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A CRITERION FOR COMPLETE REDUCIBILITY AND SOME APPLICATIONS

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Introduction

The representations of finite groups of Lie type and characteristic p over fields of the same characteristic are a natural by-product of their construction. If $\overline{G}$ is a reductive group over an algebraically closed field k with σ a non bijective endomorphism, $G := \overline{G}^{\sigma}$ is finite of Lie type, and (rational) representations of $k[\overline{G}]$ give rise to representations of the group algebra kG by restriction. Those modular representations of finite groups of Lie type in natural characteristic also provide a decisive tool in the classification of certain geometric situations through the action of known (Lie type) groups G on elementary p-groups (see references in [10], [14]), especially when the action is shown to be completely reducible. The modules one has to consider are often generated by fixed points under a Sylow p-subgroup of G (condition (A) in [14]).

In the present talk we shall use the theory of representations of modular Hecke algebras to study this type of modular representations of G. Versions in characteristic p of the Hecke algebras were introduced in the mid 70's through independent works by Cline, Parshall and Scott on cohomology of kG-modules, by Carter and Lusztig ([6]) and Sawada ([12]) on submodules of $Y := \mathrm{Ind}_U^G k$. The algebra one considers is $\mathcal{H}_k := \mathrm{End}_{kG}\mathrm{Ind}_U^G k$, a modular analogue of the standard Hecke algebra where the Borel subgroup is replaced by a Sylow p-subgroup U. Sawada showed that a decompositon of $\mathcal{H}_k$ gives

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rise to a remarkable bijection between simple modules for $\mathcal{H}_k$ and for kG, leading to a good description of the summands of $\mathrm{Ind}_U^G k$ (see also [17], [5]). J.A. Green gave in [7] a generalization where the functor $F = \mathrm{Mor}_{kG}(Y,.)$ from kG-**mod** to **mod**-$\mathcal{H}_k$ produces the bijection between irreducibles.

For representations satisfying (A), we use F to study representations not reducible *a priori*. This approach may be useful to reach representations not obtained from (rational) representations of $k[\overline{G}]$. We prove a complete reducibility criterion (Theorem 9) with applications to the reducibility problems mentioned above. For instance, one proves that if a kG-module and its dual satisfy (A) (see also condition $(*)$ of [3]), it is semi-simple if, and only if, its modules of fixed points for every radical of proper parabolic are semi-simple (see **3.1**). This applies well to the study of Ronan-Smith sheaves (**3.2**). From our study of $\mathrm{Ext}^1_{\mathcal{H}_k}$ groups ([5]) we derive a condition on composition factors of kG-modules satisfying $(*)$ which is reminiscent of the linkage principle for reductive groups ([9] II.6) : here, like in the theory of representations in characteristic zero, Hecke algebras help bringing the discussion to the level of Weyl groups. Modules of Jordan-Hölder length 2 satisfying $(*)$ (see (B) in [14]) are studied (Theorem 16). We also show (Theorem 18) that certain "selfextensions" occur only for symplectic groups over prime fields of odd characteristic (compare [8]). All those results stem easily from the representation theory of $\mathcal{H}_k$, and no case-by-case analysis is involved. We use the hypothesis that k is algebraically closed only when necessary : representations over finite fields seem more natural in geometry.

We should mention that the difficult problem of Ext^1_{kG} groups for simple modules (see [1], [2]) seems to us still beyond the reach of those methods : while the corresponding $\mathrm{Ext}^1_{\mathcal{H}_k}$ groups are known ([5]), the functor F gives informations only on a little bit of kG-**mod**, especially when k^σ is big, and concerning the whole categories of modules it seems that kG-**mod** resembles much more $k[\overline{G}]$-**mod** (see for instance [1]) than **mod**-$\mathcal{H}_k$. After this talk was given we noticed that the above "little bit" is a subcategory (see **1**) in the sense of abelian categories : F induces an equivalence between **mod**-$\mathcal{H}_k$ and the full subcategory in kG-**mod** of modules satisfying $(*)$ denoted by kG-$\mathbf{mod}_Y$ (see Theorem 2). This explains why the basic traits of **mod**-$\mathcal{H}_k$ (generation of the radical, presentation as an amalgam of the subalgebras for the rank two, Ext^1 groups) described in [5] may be found in kG-**mod**. The equivalence holds as soon as the modular Hecke algebra is self-injective, thus extending the results of [7]. Unfortunately, kG-$\mathbf{mod}_Y$

is not stable by quotient in kG-**mod**. An exception is the case of $\mathrm{SL}_2(p)$, this is treated in an Appendix.

Notations

Through the whole talk we fix p a prime, k a field of characteristic p, G a finite group with a split BN-pair of characteristic p. We recall the standard relevant notations and results.

The subgroup B of G decomposes as $B = U \rtimes T$ where U is a Sylow p-subgroup of G, T is an abelian p'-group. We denote by $W = N/T$ the Weyl group of G, R its set of generators, Φ the corresponding root system of basis Δ and same Weyl group : let $\alpha \mapsto r_\alpha \in R$ be the associated indexation. The rank of G is the cardinality of R. If α, $\beta \in \Delta$, we denote by α—β the adjacency in the Dynkin diagram. If $\alpha \in \Phi$, we denote by X_α the associated root subgroup. If S is a subset of Φ, one sets $X_S :=< X_\alpha \, ; \alpha \in S >$. Then $U = X_{\Phi^+}$, $U^- = X_{\Phi^-}$. If $I \subseteq \Delta$, one sets $P_I := BW_IB = U_I \rtimes L_I$, where $U_I = X_{\Phi^+ \setminus \Phi_I^+}$ and $L_I = TX_{\Phi_I}$ is the "Levi subgroup". Let $G_I = X_{\Phi_I}$. Then the split BN-pair of G endows G_I and L_I with split BN-pairs of rank $|I|$ by intersection.

We use the standard notations for modules. If A is a k-algebra, M a (left) A-module, S a subset of M and A' a subset of A, we denote by $A'.S$ or $A'S$ the vector space $< as; a \in A' \, , s \in S >$. We denote by $S^{A'}$ the set of fixed points $\{s \in S; \forall a \in A' \; as = s\}$. If A' is a subalgebra of A, one denotes by $\mathrm{Res}_{A'} M$ the restriction of M to A'. The radical of A is denoted by $J(A)$. All those notations have clear analogues for right modules.

Let us recall that if X is a p-group, $J(kX) = \{\sum_{x \in X} \lambda_x x; \sum \lambda_x = 0\}$ and every non zero kX-module M satisfies $M^X \neq 0$.

1. Self-injective endomorphism rings and equivalence of categories.

The main result of this talk may be proved in a more general context than the one of modular representations of split BN-pairs. The following general framework is inspired by the one set by Green in [7] : see hypothesis B below. Our result implies Green's theorem.

Let R be a finite dimensional ring over a field, we consider R-**mod** the category of (finitely generated) R-modules. Let Y be an R-module. One denotes by E the endomorphism algebra $\mathrm{End}_R Y$ and by F the functor from R-**mod** to **mod**-E defined by $F(V) = \mathrm{Mor}_R(Y, V)$, where E acts on $F(V)$

by composition on the right. Then $F(Y) = E_E$ (E considered as right E-module) and, if $l \geq 1$, $\mathrm{Mor}_E((E_E)^l, F(V)) = F(\mathrm{Mor}_R(Y^l, V))$ (see [7] 2.1 (a)). If $M \subseteq F(V)$, we denote $MY := \sum_{m \in M} m(Y) \subseteq V$.

We now define a full subcategory of R-**mod** associated to Y.

Definition 1. Let R-$\mathbf{mod}_Y$ be the full subcategory of R-**mod** whose objects are the R-modules V satisfying
(**) there exist $l \geq 1$ and $e \in \mathrm{End}_R Y^l$ such that $V \simeq e(Y^l)$.

If $V = e(Y^l)$ as above, $F(V)$ contains e composed with all the coordinate maps $Y \to Y^l$, hence $F(V)Y = V$. The condition (**) is clearly equivalent to the fact that V is isomorphic to a submodule of some power of Y and to a quotient of some power of Y. From now on, we assume the following (see [7]) :
B. E is self-injective (or quasi-Frobenius), that is, E_E is *injective.*

Theorem 2. *Let Y, E be as above and Y satisfy B. Then F is an equivalence of additive categories from R-$\mathbf{mod}_Y$ to $\mathbf{mod}$-E.*

Corollary 3. *Assume the hypotheses of the Theorem. Let V be an R-module, then :*
(i) if V satisfies (**) and $F(V) = M \oplus N$ as right E-module, then $V = MY \oplus NY$ as R-module.
(ii) if V satisfies (**), V is indecomposable if, and only if, $F(V)$ is indecomposable.

Corollary 4. *Assume moreover that every simple R-module occurs in both soc Y and hd Y. Then the simple R-modules are in R-$\mathbf{mod}_Y$. Moreover*
(i) (Green) F induces a bijection between isomorphism type of simple R-modules and simple E-modules,
(ii) an R-module V is semi-simple if, and only if, it satisfies (**) and $F(V)$ is semi-simple.

We need the following

Lemma 5. *Assume Y satisfies hypothesis B. Let $l \geq 1$ and $M \subseteq F(Y^l) = (E_E)^l$ be a right E-submodule. Then $F(MY) = M$.*

Proof of the Lemma. The proof is very similar to that of 1(ii) in [7]. One has clearly $MY \subseteq Y^l$ and $M \subseteq F(MY) \subseteq (E_E)^l$ as right E-modules. Let us assume $F(MY)/M \neq 0$. Then there exists a right E-module N such that $M \subset N \subseteq F(MY) \subseteq (E_E)^l$ and N/M is simple. Thanks to hypothesis B, N/M injects in E_E (see [7] 2.3 (b)). So there is a non-zero map $f : N \to E_E$ such that $f(M) = 0$. By injectivity of E_E, f extends to $(E_E)^l$: $\widehat{f} : (E_E)^l \to E_E$. But then $\widehat{f}$ is under the form $\widehat{f} = F(e)$ where $e \in \mathrm{Mor}_R(Y^l, Y)$, and the hypothesis on f implies $e(MY) = 0$, $e(NY) \neq 0$. But $MY \subseteq NY \subseteq F(MY)Y \subseteq MY$, so $NY = MY$, a contradiction.

Proof of the Theorem. Let M be a right E-module. Since E is self-injective, it is standard that M injects into a free module : it is true for soc M ([7] 2.3 (b)), then such an injection extends to all M by injectivity of free E-modules (B). Now, if M is a submodule of some $(E_E)^l$, Lemma 5 applies : $M = F(V)$ where $V = MY$. Moreover V is a submodule of Y^l and a quotient of some power of Y since M is finite dimensional, hence V satisfies $(**)$. There remains to check that F is faithful and full. Let V, V' be R-modules satisfying $(**)$; one must check that F induces an isomorphism of vector spaces between $\mathrm{Mor}_R(V, V')$ and $\mathrm{Mor}_E(F(V), F(V'))$. Obviously F is linear. If $f \in \mathrm{Mor}_R(V, V')$ is in its kernel, then $f(F(V)Y) = 0$ by definition of F, but $F(V)Y = V$ by $(**)$, so $f(V) = 0$ and $f = 0$: F is injective. In order to check surjectivity, one may assume that $V = e(Y^l)$, $V' = e'(Y^l)$ for $e, e' \in \mathrm{End}_R(Y^l)$. Then $F(V)$ and $F(V')$ are submodules of $(E_E)^l$. Let $g \in \mathrm{Mor}_E(F(V), F(V'))$. By injectivity of $(E_E)^l$, g extends to $\widehat{g} \in \mathrm{Mor}_E((E_E)^l, (E_E)^l) = F(\mathrm{Mor}_R(Y^l, Y^l))$, so $\widehat{g} = F(\widehat{f})$ for $\widehat{f} \in \mathrm{End}_R(Y^l)$. We have $\widehat{f}(V) \subseteq V'$ since $\widehat{f}(V) = \widehat{f}(F(V)Y) = (\widehat{g}.F(V))Y = (g.F(V))Y \subseteq F(V')Y = V'$. Therefore $g = F(f)$, where $f : V \to V'$ is the restriction of $\widehat{f}$.

Proof of the Corollaries. (i) and (ii) of Corollary 3 now come from the fact that F induces a ring isomorphism between $\mathrm{End}_R(V)$ and $\mathrm{End}_E(F(V))$, hence bijects the idempotents.

Let us assume that all simple R-modules occur in hd Y and soc Y ; they are both quotients and submodules of Y, so they satisfy $(**)$. The equivalence then implies (i). For (ii), take V such that $(**)$ holds and $F(V)$ is semi-simple. One may assume there is some l such that $V \subseteq Y^l$ and $V = F(V)Y$. Thus V is a quotient of a product of R-modules under the form SY for a simple $S \subseteq F(Y^l) = (E_E)^l$. It suffices to check that each SY is simple. First $SY \neq 0$ since $S \neq 0$. Let X be a simple submodule of SY. Then X occurs in hd Y, so $F(X) \neq 0$. But $F(X) \subseteq F(SY) = S$ by Lemma 5, hence $F(X) = S$ and $SY \subseteq F(X)Y \subseteq X$, so SY is simple.

Conversely, assume that V is semi-simple. One must check $(**)$ and that $F(V)$ is semi-simple. We may assume that V is simple. V is a quotient and a submodule of Y, so it satisfies $(**)$. Let us check that $F(V)$ is simple. If S is a simple submodule of $F(V)$, then $0 \neq SY \subseteq F(V)Y = V \subseteq F(Y) = E_E$ so $SY = V$ and $S = F(SY) = F(V)$ by Lemma 5. This completes the proof of the corollary.

Remark. R-$\mathbf{mod}_Y$ is a subcategory of R-$\mathbf{mod}$ in the sense of additive categories but not in the sense of abelian categories : a quotient of modules satisfying $(**)$ may not satisfy $(**)$. If V satisfies $(**)$, its Jordan-Hölder length is generally bigger than that of $F(V)$, (see **3.3** Remark).

Remark. (April 1989) M. Auslander and C. Riedtmann have communicated to us alternative proofs of the theorem. Auslander proves a theorem which implies both Theorem 2 and the main equivalence of [18].

2. The case of split BN-pairs.

In this section, we apply the results above to finite groups G with a split BN-pair of characteristic p. We take $R = kG$ where k is a field of characteristic p, Y is the induced kG-module $\mathrm{Ind}_U^G k = kG \otimes_{kU} k$, where k stands for the one dimensional trivial kU-module. Then E is the algebra denoted by $\mathcal{H}_k$ in [4],[5]. A k-basis is indexed by N : $(a_n)_{n \in N}$ defined by $a_n(1 \otimes 1) = \sum_{g \in C} gn \otimes 1$ where C is a representative system of $U/U \cap nUn^{-1}$(see [4]7.2). If $\alpha \in \Delta$, one may choose $n_\alpha \in G_\alpha$ such that $n_\alpha T = r_\alpha$. Then the a_t's and the a_{n_α}'s for $t \in T$ and $\alpha \in \Delta$ generate $\mathcal{H}_k$ (see [5], [12]). Moreover the hypothesis (B) above is satisfied (see [16] 3.7).

If V is a kG-module, $F(V) := \mathrm{Mor}_{kG}(\mathrm{Ind}_U^G(k), V)$, and by Frobenius reciprocity, $F(V)$ may also be identified with the subspace V^U of V. This implies first that any simple kG-module is in hd Y, hence also in soc Y since Y is isomorphic to its dual : the additional hypothesis of Corollary 4 is satisfied. Through the above identification, the action of $\mathcal{H}_k$ is given by the following (see [4]10.1 or [7] footnote p.248) :

Proposition 6. *If $n \in N$ and $x \in V^U$, then $xa_n = Tr^U_{U \cap {}^nU}(nx)$. This gives $xa_t = tx$ when $t \in T$ and $xa_{n_\alpha} = \sum_{g \in X_\alpha} gn_\alpha x = (n_\alpha)^{-1} \sum_{g \in X_{-\alpha}} gx$.*

If $M \subseteq V^U$ is an $\mathcal{H}_k$-submodule, then MY (see **1**) is clearly $kG.M$.

2.1. Condition (*).

In this section we give some conditions equivalent to (**) in this case of split BN-pairs. The following lemma is used repeatedly. The idea is standard (first appearance is [11] 3.14 where M is a line).

Lemma 7. *Let V be a kG-module and M an $\mathcal{H}_k$-submodule of $F(V) = V^U$. Then $kGM = kU^-M$.*

Proof. We have to check that kU^-M is stable under G. G is generated by U^-, T and $\{n_\alpha\ ;\alpha \in \Delta\}$. M is stable under T which acts as $\{a_t; t \in T\} \subseteq \mathcal{H}_k$. Let now $\alpha \in \Delta, y \in U^-, m \in M$ and check $n_\alpha ym \in kU^-M$. The element y decomposes as ux with $u \in U_\alpha^-$ and $x \in X_{-\alpha}$, then $n_\alpha ym \in kU^- n_\alpha xm$ since n_α normalizes U_α^- . So one may assume $y = x \in X_{-\alpha}$.

One has $x \in L_\alpha = TX_\alpha \cup X_\alpha n_\alpha^{-1} TX_\alpha$ (Bruhat decomposition in L_α), with $TX_\alpha \cap X_{-\alpha} = 1$. If $x \neq 1$ then $x \in X_\alpha n_\alpha^{-1}TX_\alpha$, so $n_\alpha x \in X_{-\alpha}TX_\alpha$ and $n_\alpha xm \in kU^-M$. If $x = 1$, then one writes $n_\alpha m = n_\alpha \sum_{g\in X_{-\alpha}} gm - \sum_{x'\in X_{-\alpha}\setminus 1} n_\alpha x'm = n_\alpha^2(ma_{n_\alpha}) - \sum_{x'\in X_{-\alpha}\setminus 1} n_\alpha x'm \in kU^-M$ by the case $x \neq 1$, the hypothesis on M and $n_\alpha^2 \in T$.

In [3] we have studied kG-modules V satisfying the condition :

$$(*) \qquad V = V^U \oplus J(kU^-)V^U.$$

In [14] other conditions are studied, for instance :

$$(A) \qquad V = kG.V^U.$$

We define (A*) as (A) for the dual V^*, we also define :

$$(D) \qquad \dim_k V^U + \dim_k J(kU)V = \dim_k V.$$

The following proposition gives connections between them and (**) of **1**.

Proposition 8. *Let V be a kG-module, then the following conditions are equivalent :*
(**): V satisfies (**) for $Y = \mathrm{Ind}_U^G k$,
(A+D) : V satisfies (A) and (D),
(*) : $V = V^U \oplus J(kU^-)V^U$,

(A+A) : V and its dual V^* satisfy (A),*
(A+I) : V satisfies (A) and the following :

$$(I) \qquad \bigcap_{g\in G} gJ(kU)V = 0.$$

Proof. (**) and (A+A*) are equivalent : since Y is isomorphic to its dual, it suffices to check that V satisfies (A) if, and only if, it is isomorphic to a quotient of a power of Y. Y satisfies (A) since $1 \otimes 1 \in Y^U$, so do any power of Y. On the other hand (A) is clearly stable by quotient. Conversely, if $V = kG.V^U$, the map $g \otimes v \mapsto gv$ is an epimorphism from $\mathrm{Ind}_U^G(V^U)$ onto V and $\mathrm{Ind}_U^G(V^U)$ is $(\mathrm{Ind}_U^G k)^l$ for $l = \dim_k V^U$.

(A+D) and (*) are equivalent : (*) implies (A) and $V = V^U \oplus J(kU^-)V$ by [3]**P2**, so (D) is satisfied. Conversely, if (A) and (D) are satisfied, then $V = kU^- V^U$ (Lemma 7), so $V = V^U + J(kU^-)V^U = V^U + J(kU^-)V$. The second sum is direct thanks to (D), so is the first. This gives (*).

Concerning (A*), one has $(kG(V^*)^U)^\perp = \bigcap_{g\in G} gJ(kU)V$, so (A*) is equivalent to (I).

(*) implies (I) : if (I) is not satisfied, the corresponding intersection is a non zero kG-module, so it contains a non zero element fixed by U^-, therefore $V^{U^-} \cap J(kU)V \neq 0$. But $V = kU.V^{U^-}$ (Lemma 7), so $V^{U^-} \cap J(kU)V^{U^-} \neq 0$, a contradiction with (*).

(A+A*) implies (*) : One has to check that $V^U \cap J(kU^-)V^U = 0$. Otherwise, there is $0 \neq x \in V^{U^-} \cap J(kU)V^{U^-}$, so $\forall g \in U^-$, $x = gx \in gJ(kU)V$. So $(kU^-(V^*)^U)^\perp \neq 0$. This contradicts (A*) by Lemma 7.

Remark. If G is a reductive group over k and V provides a *rational* representation, the above Proposition is still true, but one can prove that V satisfies (*) if, and only if, it is semi-simple (using the density of the "big cell" U^-B). On the other hand, Weyl modules satisfy (A) but are not semi-simple in general (see [9] II.5).

2.2. A criterion for complete reducibility.

We now give the version for split BN-pairs of the criterion for complete reducibility of **1** Corollary 4. ii). Using Proposition 8, one may state :

Theorem 9. *Let V be a kG-module, then it is semi-simple if, and only if, the following three conditions are fulfilled :*
(A) $V = kG.V^U$,

(D) V^U *and* $V/J(kU)V$ *have the same dimension,*
(H) $F(V)$ *is a semi-simple* $\mathcal{H}_k$*-module.*

Remark. It is possible to prove Theorem 9 in a more elementary fashion, using just Lemma 7 and Proposition 8. (A), (D) and (H) are all necessary : see last remark in **3.4**.

2.3. Truncation functor.

Take now $I \subseteq \Delta$ and $P_I = U_I \rtimes L_I$ a parabolic subgroup with its Levi decomposition. Then L_I is a split BN-pair and we denote by kL_I-$\mathbf{mod}_Y$ the category obtained as in **1** from the induced module $\mathrm{Ind}_{U\cap L_I}^{L_I} k$. The corresponding endomorphism algebra identifies with the subalgebra $\mathcal{H}_k(L_I) = \oplus_{n\in N\cap L_I} k.a_n \subseteq \mathcal{H}_k(G)$. If V is a kG-module, let us denote by $T_{L_I}^G(V)$ the kL_I-module of fixed points V^{U_I}. Then, by [3] 3.2.(ii), $T_{L_I}^G$ is a functor from kG-$\mathbf{mod}_Y$ to kL_I-$\mathbf{mod}_Y$. Since $(T_{L_I}^G(V))^{U\cap L_I} = V^U$, the following is commutative :

$$\begin{array}{ccc} kG-\mathbf{mod}_Y & \xrightarrow{T_{L_I}^G} & kL_I-\mathbf{mod}_Y \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathbf{mod}-\mathcal{H}_k(G) & \xrightarrow{\mathrm{Res}} & \mathbf{mod}-\mathcal{H}_k(L_I) \end{array}$$

where Res is the restriction functor corresponding to the inclusion $\mathcal{H}_k(L_I) \subseteq \mathcal{H}_k(G)$. Moreover, Res sends simple modules to simple modules : by [5] Theorem 23, the simple $\mathcal{H}_k$-modules remain simple when restricted to $\oplus_{t\in T} k.a_t = \mathcal{H}_k(L_\emptyset) \simeq kT$. This is also true if k is algebraically closed and one further restricts to G_I : the simple $\mathcal{H}_k$-modules are then one-dimensional (see [5]). This, carried back to the level of $\mathbf{mod}_Y$ categories by F, gives the following version of Smith's theorem :

Theorem 10. *If V is a kG-module satisfying $(*)$, then $V^{U_I} = kG_I.V^U$ and it satisfies $(*)$ as kL_I-module (resp. kG_I-module). Moreover, if V is simple, V^{U_I} is simple as kL_I-module (resp. kG_I-module if k is algebraically closed).*

Remark. We did not assume k to be algebraically closed, so this also accounts for the rationality questions (compare [10] p.332).

3. Applications

We assume from now on that k is algebraically closed.

The complete reducibility criterion proved in **2.2** is now used together with results about $J(\mathcal{H}_k)$, $\mathrm{Ext}^1_{\mathcal{H}_k}$ groups and blocks of $\mathcal{H}_k$ (see [5]).

3.1. A reduction to groups of rank 2.

We obtain first some "relative" statements : we connect complete reducibility for G with the same question for its proper parabolic subgroups.

Let V be a kG-module and $I \subseteq \Delta$.

Let $V = V_1 \supset V_2 \supset \ldots \supset V_i \supset V_{i+1} \supset \ldots V_{l+1} = 0$ be a composition series with $V_i/V_{i+1} = Q_i$ a simple kG-module. Then $V^{U_I} = V_1^{U_I} \supseteq \ldots \supseteq V_i^{U_I} \supseteq V_{i+1}^{U_I} \supseteq \ldots V_{l+1}^{U_I} = 0$. Moreover $V_i^{U_I}/V_{i+1}^{U_I}$ is a kL_I-submodule of $Q_i^{U_I}$. By Theorem 10 $Q_i^{U_I}$ is simple, so $V_i^{U_I}/V_{i+1}^{U_I}$ is either 0 or $Q_i^{U_I}$.

We have proved :

F1. *If a composition series has quotient series $(Q_i)_{1\leq i\leq l}$, there is a composition series for V^{U_I} with quotient series extracted from $(Q_i^{U_I})_{1\leq i\leq l}$.*

If $I = \emptyset$, V_i^U/V_{i+1}^U is an $\mathcal{H}_k$-submodule of Q_i^U, so it is 0 or the line Q_i^U. Thus :

F2. *$F(V)$ has a composition series with quotient series extracted from $(F(Q_i))_{1\leq i\leq l}$.*

Theorem 11. *Let V be a kG-module, then V is semi-simple if, and only if, it satisfies (A+D) and*

$\forall \alpha, \beta \in \Delta$ such that $\alpha = \beta$ or α—β, $kG_{\alpha\beta}.V^U$ is a semi-simple $kG_{\alpha\beta}$-module.

Proof. If V is simple, the above conditions are satisfied thanks to Theorem 9 and Smith's Theorem (see Theorem 10 above), this proves the direct part of the Theorem.

Let us prove the converse. If V satisfies the above conditions, we have (A) and (D), so it remains to check that (H) holds. The right $\mathcal{H}_k$-module $F(V)$ is semi-simple if, and only if, $\forall \alpha, \beta \in \Delta$ such that $\alpha = \beta$ or α—β, $\mathrm{Res}_{\mathcal{H}_k(G_{\alpha\beta})} F(V)$ is semi-simple ([5] Corollary 6). But $\mathrm{Res}_{\mathcal{H}_k(G_{\alpha\beta})} F(V)$ is $F(kG_{\alpha\beta}.V^U)$ which is semi-simple (**2.3**) since $kG_{\alpha\beta}.V^U$ is semi-simple. This proves the Theorem.

Proposition 12. *Let V be a kG-module with all composition factors isomorphic, then V is semi-simple if, and only if, (A+D) is satisfied and $\forall \alpha \in \Delta$, $kG_\alpha.V^U$ is a semi-simple kG_α-module.*

Proof of the Proposition. The proof is similar to that of Theorem 10, except that instead of [5] Corollary 6 one may invoke [5] Proposition 7 since $F(V)$ has all its composition factors isomorphic (**F2**).

3.2. Ronan-Smith sheaves.

We now give some connections with the approach of Ronan-Smith ([10]). One considers the Tits building associated to G, with simplexes σ and associated proper parabolics (P_σ). If τ is a face of σ then $P_\sigma \subseteq P_\tau$. A "sheaf" $\mathcal{F}$ is then defined as a coefficient system on the building, i.e. a set of k-vector spaces $(\mathcal{F}_\sigma)_\sigma$ together with connecting homomorphisms $\phi_{\sigma\tau} : \mathcal{F}_\sigma \to \mathcal{F}_\tau$ (τ is a face of σ) making the system equivariant for the action of G on the building (see [10] 1). As a consequence, P_σ acts linearly on $\mathcal{F}_\sigma$ and $\phi_{\sigma\tau} \in \mathrm{Mor}_{kP_\sigma}(\mathcal{F}_\sigma, \mathrm{Res}^{P_\tau}_{P_\sigma}\mathcal{F}_\tau)$. One has obvious notions of sheaf morphisms, isomorphisms, subsheaf, irreducible sheaf, etc... Ronan-Smith also define a sheaf associated to each kG-module V in the following way : $\mathcal{F}_V$ is the Tits building endowed with the kL_I-module V^{U_I} at the parabolic P_I and natural inclusions as connecting maps (see [10]). Then Smith's theorem says that $\mathcal{F}_V$ is irreducible (resp. completely reducible) when V is simple (resp. semi-simple). The statements of **3.1** allow to reach a converse.

Proposition 13. *Let G be a finite group with a split BN-pair of characteristic p and rank $n \geq 2$, V a kG-module for k an algebraically closed field of characteristic p. We assume that $n \geq 3$ or that V has all its composition factors isomorphic. Then V is semi-simple if, and only if, it satisfies (A+D) and $\mathcal{F}_V$ is completely reducible.*

Proof. There just remains to prove that (A+D) and the complete reducibility of $\mathcal{F}_V$ imply V is semi-simple. Since $\mathcal{F}_V$ is completely reducible, $\forall \alpha \in \Delta$ V^{U_α} is semi-simple and it is also the case for any $V^{U_{\alpha\beta}}$ when $n \geq 3$. Therefore, we may apply Theorem 11 and Proposition 12 to get our claim.

Remark. If $G = \mathrm{SL}_3(p)$ and P is a proper parabolic subgroup, then $\mathrm{Ind}_P^G k = k \oplus V$ where V is an indecomposable extension of $L(0, p-1)$ by $L(p-1, 0)$. Then $F(V)$ becomes semi-simple when restricted to any $\mathcal{H}_k(G_\alpha)$, so $\mathcal{F}_V$ is semi-simple. This also provides a counterexample to Proposition 12 when the composition factors are not all isomorphic.

Homology is the standard way to recover a kG-module from a sheaf $\mathcal{F}$, but $H_0(\mathcal{F}_V)$ for instance is not always equal to V, even if V is simple (see [10] 2). Another method would be to use the equivalence of Theorem 2. The problem of "gluing" together the local information given at each proper parabolic is then solved by reducing to the same problem for Hecke algebras.

Proposition 14. *Let G be a finite group with a split BN-pair of characteristic p and rank≥ 3, let k be a field of characteristic p. Let $\mathcal{F}$ be a sheaf for k and G in the sense of above. For each simplex σ assume the following hypotheses :*
– $\mathcal{F}_\sigma$ satisfies $(*)$ as kP_σ-module,
– for each face τ of σ, $\phi_{\sigma\tau}$ induces a bijection between fixed points in $\mathcal{F}_\sigma$ and in $\mathcal{F}_\tau$ under a Sylow p-subgroup of P_σ.
Then there is a kG-module V such that $\mathcal{F} \simeq \mathcal{F}_V$.

Proof. When $I \subset \Delta$, let σ_I correspond to P_I. $\mathcal{F}$ is equivariant for the action of G and each σ is under the form $g\sigma_I$ with all faces of σ under the form $g\sigma_J$ for $I \subseteq J$, so it suffices to find a kG-module V and maps $\theta_I : \mathcal{F}_{\sigma_I} \to V^{U_I}$ such that θ_I is a kP_I-isomorphism and $\forall I \subseteq J \subset \Delta$, $\theta_J \circ \phi_{\sigma_I\sigma_J} = \theta_I$.

If $I \subseteq \Delta$, let $M_I := (\mathcal{F}_{\sigma_I})^U$ considered as $\mathcal{H}_k(L_I)$-module. This comes from the fact that $\mathcal{F}_{\sigma_I}$ is a kP_I-module and $\mathcal{H}_k(L_I) = \mathcal{H}_k(P_I) = \mathrm{End}_{kP_I}(\mathrm{Ind}_U^{P_I} k)$ since U_I acts trivially on $\mathrm{Ind}_U^{P_I} k$; M_I also identifies with $F(\mathcal{F}_{\sigma_I})$ for kP_I. When $I \subseteq J \subset \Delta$, let $\phi'_{IJ} : M_I \to M_J$ be the restriction of $\phi_{\sigma_I\sigma_J}$ to $(\mathcal{F}_{\sigma_I})^U$. Then, by the second hypothesis, ϕ'_{IJ} is an $\mathcal{H}_k(L_I)$-isomorphism between M_I and $\mathrm{Res}_{\mathcal{H}_k(L_I)} M_J$. Moreover, $I \subseteq J \subseteq K \subset \Delta$ implies $\phi'_{IK} = \phi'_{JK} \circ \phi'_{IJ}$. On the other hand, since the rank of G is ≥ 3 and $\mathcal{H}_k(G)$ has a presentation with generators $(a_t)_{t\in T}$, $(a_{n_\alpha})_{\alpha\in\Delta}$ subject to relations written in the subalgebras $\mathcal{H}_k(L_{\alpha\beta})$ for $\alpha, \beta \in \Delta$ (see [5] 2), $\mathcal{H}_k(G)$ is an amalgam of the $\mathcal{H}_k(L_I)$ $(I \subset \Delta)$ with respect to the inclusions for $I \subseteq J$. So there exists an $\mathcal{H}_k(G)$-module M and maps $\phi'_I : M_I \to M$ such that ϕ'_I is an isomorphism between M_I and $\mathrm{Res}_{\mathcal{H}_k(L_I)} M$, and $\phi'_I = \phi'_J \circ \phi'_{IJ}$ for all $I \subseteq J \subset \Delta$.

Now, by Theorem 2, $M = F(V)$ for a kG-module V satisfying $(*)$. On the other hand, each $\mathcal{F}_{\sigma_I}$ satisfies $(*)$ as kP_I-module by the first hypothesis of the proposition, so do V^{U_I} by Theorem 10, while the images by F are respectively M_I and $\mathrm{Res}_{\mathcal{H}_k(L_I)}M$. Then $\phi'_I = F(\phi_I)$ for some isomorphism $\phi_I \in \mathrm{Mor}_{kP_I}(\mathcal{F}_{\sigma_I}, V^{U_I})$ (Theorem 2). Moreover, $\theta_J \circ \phi_{\sigma_I\sigma_J} = \theta_I$ since the images by F are $\phi'_J \circ \phi'_{IJ}$ and ϕ'_I. This completes the proof of the proposition.

Remark. The second condition of Proposition 14 is clearly necessary for $\mathcal{F}$ to be a fixed point sheaf. The first is not necessary in general, it is if one seeks some V satisfying $(*)$, then such a V is unique.

3.3. A linkage principle.

Until the end of this talk, we assume k to be algebraically closed.

In this section we just apply our results on blocks of $\mathcal{H}_k$ ([5] 5.) to get restrictions on the possible isomorphism types of composition factors of indecomposable kG-modules satisfying $(*)$. The result is strikingly analogous to the linkage principle ([9] II 6.17) for rational representations of reductive groups : if $\overline{G}$ is reductive over k and λ, μ are dominant weights with associated simple rational kG-modules $L(\lambda)$ and $L(\mu)$, if $L(\lambda)$ and $L(\mu)$ are composition factors of an indecomposable rational $k\overline{G}$-module, then $\lambda \in W_p.\mu$, where W_p is the affine Weyl group (with respect to p).

Here we just consider the set $\mathrm{Mor}(T, k^*)$ and the usual action of W on $\mathrm{Mor}(T, k^*)$: $w.\chi(t) = \chi(t^w)$. If L is a simple kG-module, L^U is a line (see [11]), we denote by $\chi_L \in \mathrm{Mor}(T, k^*)$ the corresponding action of T on this line.

Theorem 15. *Let V be an indecomposable non simple kG-module satisfying (A^*), let L, L' be composition factors of V with fixed points under U covered by V^U, then*
(b) $\chi_L \in W.\chi_{L'}$ and none of L, L' is projective nor one-dimensional.
In particular, (b) is fulfilled if V satisfies $(*)$, $L \subseteq hd\ V$ and $L' \subseteq soc\ V$.

Proof. L and L' being composition factors of $kG.F(V)$ by the hypothesis, one may assume that V satisfies (A), hence $(*)$. By Corollary 3, $F(V)$ is indecomposable, so all its composition factors are in the same block of $\mathcal{H}_k$. The hypothesis on the considered composition factors of V implies $F(V)$ admits $F(L)$ and $F(L')$ as composition factors. Then [5] Theorem 21 gives condition (b) above since χ_L corresponds to the action of the a_t's for $t \in T$, and projective or one-dimensional L corresponds to $\chi_L(T \cap G_R) = 1$.

There remains to check that, if S is a simple component of hd V or soc V, S meets V^U. If $S = V/V'$ then $V^U \not\subseteq V'$ since otherwise $V \subseteq V'$ by condition (A). On the other hand, if $S \subseteq V$, $V^U \cap S \supseteq S^U \neq 0$.

Remark. Not all composition factors of V are concerned by the theorem, and it is not possible to remove the hypothesis of intersection with V^U : see [6] p.378 where some $\mathrm{Ind}_U^G(\chi)$ has composition factors corresponding to $\chi' \in \mathrm{Mor}(T, k^*) \setminus W.\chi$.

3.4 kG-modules of Jordan-Hölder length 2.

The last assertion of the theorem above applies to extensions $0 \to S' \to V \to S \to 0$ satisfying $(*)$ with simple S and S' (see also condition (B) of [14]). In the following theorem we get some more restrictive conditions (see also the Remark following it). Unlike the case of $\mathrm{SL}_2(p)$ (see Appendix), this type of extensions seems rather exceptional.

We take $G = G(q)$ a Chevalley group over $\mathrm{GF}(q) \subseteq k$. Simple kG-modules are indexed by weights λ such that $\forall \alpha \in \Delta \ \langle \lambda, \alpha\check{\ } \rangle \in [0, q-1]$ (*restricted weights*) :

$$\lambda \mapsto L(\lambda),$$

(see [15]). For brevity we write $\alpha(\lambda) := \langle \lambda, \alpha\check{\ } \rangle$. We denote by χ_λ the linear character of $T(q)$ with values in k^* obtained by restriction of λ (this was $\chi_{L(\lambda)}$ in **3.3**).

Theorem 16. *Let G be a Chevalley group over* $\mathrm{GF}(q)$ *with irreducible root system of rank ≥ 2, let $k \supseteq \mathrm{GF}(q)$ be algebraically closed. Let $L(\lambda)$, $L(\mu)$ be irreducible kG-modules and V be a non trivial extension of $L(\lambda)$ by $L(\mu)$ satisfying (A). Then one of the following holds :*
i) V^* does not satisfy (A).
ii) q is an odd prime ; there is $\alpha \in \Delta$ such that $\alpha(\lambda) \neq 0, q-1$ and $\forall \beta \in \Delta$ $\beta(\mu) \equiv \beta(r_\alpha.\lambda)$ (mod.$q-1$).
iii) $\forall \alpha \in \Delta$ $\alpha(\lambda) = \alpha(\mu)$ or $\{\alpha(\lambda), \alpha(\mu)\} = \{0, q-1\}$; $X := \{(\beta, \gamma) \in \Delta \times \Delta$; $(\beta(\lambda), \beta(\mu)) = (\gamma(\mu), \gamma(\lambda)) = (0, q-1)\}$ is non empty and $\forall (\beta, \gamma) \in X$ β—γ.

The following lemma will help us to compare conditions coming from representations of $\mathcal{H}_k$ with the results on representations of $\mathrm{SL}_2(q)$ as drawn from [2].

Lemma 17. *Assume that λ, μ are restricted weifhts of $SL_2(q)$ and that one has both $\mathrm{Ext}^1_{\mathrm{SL}_2(q)}(L(\lambda), L(\mu)) \neq 0$ and $\mathrm{Ext}^1_{\mathcal{H}_k(\mathrm{SL}_2(q))}(F(L(\lambda)), F(L(\mu))) \neq 0$. Then q is a prime $\neq 2$.*

Proof of the lemma. One assimilates λ, μ to integers in $[0, q-1]$ with p-adic expansions $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$ and $\mu = \sum_{i=0}^{n-1} \mu_i p^i$ where $q = p^n$. The condition on $\mathrm{Ext}^1_{\mathcal{H}_k}$ forces $r.\chi_\lambda = \chi_\mu$, r denoting the non trivial element of the Weyl group (Theorem 16 of [5], where case 1 is impossible since $|R| = 1$). This amounts to $\lambda + \mu = q - 1$, i.e. $\forall i \in [0, n-1]$ $\lambda_i + \mu_i = p - 1$. On the other hand, [2] 4.5 (a) tells us that the condition on Ext^1_G when $n \geq 2$ implies there is j such that $\lambda_j = p - \mu_j - 2$, a contradiction. So $n = 1$, $q = p$; the case $q = 2$ is also impossible : the projectivity of $L(1)$ would force $\lambda = \mu = 0$.

Proof of the theorem. Let us show that if V and V^* satisfy (A), then ii) or iii) holds. By Theorem 9, $F(V)$ is not semi-simple, so

$$(E) \qquad 0 \to F(L(\mu)) \to F(V) \to F(L(\lambda)) \to 0$$

is exact (**F2**) and non-split as $\mathcal{H}_k$-sequence. Before we apply the discussion of $\mathrm{Ext}^1_{\mathcal{H}_k}$ groups of [5], we must find the isomorphic type of $F(L(\lambda))$. Let $I_\lambda := \{\alpha \in \Delta\, ; \alpha(\lambda) = q-1\}$. Then, using Proposition 6 and $L(\lambda)^U = L(\lambda)_\lambda$ (weight space), one sees that $F(L(\lambda))$ is $\psi(\chi_\lambda, I_\lambda)$ in the notation of [5] : the a_t's act by χ_λ and a_{n_α} acts by -1 (resp. 0) if $\alpha \in I_\lambda$ (resp. $\alpha \in \Delta \setminus I_\lambda$). The discussion in [5] **4** now implies that either

(1) $\chi_\lambda = \chi_\mu$, $I_\lambda \setminus I_\mu \neq \emptyset$, $I_\mu \setminus I_\lambda \neq \emptyset$ and $\forall \alpha \in I_\lambda \setminus I_\mu$ $\forall \beta \in I_\mu \setminus I_\lambda$ α—β, (case (1) of [5] 4),

or

(2) there is $\alpha \in \Delta$ such that the restriction to $\mathcal{H}_k(L_\alpha)$ of (E) is non-split (case (2) of [5] 4).

Case 1 above implies condition iii) since $\chi_\lambda = \chi_\mu$ restricted to T_α means $\alpha(\lambda) \equiv \alpha(\mu)$ (mod. $q-1$) with $\alpha(\lambda), \alpha(\mu) \in [0, q-1]$.

Let us concentrate on case 2. Applying Theorem 16 of [5] when the rank is one, we get $\chi_\lambda(T_\alpha) \neq 1$ and $\chi_\lambda = r_\alpha.\chi_\mu$. This implies $\alpha(\lambda) \neq 0, q-1$ and $\forall \beta \in \Delta$ $\beta(\mu) \equiv \beta(r_\alpha.\lambda)$ (mod.$q-1$). Since G_α is $\mathrm{SL}_2(q)$, there remains to check that the hypotheses of Lemma 17 are fulfilled by $\alpha(\lambda)$ and $\alpha(\mu)$ to get that q is an odd prime. The restriction of (E) to $\mathcal{H}_k(G_\alpha)$ is non-split either ([5] Corollary 6), so $\mathrm{Ext}^1_{\mathcal{H}_k(G_\alpha)}(F(L(\alpha(\mu)), L(\alpha(\lambda))) \neq 0$. This restriction of (E) to $\mathcal{H}_k(G_\alpha)$ is also the image by F of the exact sequence of kG_α-modules $0 \to L(\mu)^{U_\alpha} \to V^{U_\alpha} \to L(\lambda)^{U_\alpha} \to 0$ (**F1-2**). It is necessarily non-split, otherwise its image would be split. Therefore $\mathrm{Ext}^1_{kG_\alpha}(L(\alpha(\mu)), L(\alpha(\lambda))) \neq 0$ and Lemma 17 applies. This completes the proof of the theorem.

Remark. (April 1989) Since this was written we got from two sources several evidences that the statement of Theorem 16 could be greatly strengthened.

Unpublished work of H.H. Andersen using the methods of [1] shows that, if $G(q)$ is Chevalley of rank two and $q > p$, then $\mathrm{Ext}^1_{kG(q)}(L(0,q-1), L(q-1,0)) = 0$. So the same proof as above implies that if G is Chevalley over GF(q) and $q > p$, an extension

$$0 \to L(\mu) \to V \to L(\lambda) \to 0$$

splits if, and only if, V satisfies $(*)$.

Independent work by H. Völklein ([19]) also deals with this kind of extensions ; the condition obtained there is a bit weaker than our ii)-iii) but twisted groups are considered. Some interesting applications to Ext^1_{kG} groups are given.

Remark. One may now see that conditions (A), (D), (H) of Theorem 9 are all necessary. The summands of Y (see for instance the case of $\mathrm{SL}_2(p)$ in the Appendix below) show there are indecomposable non-simple modules satisfying $(*)$ or equivalently (A+D). By [2] 4.5 there exist non-trivial extensions $0 \to L(\mu) \to V \to L(\lambda) \to 0$ for $\mathrm{SL}_2(p)$ with $\lambda + \mu \neq p - 1$. Then, by the Appendix, they do not satisfy (A), nor (A*). So $F(V)$ and $F(V^*)$ are lines, therefore V satisfies (D) and (H) but is not semi-simple. This also proves that F is not "injective" (up to isomorphisms) from kG-**mod** to **mod**-$\mathcal{H}_k$.

One may find indecomposable non-simple kG-modules satisfying (A) and (H) as follows. Let $G(q)$ be a Chevalley group for $q = p^n$ and $n \geq 2$, let λ be a weight such that $\forall \alpha \in \Delta\ 0 < \alpha(\lambda) < q-1$. Then $F(L(\lambda)) = \psi(\chi_\lambda, \emptyset)$, so $L(\lambda) = \mathrm{hd\ Ind}_B^G(\chi_\lambda)$ ([17] 3.5) and $\mathrm{Ind}_B^G(\chi_\lambda)$ is indecomposable but non-simple : its dimension is $(G : B) > |U|$. Thus, there is a quotient V of $\mathrm{Ind}_B^G(\chi_\lambda)$ with a non split exact sequence $0 \to L(\mu) \to V \to L(\lambda) \to 0$. V being a quotient of $\mathrm{Ind}_B^G(\chi_\lambda)$, it satisfies (A), so the image by F of the above sequence is exact. Using [5]4, Lemma 17 and $I_\lambda = \emptyset$, one gets the splitting of this $\mathcal{H}_k$-sequence, thus V satisfies (H). Also V^* statisfies (A*) and (H) but is not semi-simple.

3.5. Modules with all composition factors isomorphic.

The study of certain geometries involves modules satisfying (A) with all composition factors isomorphic. We give next a partial answer to a question of Smith ([14] 5) about them. It somehow provides evidence that non trivial self-extensions are scarce (see [1]). As in the above statements, it singles out the cases $p = 2$ and $q \neq p$ as more likely to split (see [14] and its references for $p = 2$), but we also obtain restrictions on the type of the group : symplectic groups may cause exceptions (see [1], [8]).

Theorem 18. *Let G be a Chevalley group over $GF(q)$ (q a power of p) with irreducible root system of rank ≥ 2, let V be an indecomposable, non simple kG-module satisfying conditions (A) and (D) with all composition factors isomorphic to a given $L(\lambda)$. Then $q = p \neq 2$, G is of type C_l and $\alpha_l(\lambda) = \frac{p-1}{2}$.*

Proof. Let's check q is a prime. By Proposition 10, there is $\alpha \in \Delta$ such that V^{U_α} is not semi-simple as kG_α-module. The group G_α is isomorphic to $\mathrm{SL}_2(q)$ and V^{U_α} has all its composition factors isomorphic to $L(\lambda)^{U_\alpha} = L(\alpha(\lambda))$ (**F1**), so $\mathrm{Ext}^1_{kG_\alpha}(L(\alpha(\lambda)), L(\alpha(\lambda))) \neq 0$. On the other hand, V^{U_α} satisfies $(*)$ (Proposition 8) and is non semi-simple (Theorem 9), so $F(V^{U_\alpha})$ is non semi-simple and has all its composition factors isomorphic to $F(L(\alpha(\lambda)))$, so $\mathrm{Ext}^1_{\mathcal{H}_k(G_\alpha)}(F(L(\alpha(\lambda))), F(L(\alpha(\lambda)))) \neq 0$. Then Lemma 17 applies : $q = p \neq 2$.

By Theorem 9, $F(V)$ is not semi-simple as $\mathcal{H}_k$-module. By **F2**, all its composition factors are isomorphic to $F(L(\lambda))$, so $\mathrm{Ext}^1_{\mathcal{H}_k}(F(L(\lambda)), F(L(\lambda))) \neq 0$. Then Theorem 16 in [5] tells us there is $\alpha \in \Delta$ such that $(\chi_\lambda)^{r_\alpha} = \chi_\lambda$ and $\chi_\lambda(T_\alpha) \neq 1$ (case (2) of [5] is impossible since $I = J = I_\lambda$). So $\chi_\lambda([T, n_\alpha]) = 1$ while $\chi_\lambda(T_\alpha) \neq 1$. The only case when the inclusion $[T, n_\alpha] \subset T_\alpha$ is strict occurs when $p \neq 2$, G is of type C_l and α is the longest fundamental root : this is because $T = \Pi_{\beta \in \Delta} T_\beta$ and $[T_\beta, n_\alpha]$ is the image of T_α by $t \mapsto t^{A_{\beta\alpha}}$ (where $(A_{\alpha\beta})$ is the Cartan matrix) and g.c.d.$(A_{\beta\alpha})_\beta = 1$, except in the case we mention. Then the index is two, so $\mathrm{Res}_{T_\alpha}(\chi_\lambda)$ is of order 2, forcing $\alpha(\lambda) = \frac{p-1}{2}$. This finishes the proof of the theorem.

Appendix. The case of $\mathrm{SL}_2(p)$.

We take $G = \mathrm{SL}_2(p)$. Since its Sylow p-subgroup is cyclic, there is only a finite number of indecomposable kG-modules. So one may expect to classify the ones satisfying (A). Let us enumerate some. First, the simple kG-modules. They are p with dimensions $1, 2, \ldots, p$, the last being the Steinberg module St. Another example is $Y := \mathrm{Ind}_U^G k$ since it is generated by $1 \otimes 1$ which is fixed by U. Then all indecomposable summands of Y satisfy (A). These are the following (see for instance [4] A.5) : $Y = k \oplus \mathrm{St} \oplus \oplus_\chi \mathrm{Ind}_B^G(\chi)$ where χ ranges $\mathrm{Mor}(B, k^*) \setminus \{1\}$. Head and socle of $\mathrm{Ind}_B^G(\chi)$

are known (see for instance [12] 3.10) : they are simple and the sum of their dimensions is $p+1$, so $\mathrm{Ind}_B^G(\chi)$ has Jordan-Hölder length 2. Moreover $\mathrm{Ind}_U^G(\chi) \simeq \mathrm{Ind}_B^G(\chi')$ implies $\chi = \chi'$ (see [12]). F induces a bijection between indecomposable summands of Y and principal indecomposable modules for $\mathcal{H}_k$ (see [7] 2.1b). We prove :

Theorem 19. *The indecomposable $k\mathrm{SL}_2(p)$-modules satisfying (A) are the p simple modules and the $(p-2)$ modules $\mathrm{Ind}_B^G(\chi)$ for $\chi \in Mor(B, k^*) \setminus \{1\}$. The $k\mathrm{SL}_2(p)$-modules satisfying (A) all satisfy $(*)$, they form an abelian subcategory of $k\mathrm{SL}_2(p)$-**mod** and F makes it equivalent to **mod**-$\mathcal{H}_k(\mathrm{SL}_2(p))$.*

Proof. Once the first assertion is proved, the others are easy : (A*) comes from the fact that the family of indecomposable modules in the theorem is clearly stable by duality. The equivalence is clear by Theorem 2. Notice that the argument below for the first assertion of the theorem is very similar to the standard one for uniserial rings.

Let V be an indecomposable kG-module satisfying (A), then (see proof of Proposition 8) V is a quotient of a power of Y. We have an isomorphism

$$(Q) \qquad V \simeq \frac{I_1 \times \cdots \times I_l}{N}$$

with $(I_i)_i$ some indecomposable summands of Y. We assume moreover that l is minimal. It is enough to prove that $l = 1$.

Let $\pi_i \ : N \to I_i$ be the projection on I_i. Then $0 \neq \pi_i(N) \subseteq \mathrm{rad}\, I_i$: otherwise $\pi_i(N) = I_i$ or 0, so $I_i \subseteq N + (I_1 \times \cdots \widehat{I_i} \cdots \times I_l)$ or $N \subseteq I_1 \times \cdots \widehat{I_i} \cdots \times I_l$, then in both cases $(I_1 \times \cdots \times I_l)/N = (I_1 \times \cdots \widehat{I_i} \cdots \times I_l)/N$, a contradiction with minimality of l. Now, since $\mathrm{rad}\, I_i \neq 0$, each I_i in (Q) is an $\mathrm{Ind}_B^G(\chi)$ with rad = soc and Jordan-Hölder length 2.

All I_i's in (Q) are isomorphic : it suffices to check that all socles $S_i := \mathrm{soc}\, I_i$ are isomorphic. We have $N \subseteq \Pi_i S_i$, so $N = \oplus_J N_J$ where J ranges the isomorphism classes of simple kG-modules and $N_J := N \cap \Pi_{S_i \simeq J} S_i$ (isotypic components). Then $V \simeq \Pi_J I_J / N_J$ where $I_J = \Pi_{S_i \simeq J} I_i$. Indecomposability of V implies only one I_J is non zero.

(Q) becomes $V \simeq (I_1)^l/N$ with simple $S_1 = \mathrm{soc}\, I_1$ and $N \subseteq (S_1)^l$. N is isomorphic to some power of S_1, so there is some automorphism a of $(S_1)^l$ such that $N = a((S_1)^m \times 0)$. Since $\mathrm{Aut}\, S_1 = k$, a is an element of $\mathrm{GL}_l(k)$, so it extends to an automorphism of $(I_1)^l$. Then $V \simeq (a((I_1)^m \times 0)/N) \times (I_1)^{l-m} \simeq (\mathrm{hd}\, I_1)^m \times (I_1)^{l-m}$. So $l = 1$.

References

[1]. H.H. Andersen, Extensions of simple modules for finite Chevalley groups, *J. Algebra,* **111-2** (1987), 388-403.

[2]. H.H. Andersen, J. Jorgensen and P. Landrock, The projective indecomposable modules of SL(2,p^n), *Proc. London Math. Soc. (3),* **46** (1983), 38-52.

[3]. M. Cabanes, Irreducible modules and Levi supplements, *J. Algebra,* **90-1** (1984), 84-97.

[4]. M. Cabanes, Brauer morphism between modular Hecke algebras, *J. Algebra,* **115-1** (1988), 1-31.

[5]. M. Cabanes, Extension groups for modular Hecke algebras, *J. Fac. Sci. Univ. Tokyo, Sect. IA*, 36(2) (1989), 347-362.

[6]. R. Carter and G. Lusztig, Modular representations of finite groups of Lie type, *Proc. London Math. Soc.*, **(3)32** (1976), 347-384.

[7]. J.A. Green, On a theorem of Sawada, *J. London Math. Soc.* (2), **18** (1978), 247-252.

[8]. J.E. Humphreys, Non-zero Ext^1 for Chevalley groups (via algebraic groups), *J. London Math. Soc. (2)* **31** (1985), 463-467.

[9]. J.C. Jantzen, ***Representations of algebraic groups,*** Pure and applied mathematics **131**, Academic Press, 1987.

[10]. M.A. Ronan and S.D. Smith, Sheaves on buildings and modular representations of Chevalley groups, *J. Algebra,* **95-2** (1985), 319-347.

[11]. F. Richen, Modular representations of split (B,N)-pairs, *Trans. Amer. Math. Soc.,* **140** (1969), 435-460.

[12]. H. Sawada, A characterization of the modular representations of finite groups with split (B,N)-pairs, *Math. Z.,* **155** (1977), 29-41.

[13]. S. Smith, Irreducible modules and parabolic subgroups, *J. Algebra,* **75** (1982), 286-289.

[14]. S. Smith, Sheaf homology and complete reducibility, *J. Algebra,* **95-1** (1985), 72-80.

[15]. R. Steinberg, Representations of algebraic groups, *Nagoya Math. J.,* **22** (1963), 33-56.

[16]. N. Tinberg, Modular representations of finite groups with unsaturated split (B,N)-pairs, *Can. J. Math.*, Vol **XXXII (3)**, (1980), 714-733.

[17]. N. Tinberg, Some indecomposable modules of groups with split BN-pairs, *J. Algebra,* **61** (1979), 508-526.

[18] M. Auslander, Representation theory of Artin algebras I, *Comm. Algebra*, **1(3)** (1974), 177-268.
[19] H. Völklein, On extensions between simple modules of a Chevalley group, to appear in *Proceedings of the Brixen group theory meeting.*

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