppose inclusion induces an isomorphism $H^i(BG;\mathbb{F}_p) \to H^i(BG_p;\mathbb{F}_p)$ for all sufficiently large i. Then G is p–nilpotent.*

Proof. By a transfer argument (cf. [Cl] for the existence of a stable transfer map) there is a p–local stable splitting $BG_p \underset{(p)}{\simeq} BG \vee X$ for some p–local connected X with bounded above and finite type mod p homology. Now G_p is a finite torus extension. Let F be a finite subgroup of G_p as in Lemma 1.4. If T_{p^∞} denotes the subgroup of T consisting of all torsion elements of order a power of p, then the inclusion $\langle T_{p^\infty}, F\rangle \hookrightarrow G_p$ induces a mod p homology equivalence and therefore there is for each n a finite p–subgroup F_n of $\langle T_{p^\infty}, F\rangle$ such that inclusion induces an epimorphism $H_i(BF_n; \mathbb{F}_p) \rightarrow H_i(BG_p; \mathbb{F}_p)$ for all $i \leq n$. In particular, there exists n such that there is a stable map $BF_n \rightarrow X$ (after localizing at p) which is onto in mod p homology. Now the solution of the Segal conjecture [Ca] forces X to be trivial because there are no nontrivial stable maps from BF_n to any positive dimensional sphere. We conclude that $H^i(BG; \mathbb{F}_p) \rightarrow H^i(BG_p; \mathbb{F}_p)$ is an isomorphism for all i.

For $i = 1$ we get

(2.6)

$$H^1(BG; \mathbb{F}_p) \cong \mathrm{Hom}(H_1(BG); \mathbb{F}_p) \cong \mathrm{Hom}(\pi_1(BG); \mathbb{F}_p) \cong \mathrm{Hom}(\pi_0(G); \mathbb{F}_p)$$

and therefore we have a bijection

$$\mathrm{Hom}(\pi_0(G); \mathbb{F}_p) \rightarrow \mathrm{Hom}(\pi_0(G_p); \mathbb{F}_p). \tag{2.7}$$

Because of Theorem 2.1 (cf. remark 2.2) we may assume $p = 2$. The determinant of the adjoint representation of a 2–Sylow normalizer G_2 on the Lie algebra LT defines a homomorphism $\pi_0(G_2) \xrightarrow{\varphi} \mathbb{F}_2$. If T is properly contained in $G_{(1)}$, the connected component of $1 \in G$, then the reflections in the Weyl group $W(G_{(1)})$ show that φ restricts nontrivially to $\pi_0(G_2 \cap G_{(1)})$ and can therefore not come from $\pi_0(G)$. It follows that $T = G_{(1)}$ and G is a finite torus extension.

Now (2.6), (2.7) and Tate's H^1–criterion imply that $\pi_0(G) = G/T$ is 2–nilpotent. By Proposition 1.3.(f) it suffices therefore to show that odd order elements of $\pi_0(G)$ act trivially on T.

Our hypothesis implies certainly that $H^*(BG; \mathbb{F}_p) \rightarrow H^*(BG_p; \mathbb{F}_p)$ is an F–isomorphism, hence $\mathrm{Rep}(V, G_p) \rightarrow \mathrm{Rep}(V, G)$ is bijective for all elementary abelian p–groups V and therefore $N_G(T_p)/C_G(T_p)$ is a p–group by the proof of Proposition 1.3.((b) $\Rightarrow$ (c)). For $p = 2$ it follows that odd order elements of $\pi_o(G)$ act trivially on T_2 and hence on T (cf. proof of Proposition 1.3. ((e) $\Rightarrow$ (f))). This finishes the proof of 2.5.

□

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