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Introduction

Throughout this paper G denotes a finite group, $R(G)$ the character ring of G and $(-,-)$ the usual inner product of $R(G)$.

In 1946 Richard Brauer proved (cf. [Br1]) that each virtual character χ of G can be expressed as a linear combination

$$\chi = \sum_i z_i \mathrm{ind}_{H_i}^G \varphi_i$$

where $z_i \in \mathbf{Z}$, $H_i \leq G$ and $\varphi_i \in \hat{H}_i = \mathrm{Hom}(H_i, \mathbf{C}^*)$. Brauer was motivated by the question whether Artin L-functions of any virtual character have a meromorphic extension to the entire complex plane. This was known for one-dimensional characters, and it was also known that the Artin L-functions are invariant under induction. So Brauer's induction theorem gave a positive answer to the above question, and this is a very typical example for the applications of the theorem in number theory. However, Brauer's theorem is a mere existence theorem, and it remained the question for an explicit formula, associating to each virtual character χ an integral linear combination as above. A first result in this direction is again due to Brauer, who gave in 1951, cf. [Br2], an explicit formula to Artin's induction theorem, i.e. a formula which induces from cyclic subgroups and has rational coefficients. It was not before 1986, that there appeared V. Snaith's explicit version of Brauer's induction theorem, cf. [Sn]. His formula is based on topological invariants, in particular on Euler characteristics of quotient spaces of the unitary group $U(n)$.

Here we give an explicit and canonical formula for Brauer's induction theorem by algebraic and combinatorial methods. 'Canonical' means that this formula is unique among all the expressions for χ as above, if a certain functorial behaviour with respect to G is required. To state this functorial property it is convenient to introduce the free abelian group $R_+(G)$ whose basis is given by the G-conjugacy classes of pairs (H, φ), where $H \leq G$ and $\varphi \in \hat{H}$, cf. [De], p. 11. We consider a formula as a map from $R(G)$ to $R_+(G)$, such that it becomes the identity, if the symbols (H, φ) are replaced by $\mathrm{ind}_H^G \varphi \in R(G)$. It turns out that $R_+(G)$ carries a lot of structures, which we investigate in section 1. Using the results about $R_+(G)$ we define the formula $a_G : R(G) \longrightarrow R_+(G)$ and prove its natural properties, cf. theorem (2.1) and cor. (2.12). In section 3 we apply the methods developed in the previous sections to obtain an induction formula which induces only from subgroups of a fixed type $\mathcal{T}$, cf. theorem (3.2). In this case however, we don't have integral coefficients any longer. For the type of cyclic groups we obtain again Brauer's explicit version [Br2] of Artin's induction theorem. The cases in which the formula is integral are determined in (3.12) and (3.13). Unfortunately the formula is not integral for the type of elementary groups. For the type of cyclic groups we obtain that the "worst" denominator in the formula for the characters of G coincides with the Artin exponent of G.

The formula a_G we introduce in section 2 is different from Snaith's formula in [Sn], but there is a relation between them which can be found in [Bo], chap. IV.

I am grateful to G.-M. Cram for his proof of proposition (2.24).

1. The ring $R_+(G)$

For a finite group G we consider the set $\mathcal{M}_G$ of all pairs (H, φ) where $H \leq G$ and $\varphi \in \hat{H} = \mathrm{Hom}(H, \mathbf{C}^*)$. G acts from the left on $\mathcal{M}_G$ by componentwise conjugation: ${}^g(H, \varphi) := ({}^gH, {}^g\varphi)$ where ${}^gH = gHg^{-1}$, ${}^g\varphi := \varphi(g^{-1}.g)$, for $g \in G$. We denote the G-orbit of (H, φ) by $\overline{(H, \varphi)}^G$ and the set of G-orbits by $\mathcal{M}_G/G$. Let $R_+(G)$ be the free abelian group with the basis $\mathcal{M}_G/G$, then we have the well-defined map into the character ring $R(G)$

$$(1.1) \qquad b_G : R_+(G) \longrightarrow R(G), \quad \overline{(H, \varphi)}^G \mapsto \mathrm{ind}_H^G \varphi.$$

b_G is surjective by Brauer's induction theorem [Br1]. We want to construct a map

$$(1.2) \qquad a_G : R(G) \longrightarrow R_+(G), \quad \chi \mapsto \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) \overline{(H, \varphi)}^G$$

with $b_G a_G = \mathrm{id}_{R(G)}$, i.e.

$$(1.3) \qquad \chi = \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) \mathrm{ind}_G^H \varphi$$

for all $\chi \in R(G)$. Moreover we want a_G to have a good functorial behaviour with respect to the structures carried by $R(G)$ and $R_+(G)$.

(1.4) Remark. We may consider $R_+(G)$ as the Grothendieck group of the category of *monomial representations of G*. Its objects are finite dimensional $\mathbf{C}G$-modules V ($\mathbf{C}G$ denotes the group ring) with a fixed decomposition $V = V_1 \oplus \ldots \oplus V_n$ into one-dimensional subspaces, called the *lines* of V, such that G permutes the lines. V is called *simple*, if its lines are permuted transitively by G. A *morphism* $F : V = V_1 \oplus \ldots \oplus V_n \longrightarrow W = W_1 \oplus \ldots \oplus W_m$ of two monomial representations of G is a $\mathbf{C}G$-linear map such that for each $i \in \{1, \ldots, n\}$ there is some $j \in \{1, \ldots, m\}$ with $F(V_i) \subseteq W_j$. For monomial representations we may define in an obvious way direct sums, tensor products, duals, restriction maps along group homomorphisms and induction maps along subgroup relations. Every monomial representation of G is a unique direct sum of simple ones and the isomorphism classes of simple monomial representations are in a bijective correspondence to $\mathcal{M}_G/G$ by the following construction: For simple $V = V_1 \oplus \ldots \oplus V_n$ define H to be the stabilizer of V_1 and $\varphi \in \hat{H}$ to be the action of H on V_1. The choice of another line V_i gives a conjugated pair ${}^g(H, \varphi)$. b_G is induced from the forgetful functor which associates to every monomial representation of G the underlying $\mathbf{C}G$-module. For more details of the above statements see [Bo] chap.I §1.

The constructions described above provide $R_+(G)$ with the following structures:

Multiplication. The tensor product on monomial representations is translated into a commutative ring structure on $R_+(G)$ given by

$$\overline{(H,\varphi)}^G \cdot \overline{(K,\psi)}^G = \sum_{s \in H\backslash G/K} \overline{(H \cap {}^sK, \varphi \cdot {}^s\psi)}^G . \tag{1.5}$$

The unity is $\overline{(G,1)}^G$. $R_+(G)$ contains the group ring $\mathbf{Z}\hat{G} \cong \oplus_{\varphi \in \hat{G}} \mathbf{Z}\overline{(G,\varphi)}^G$ as a subring. Note that the G-orbit of (G, φ) consist only of this single pair. So $R_+(G)$ is a $\mathbf{Z}\hat{G}$-algebra and b_G is a $\mathbf{Z}\hat{G}$-algebra map. We have the $\mathbf{Z}\hat{G}$-module decomposition

$$R_+(G) = \mathbf{Z}\hat{G} \oplus \bigoplus_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G, H<G} \mathbf{Z}\overline{(H,\varphi)}^G,$$

with the corresponding projection map

$$\pi_G : R_+(G) \longrightarrow \mathbf{Z}\hat{G}, \quad \overline{(H,\varphi)}^G \mapsto \begin{cases} \varphi, & \text{if } H = G; \\ 0, & \text{if } H < G. \end{cases} \tag{1.6}$$

Also $R(G)$ is a $\mathbf{Z}\hat{G}$-algebra, since it contains $\mathbf{Z}\hat{G}$ as a subring. This gives rise to the $\mathbf{Z}\hat{G}$-module decomposition

$$R(G) = \mathbf{Z}\hat{G} \oplus \bigoplus_{\chi \in \mathrm{Irr}G \setminus \hat{G}} \mathbf{Z}\chi$$

where $\mathrm{Irr}G$ is the set of irreducible characters of G. We obtain the corresponding projection

$$p_G : R(G) \longrightarrow \mathbf{Z}\hat{G}, \quad \mathrm{Irr}G \ni \chi \mapsto \begin{cases} \chi, & \text{if } \chi \in \hat{G}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Note that π_G is multiplicative, which is in general not true for p_G.

Restriction. The restriction of monomial representations of G along a group homomorphism $f : G' \longrightarrow G$ gives rise to the ring homomorphism

$$\mathrm{res}_{+f} : R_+(G) \longrightarrow R_+(G') : \quad \overline{(H,\varphi)}^G \mapsto \sum_{s \in f(G') \setminus G/H} \overline{(f^{-1}({}^sH),\, {}^s\varphi \circ f)}^{G'}. \tag{1.8}$$

The diagram

$$\begin{array}{ccc} R_+(G) & \xrightarrow{b_G} & R(G) \\ {\scriptstyle \mathrm{res}_{+f}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_f} \\ R_+(G') & \xrightarrow{b_{G'}} & R(G') \end{array} \tag{1.9}$$

commutes, since the corresponding diagram on the level of the categories of (monomial) representations commutes. For the same reason we have

$$\mathrm{res}_{+ff'} = \mathrm{res}_{+f'}\mathrm{res}_{+f} \tag{1.10}$$

for another group homomorphism $f : G'' \longrightarrow G'$. If f is given as the inclusion of a subgroup $H \le G$, we write res^G_{+H} instead of res_{+f} and obtain from (1.8)

$$\mathrm{res}^G_{+H} : R_+(G) \longrightarrow R_+(H), \quad \overline{(K,\psi)}^G \mapsto \sum_{s \in H\setminus G/K} \overline{(H \cap {}^sK,\, {}^s\psi)}^H. \tag{1.11}$$

If $f : G \longrightarrow G/N =: \overline{G}$ is the canonical surjection for a normal subgroup N of G, we obtain

$$\mathrm{res}_{+f}\overline{(H/N,\overline{\varphi})}^{\overline{G}} = \overline{(H,\varphi)}^G \tag{1.12}$$

where $N \le H \le G$ and $\varphi \in \hat{H}$ vanishes on N. Thus res_{+f} maps the basis $\mathcal{M}_{\overline{G}}/\overline{G}$ injectively into the basis $\mathcal{M}_G/G$. We use the restriction maps to define the ring homomorphism

$$\rho_G : R_+(G) \longrightarrow \prod_{H \le G} \mathbf{Z}\hat{H}, \quad x \mapsto \left(\pi_H \mathrm{res}^G_{+H} x\right)_{H \le G}. \tag{1.13}$$

We define for $(H,\varphi),(H',\varphi')\in\mathcal{M}_G$ the natural number

(1.14) $\gamma^G_{(H,\varphi),(H',\varphi')} :=$ coefficient of the basis element $\overline{(H,\varphi)}^H \in R_+(H)$ in $\mathrm{res}_{+H}^{G}\overline{(H',\varphi')}^G$.

Moreover we define natural poset structures on $\mathcal{M}_G$ and $\mathcal{M}_G/G$ by

$$(1.15)\qquad \begin{aligned} (K,\psi)\le(H,\varphi) &\iff K\le H \text{ and } \psi=\varphi|_K \\ \overline{(K,\psi)}^G\le\overline{(H,\varphi)}^G &\iff (K,\psi)\le {}^g(H,\varphi) \text{ for some } g\in G \end{aligned}$$

Note that infima exist in $\mathcal{M}_G$ but in general not in $\mathcal{M}_G/G$. With the notation (1.14) we have

$$(1.16)\qquad \begin{aligned} \gamma^G_{(H,\varphi),(H',\varphi')} &= \#\{s\in H\backslash G/H' \mid (H,\varphi)\le {}^s(H',\varphi')\} \\ &= \#\{s\in G/H' \mid (H,\varphi)\le {}^s(H',\varphi')\}. \end{aligned}$$

In fact, the first equation is clear from (1.11) and the second equation follows from the fact, that if s satisfies $(H,\varphi)\le {}^s(H',\varphi')$, then $HsH' = H\,{}^sH's = {}^sH's = sH'$. From (1.16) we deduce

$$(1.17)\qquad \begin{aligned} &\text{a) } \gamma^G_{(H,\varphi),(H',\varphi')} = \gamma^G_{{}^s(H,\varphi),\,{}^t(H',\varphi')} \text{ for all } s,t\in G, \\ &\text{b) } \gamma^G_{(H,\varphi),(H',\varphi')}\neq 0 \iff \overline{(H,\varphi)}^G\le\overline{(H',\varphi')}^G, \\ &\text{c) } \gamma^G_{(H,\varphi),(H,\varphi)} = (N_G(H,\varphi):H), \\ &\text{d) } \gamma^G_{(H',\varphi'),(H',\varphi')} \text{ divides } \gamma^G_{(H,\varphi),(H',\varphi')}. \end{aligned}$$

Here $N_G(H,\varphi)$ denotes the stabilizer of (H,φ) in G. Note that $\gamma^G_{(H,\varphi),(H',\varphi')}/\gamma^G_{(H',\varphi'),(H',\varphi')}$ is the number of elements of $\mathcal{M}_G$ in the G-orbit of (H',φ') which are greater or equal to (H,φ). With the definition (1.14) we can express ρ_G as follows:

$$(1.18)\qquad \rho_G\overline{(H',\varphi')}^G = \Big(\sum_{\varphi\in\hat H}\gamma^G_{(H,\varphi),(H',\varphi')}\varphi\Big)_{H\le G}.$$

G acts on the ring $\prod_{H\le G}\mathbf{Z}\hat H$ by conjugation, and from (1.17) a) we see that the immage of ρ_G is actually contained in the subring $\big(\prod_{H\le G}\mathbf{Z}\hat H\big)^G$ of G-invariant elements. Moreover ρ_G is injective. In fact, let

$$x = \sum_{\overline{(H',\varphi')}^G\in\mathcal{M}_G/G} \alpha_{\overline{(H',\varphi')}^G}\overline{(H',\varphi')}^G \in \ker\rho_G$$

and assume that $H\le G$ is a subgroup which is maximal with the property that $\alpha_{\overline{(H,\varphi)}^G}\neq 0$ for some $\varphi\in\hat H$. Then we have

$$\pi_H\mathrm{res}_{+H}^{G} = \sum_{\varphi\in\hat H}\Big(\sum_{\overline{(H',\varphi')}^G\in\mathcal{M}_G/G}\gamma^G_{(H,\varphi),(H',\varphi')}\alpha_{\overline{(H',\varphi')}^G}\Big)\cdot\varphi$$

and by (1.17) b) and the maximality of H we obtain for the coefficient of φ in the last expression the number $\gamma^G_{(H,\varphi),(H,\varphi)} \neq 0$. This contradicts the hypothesis $x \in \ker\rho_G$. Since $\left(\prod_{H\leq G}\mathbf{Z}\hat{H}\right)^G$ has the same $\mathbf{Z}$-rank as $R_+(G)$, we obtain

(1.19) Proposition. *ρ_G is an injective ring homomorphism with finite cokernel.* □

Besides the $\gamma^G_{(H,\varphi),(H',\varphi')}$'s we will need a more general constant. For $U \leq G$, $(H,\varphi) \in \mathcal{M}_G$ and $(K,\psi) \in \mathcal{M}_U$ we define the nonnegative integer

$$(1.20) \qquad \delta^{\overline{(H,\varphi)}^G}_{\overline{(K,\psi)}^U} := \text{the coefficient of } \mathrm{res}_{+U}^G\overline{(H,\varphi)}^G \text{ at } \overline{(K,\psi)}^U.$$

Observe from (1.11) that

$$(1.21) \qquad \delta^{\overline{(H,\varphi)}^G}_{\overline{(K,\psi)}^U} = 0 \quad \text{for } \overline{(K,\psi)}^G \not\leq \overline{(H,\varphi)}^G.$$

Moreover for $(K,\psi) \in \mathcal{M}_U, U \leq G$ we have

$$(1.22) \qquad \delta^{\overline{(K,\psi)}^G}_{\overline{(K,\psi)}^U} = \frac{|U \cdot N_G(K,\psi)|}{|U|}.$$

In fact,

$$\delta^{\overline{(K,\psi)}^G}_{\overline{(K,\psi)}^U} = \#\{s \in U\backslash G/K \mid \overline{(K,\psi)}^U = \overline{(U\cap {}^sK, {}^s\psi)}^U\}.$$

And for $s \in G$ satisfying this condition, there is some $u \in U$ with $(K,\psi) = (U \cap {}^{us}K, {}^{us}\psi)$, implying $K = {}^{us}K$ and $\psi = {}^{us}\psi$. This shows that $us \in N_G(K,\psi)$. Hence the double coset $UsK = UusK = U\,{}^{us}Kus = Uus = Us$ is just a left coset. Conversely, every $s \in N_G(K,\psi)$ satisfies the above condition.

Induction. For $H \leq G$ we define

$$(1.23) \qquad \mathrm{ind}_{+H}^G : R_+(H) \longrightarrow R_+(G), \quad \overline{(K,\psi)}^H \mapsto \overline{(K,\psi)}^G.$$

Note that we have $b_G \mathrm{ind}_{+H}^G = \mathrm{ind}_H^G b_H$.

Duals. Taking duals of monomial representations yields the map

$$(1.24) \qquad \sigma_G : R_+(G) \longrightarrow R_+(G), \quad \overline{(H,\varphi)}^G \mapsto \overline{(H,\varphi^{-1})}^G$$

which commutes with restrictions to subgroups. Moreover b_G commutes with the constructions of taking duals on $R_+(G)$ and $R(G)$.

Adams operations. For $k \in \mathbf{Z}$ we define the k-th Adams operator to be the ring homomorphism

$$\Psi_+^k : R_+(G) \longrightarrow R_+(G), \quad \overline{(H,\varphi)}^G \mapsto \overline{(H,\varphi^k)}^G. \tag{1.25}$$

Obviously Ψ_+^k commutes with restrictions to subgroups.

Bilinear form. We define a bilinear form $[-,-]_G = [-,-]$ on $R_+(G)$ by

$$[\overline{(H,\varphi)}^G, \overline{(H',\varphi')}^G]_G := \gamma^G_{(H,\varphi),(H',\varphi')}. \tag{1.26}$$

If we arrange the basis $\mathcal{M}_G/G$ of $R_+(G)$ in a sequence such that $\overline{(H,\varphi)}^G$ precedes $\overline{(H',\varphi')}^G$ whenever $\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G$, then the quadratic matrix corresponding to this bilinear form is an upper triangular matrix

$$\Gamma(G) = \left(\gamma^G_{(H,\varphi),(H',\varphi')}\right)_{\overline{(H,\varphi)}^G, \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \tag{1.27}$$

with the non-zero entries $(N_G(H,\varphi) : H)$ in the diagonal. Thus $\Gamma(G)$ is non-singular and the bilinear form $[-,-]$ is non-degenerate in both arguments.

(1.28) Proposition.

a) *Let $U \leq G$, $x \in R_+(U)$ and $y \in R_+(G)$, then we have*

$$[ind_{+U}^{\,G}(x), y]_G = [x, res_{+U}^{\,G}(y)]_U. \tag{1.29}$$

b) *Let $x, y \in R_+(G)$ and let $\varphi \in \hat{G}$, then we have*

$$[\overline{(G,\varphi)}^G \cdot x, \overline{(G,\varphi)}^G \cdot y]_G = [x,y]_G. \tag{1.30}$$

Proof. a) It is enough to prove the assertion for $x = \overline{(K,\psi)}^U$ and $y = \overline{(H,\varphi)}^G$. The left hand side of (1.29) equals the number $\#\{s \in H\backslash G/H' \mid (H,\varphi) \leq {}^s(H',\varphi')\}$, cf. (1.16). Using (1.11) and (1.16) the right hand side is given by

$$\sum_{s \in U\backslash G/H} \#\{t \in U/U \cap {}^sH \mid (K,\psi) \leq {}^t(U \cap {}^sH, {}^s\varphi)\} = \#\{t \in U/U \cap {}^sH \mid (K,\psi) \leq {}^{ts}(H,\varphi)\}.$$

Now the decomposition $UsH = \dot{\bigcup}_{t \in U/U \cap {}^sH} tsH$ completes the proof.

b) Assuming $x, y \in \mathcal{M}_G/G$ equation (1.30) becomes an easy consequence of (1.16). $\square$

2. The Brauer induction formula

In section 1 we have seen that the family of maps b_G, indexed by all finite groups G, is a natural transformation between the ring valued functors R_+ and R on the category of finite groups. We are interested not only in a section a_G of b_G for each finite group G separately but in a family of such sections with functorial properties. For a family of maps $a_G : R(G) \longrightarrow R_+(G)$ we consider the following two conditions:

$$(*) \qquad \begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ {\scriptstyle \mathrm{res}^G_H}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{+}{}^G_H} \\ R(H) & \xrightarrow{a_H} & R_+(H) \end{array} \qquad \begin{array}{l} \text{commutes for all} \\ \text{subgroup relations } H \leq G, \end{array}$$

$$(**) \qquad \begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ & {\scriptstyle p_G}\searrow & \downarrow{\scriptstyle \pi_G} \\ & & \mathbf{Z}\hat{G} \end{array} \qquad \begin{array}{l} \text{commutes for all} \\ \text{groups } G. \end{array}$$

For the rest of this section we will be concerned with the proof of the following main theorem:

(2.1) Theorem. *There is one and only one family of maps $a_G : R(G) \longrightarrow R_+(G)$ satisfying the conditions $(*)$ and $(**)$. Using the notation*

$$a_G(\chi) = \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\overline{(H,\varphi)}^G, \quad \chi \in R(G),$$

this family has the following properties:

a) *Discriptions of a_G:*

(i) *The coefficients $\alpha_{\overline{(H,\varphi)}^G}(\chi)$ are the unique solution of the linear equation system*

$$\text{(2.2)} \qquad \Gamma(G)\cdot\left(\alpha_{\overline{(H,\varphi)}^G}(\chi)\right)_{\overline{(H,\varphi)}^G} = \left((\varphi, \chi|_H)\right)_{\overline{(H,\varphi)}^G}$$

or equivalently of the following equations, indexed by $\overline{(H,\varphi)}^G \in \mathcal{M}_G/G$,

$$\text{(2.3)} \qquad (\varphi, \chi|_H) = \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \gamma^G_{(H,\varphi),(H',\varphi')}\alpha_{\overline{(H',\varphi')}^G}(\chi).$$

(ii) *a_G is the unique map such that the following diagram commutes*

$$\text{(2.4)} \qquad \begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ & {\scriptstyle (p_H \mathrm{res}^G_H)_{H\leq G}}\searrow & \downarrow{\scriptstyle \rho_G} \\ & & \left(\prod_{H\leq G} \mathbf{Z}\hat{H}\right)^G \end{array}$$

(iii) a_G *is the right adjoint map of* b_G *with respect to* $[-,-]$ *and* $(-,-)$.

(iv) Let $\mu_{(H,\varphi),(H',\varphi')}$ *denote the Möbius function of the poset* $\mathcal{M}_G$, *i.e. (cf. [R])*

$$\mu_{(H,\varphi),(H',\varphi')} = \sum_{i\le 0}(-1)^i\#\{\mathcal{M}_G\text{-chains } (H,\varphi)=(H_0,\varphi_0)<\ldots<(H_i,\varphi_i)=(H',\varphi')\},$$

and let $c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}$ *be given by*

$$c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G} = \sum_{i\ge 0}(-1)^i\#\{i\text{-chains in } \mathcal{M}_G \text{ from } (H,\varphi) \text{ to the orbit } \overline{(H',\varphi')}^G\},$$

then we have for all $\chi \in R(G)$ *the explicit formulae*

$$(2.5)\qquad \begin{aligned} a)\quad & \alpha_{\overline{(H,\varphi)}^G}(\chi) = \frac{|H|}{|N_G(H,\varphi)|}\sum_{\overline{(H,\varphi)}^G\le\overline{(H',\varphi')}^G\in\mathcal{M}_G/G} c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}(\varphi',\chi|_{H'}) \\ b)\quad & a_G(\chi) = \frac{1}{|G|}\sum_{\overline{(H,\varphi)}^G\le\overline{(H',\varphi')}^G\in\mathcal{M}_G/G} |H|\,\mu_{(H,\varphi),(H',\varphi')}(\varphi',\chi|_{H'})\overline{(H,\varphi)}^G,. \end{aligned}$$

b) $b_Ga_G = id_{R(G)}$, *i.e. for all* $\chi\in R(G)$ *we have*

$$(2.6)\qquad \chi = \sum_{\overline{(H,\varphi)}^G\in\mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\mathrm{ind}_H^G\varphi.$$

c) a_G *is* $\mathbf{Z}\hat{G}$*-linear and for each* $\varphi\in\hat{G}$ *we have* $a_G(\varphi)=\overline{(G,\varphi)}^G$. *In particular,* a_G *is trivial for abelian groups* G.

d) For each group homomorphism $f: G'\longrightarrow G$, *the following diagram is commutative*

$$(2.7)\qquad \begin{array}{ccc} \mathbf{Q}R(G) & \xrightarrow{a_G} & \mathbf{Q}R_+(G) \\ {\scriptstyle res_f}\downarrow & & \downarrow{\scriptstyle res_{+f}} \\ \mathbf{Q}R(G') & \xrightarrow{a_{G'}} & \mathbf{Q}R_+(G') \end{array}$$

e) a_G *commutes with taking duals, i.e.* $a_G(\overline{\chi}) = \sigma_G a_G(\chi)$ *for all* $\chi\in R(G)$.

f) For all $k\in\mathbf{Z}$ *and* $\chi\in R(G)$ *we have*

$$(2.8)\qquad \Psi^k(\chi) = \sum_{\overline{(H,\varphi)}^G\in\mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\mathrm{ind}_H^G\varphi^k,$$

and if $(k,|G|)=1$, *then* Ψ^k *and* Ψ^k_+ *commute with* a_G.

g) For $H\le G$ *let* $tr_H^G: G^{ab}\longrightarrow H^{ab}$ *be the the transfer map. Then we have for all* $\chi\in R(G)$

$$(2.9)\qquad \det\chi = \prod_{\overline{(H,\varphi)}^G\in\mathcal{M}_G/G} (\varphi\circ tr_H^G)^{\alpha_{\overline{(H,\varphi)}^G}(\chi)}.$$

h) *For all* $\chi \in R(G)$ *with* $\chi(1) = 0$ *we have*

$$(2.10) \qquad \chi = \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) \mathrm{ind}_H^G(\varphi - 1_H).$$

j) *For all* $\chi \in R(G)$ *we have*

$$(2.11) \qquad \begin{aligned} &a) \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) = \chi(1), \\ &b) \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} (G:H)\alpha_{\overline{(H,\varphi)}^G}(\chi) = \chi(1), \\ &c) \sum_{\overline{(H,1_H)} \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) = (\chi, 1_G), \\ &d) \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G, \varphi^k=1} \alpha_{\overline{(H,\varphi)}^G}(\chi) = (\Psi^k(\chi), 1_G). \end{aligned}$$

k) *Let* χ *be an actual character of* G *and let* $(H,\varphi) \in \mathcal{M}_G$. *Then* $\alpha_{\overline{(H,\varphi)}^G}(\chi) = 0$ *in each of the following cases:*

(i) $(\varphi, \chi|_H) = 0$,

(ii) *there is some* $(\tilde{H}, \tilde{\varphi}) > (H,\varphi)$ *in* $\mathcal{M}_G$ *with* $(\tilde{\varphi}, \chi|_{\tilde{H}}) = (\varphi, \chi|_H)$,

(iii) $(H,\varphi) \not\geq (Z(\chi), \psi)$, *where* $Z(\chi)$ *denotes the center of* χ, *i.e. the biggest subgroup* $U \leq G$ *such that* $\chi|_U$ *is a multiple of an element of* $\hat{U}$, *and where* $\chi|_{Z(\chi)} = \chi(1)\cdot\psi$, $\psi \in \widehat{Z(\chi)}$,

(iv) $H \not\geq Z(G)$, *where* $Z(G)$ *denotes the center of* G.

From f) and g) we have the following result:

(2.12) Corollary. *Every* $\chi \in R(G)$ *can be written as*

$$\chi = \sum_i z_i \mathrm{ind}_{H_i}^G \varphi_i$$

with $z_i \in \mathbf{Z}$, $H_i \leq G$ *and* $\varphi_i \in \hat{H}_i$ *such that*

$$\Psi^k(\chi) = \sum_i z_i \mathrm{ind}_{H_i}^G \varphi_i^k \quad \text{and} \quad \det(\chi) = \prod_i \left(\varphi \circ tr_{H_i}^G\right)^{z_i}. \qquad \square$$

(2.13) Remark. a) Warning: In general the formula (2.8) is not the canonical one coming from $a_G(\Psi^k(\chi))$.

b) In order to calculate $a_G(\chi)$ for some $\chi \in \mathrm{Irr}G$ we first determine those $\overline{(H,\varphi)}^G \in \mathcal{M}_G/G$ for which $\alpha_{\overline{(H,\varphi)}^G}(\chi)$ is trivial by k). Then we use equation (2.3) for the maximal remaining pairs $\overline{(H,\varphi)}^G$. This amounts simply to

$$\alpha_{\overline{(H,\varphi)}^G}(\chi) = (\varphi, \chi|_H)/(N_G(H,\varphi) : H). \tag{2.14}$$

Next we try to determine the remaining coefficients by the equations (2.11) a)-d). If these equations are not sufficient to determine all the remaining coefficients we use (2.3) to get more equations. We go downwards in the poset $\mathcal{M}_G/G$ looking for maximal pairs $\overline{(H,\varphi)}^G$ with unknown coefficients and compute the $\gamma^G_{(H,\varphi),(H',\varphi')}$'s occuring in equation (2.3) as long as necessary. In this procedure the computation of the $\gamma^G_{(H,\varphi),(H',\varphi')}$'s is the part which requires the most calculations. So it should be avoided if possible. On the other hand there should be no difficulty for a computer to calculate the $\gamma^G_{(H,\varphi),(H',\varphi')}$'s and, since $\Gamma(G)$ is an upper triangular matrix, to calculate the $\alpha_{\overline{(H,\varphi)}^G}(\chi)$'s by the matrix equation (2.2).

c) By a) (ii) and the multiplicativity of ρ_G we see that a_G is a ring homomorphism if, and only if, $p_H\mathrm{res}^G_H$ is multiplicative for all H. Thus a_G is a ring homomorphism if, and only if, G is abelian.

d) Combining the equations (2.11) a) and b) we obtain that all the coefficients $\alpha_{\overline{(H,\varphi)}^G}(\chi)$ are nonnegative if, and only if, $\chi \in \mathbf{Z}\hat{G} \subseteq R(G)$. In fact, if all the $\alpha_{\overline{(H,\varphi)}^G}(\chi)$'s are nonnegative then (2.11) a) and b) imply that $\alpha_{\overline{(H,\varphi)}^G}(\chi) = 0$ for all $H < G$. Thus $a_G(\chi) = a_G\big(\sum_{\varphi\in\hat{G}} \alpha_{\overline{(G,\varphi)}^G}(\chi)\varphi\big)$, and the injectivity of a_G shows that $\chi \in \mathbf{Z}\hat{G}$.

Proof of the uniqueness statement: Let $(a_G)_G$ be a family satisfying $(*)$ and $(**)$. Then diagram (2.4) commutes for all G:

$$\pi_H \mathrm{res}^G_{+H} a_G \overset{(*)}{=} \pi_H a_H \mathrm{res}^G_H \overset{(**)}{=} p_H \mathrm{res}^G_H.$$

However, a_G is uniquely determined by (2.4), since ρ_G is injective. □

Proof of the existence statement: From the proof of the uniqueness we already know how to define a_G, namely by the commutative diagram (2.4) using the injectivity of ρ_G. It is obvious from (1.18) that the equations (2.2) or (2.3) and the commutativity of (2.4) are equivalent. But we have problems with the integrality of the solutions $\alpha_{\overline{(H,\varphi)}^G}(\chi)$ of (2.2), or in other words we don't know yet, whether the image of the diagonal map in (2.4) is contained in the image of ρ_G. In order to avoid these troubles for the moment we tensor all the occuring free abelian groups and the maps with $\mathbf{Q}$ over $\mathbf{Z}$. Then we obtain the diagram

$$\begin{array}{ccc} \mathbf{Q}R(G) & \xrightarrow{a_G} & \mathbf{Q}R_+(G) \\ & {\scriptstyle (p_H\mathrm{res}^G_H)_{H\le G}} \searrow & \downarrow {\scriptstyle \rho_G} \\ & & \big(\prod_{H\le G}\mathbf{Q}\hat{H}\big)^G \end{array} \tag{2.15}$$

where ρ_G is now a ring isomorphism, cf. (1.19). Thus we may define a_G as the unique map $\mathbf{Q}R(G) \longrightarrow \mathbf{Q}R_+(G)$ which makes the diagram (2.14) commutative. For this map a_G we will first prove some of the properties listed in the theorem and then we will use these properties to show that $a_G(R(G)) \subseteq R_+(G)$, i.e. $\alpha_{\overline{(H,\varphi)}^G}(\chi) \in \mathbf{Z}$.

Proof of a) (i),(ii): The commutativity of the $\mathbf{Q}$-tensored version of diagram (2.4) is true by the definition of a_G. (1.18) shows that each of the equations (2.2) and (2.3) is equivalent to the commutativity of (2.4). $\square$

Proof of a) (iii): For $x = \overline{(H,\varphi)}^G \in \mathcal{M}_G/G$ and $\chi \in \mathbf{Q}R_+(G)$ we have

$$\begin{aligned}[x, a_G(\chi)] &= \sum_{\overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H',\varphi')}^G}(\chi)[\overline{(H,\varphi)}^G, \overline{(H',\varphi')}^G] \\ &= \sum_{\overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H',\varphi')}^G}(\chi)\gamma^G_{(H,\varphi),(H',\varphi')} \overset{(2.3)}{=} (\varphi, \chi|_H) \\ &= (\mathrm{ind}_H^G \varphi, \chi) = (b_G\overline{(H,\varphi)}^G, \chi) = (b_G(x), \chi).\end{aligned}$$ $\square$

Proof of ($*$) and ($**$): The commutativity of diagram (2.15) implies the commutativity of the tensored version of the diagram in ($**$) by looking at the G-component of $\left(\prod_{H\leq G} \mathbf{Q}\hat{H}\right)^G$. The commutativity of the tensored diagram of ($*$) is shown by the following equation, which holds for all $\chi \in \mathbf{Q}R(G)$ and for all $x \in \mathbf{Q}R_+(H)$:

$$\begin{aligned}[x, \mathrm{res}_{+H}^{\,G}(a_G(\chi))]_H &\overset{(1.29)}{=} [\mathrm{ind}_{+H}^{\,G}(x), a_G(\chi)]_G \overset{a)(iii)}{=} \left(b_G(\mathrm{ind}_{+H}^{\,G}(x)), \chi\right)_G \\ &= \left(\mathrm{ind}_H^G(b_H(x)), \chi\right)_G = \left(b_H(x), \chi|_H\right)_H \overset{a)(iii)}{=} [x, a_H(\chi|_H)]_H.\end{aligned}$$

This implies $\mathrm{res}_{+H}^{\,G}(a_G(\chi)) = a_H(\chi|_H)$, as $[-,-]$ is non-degenerate. $\square$

Proof of c): As the adjoint map of b_G also a_G is additive. Using the $\mathbf{Q}\hat{G}$-linearity of b_G and the obvious formula $(\chi, \nu) = (\chi\varphi, \nu\varphi)$ for $\chi, \nu \in \mathbf{Q}R(G)$, $\varphi \in \hat{G}$, we get for all $\varphi \in \hat{G}$, $\chi \in \mathbf{Q}R(G)$ and $x \in \mathbf{Q}R_+(G)$:

$$\begin{aligned}[x, \overline{(G,\varphi)}^G a_G(\chi)] &\overset{(1.30)}{=} [\overline{(G,\varphi^{-1})}^G x, a_G(\chi)] \overset{a)(iii)}{=} (b_G(\overline{(G,\varphi^{-1})}^G x), \chi) \\ &= (\varphi^{-1} b_G(x), \chi) = (b_G(x), \varphi\chi) \overset{a)(iii)}{=} [x, a_G(\varphi\chi)].\end{aligned}$$

This proves the $\mathbf{Z}\hat{G}$-linearity of a_G.

In order to prove the remaining statement in c) we observe for all $(K,\psi) \in \mathcal{M}_G$:

$$\begin{aligned}[\overline{(K,\psi)}^G, a_G(\varphi)]_G &\overset{a)(iii)}{=} (b_G\overline{(K,\psi)}^G, \varphi)_G = (\mathrm{ind}_K^G \psi, \varphi)_G = (\psi, \varphi|_K)_K, \\ [\overline{(K,\psi)}^G, \overline{(G,\varphi)}^G]_G &= \gamma^G_{(K,\psi),(G,\varphi)} = \#\{s \in K \setminus G/G \mid (K,\psi) \leq {}^s(G,\varphi)\} \\ &= \begin{cases} 1, & \text{if } \psi = \varphi|_K; \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$ $\square$

Proof of b): By c) we have already proved b) for abelian groups G. Since $\chi \in \mathbf{Q}R(G)$ is uniquely determined by its restrictions to cyclic subgroups (χ is a function on G), the proof of b) is completed by

$$\mathrm{res}^G_H b_G a_G(\chi) = b_H a_H \mathrm{res}^G_H \chi = \mathrm{res}^G_H \chi$$

which holds for all cyclic subgroups H. □

Proof of d): Since R and R_+ are functors and $f : G' \longrightarrow G$ can be written as the composition $G' \xrightarrow{f} f(G') \leq G$ and since diagram (2.7) commutes for subgroup inclusions, we may assume that f is surjective. In this case we prove the commutativity of the above diagram by induction on $|G'|$.

For $|G'| = 1$ this is trivial. So let $|G'| > 1$ and $\chi \in \mathrm{Irr}(G)$. If χ is linear then c) and (1.8) imply the commutativity. So assume that $\chi \in \mathrm{Irr}(G)$ is not linear. By the injectivity of ρ_G, it is enough to show for each $H' \leq G'$ that

$$\pi_{H'} \mathrm{res}_{+H'}^{G'} \left(a_{G'} \mathrm{res}_f(\chi) - \mathrm{res}_{+f} a_G(\chi) \right) = 0.$$

For $H' = G'$ we get $\pi_{G'} a_{G'}(\mathrm{res}_f \chi) = \sum_{\varphi' \in \widehat{G'}} \alpha_{\overline{(G',\varphi')}^{G'}}(\mathrm{res}_f \chi)\varphi' = 0$ since by (**) we have $\alpha_{\overline{(G',\varphi')}^{G'}}(\mathrm{res}_f \chi) = (\mathrm{res}_f \chi, \varphi')_{G'} = 0$ (note that $\mathrm{res}_f \chi$ is non-linear irreducible as χ is). On the other hand we have $\alpha_{\overline{(G,\varphi)}^G}(\chi) = 0$ for all $\varphi \in \hat{G}$ and we know by (1.8) that $\mathrm{res}_{+f} \overline{(H,\varphi)}^G = \overline{(f^{-1}(H), \varphi \circ f)}^{G'}$. So if $H < G$, then $f^{-1}(H) < G'$ and $\mathrm{res}_{+f} a_G(\chi)$ has vanishing coefficients at $\overline{(G',\varphi')}^{G'}$ for all $\varphi' \in \widehat{G'}$, i.e. $\pi_{G'} \mathrm{res}_{+f} a_G(\chi) = 0$.

Now let $H' < G'$. Then we have:

$$\begin{aligned} \mathrm{res}_{+H'}^{G'} a_{G'}(\mathrm{res}_f \chi) &\overset{(*)}{=} a_{H'} \mathrm{res}_{H'}^{G'} \mathrm{res}_{G' \xrightarrow{f} G}(\chi) = a_{H'} \mathrm{res}_{H' \xrightarrow{f} f(H')} \mathrm{res}^G_{f(H')}(\chi) \\ &\overset{(*)}{=} \mathrm{res}_{+(H' \xrightarrow{f} f(H'))} \mathrm{res}_{+f(H')}^G a_G(\chi) = \mathrm{res}_{+H'}^{G'} \mathrm{res}_{+f} a_G(\chi), \end{aligned}$$

where we again used the functoriality of R and R_+ and the induction hypothesis for the map $f : H' \longrightarrow f(H')$. □

(2.16) Corollary. *Let $N \triangleleft G$, $\overline{G} = G/N$ and let $\chi \in \mathbf{Q}R(G)$ come by inflation along $G \longrightarrow \overline{G}$ from some $\overline{\chi} \in \mathbf{Q}R(\overline{G})$. Then we have*

$$a_G(\chi) = \sum_{\overline{(\overline{H},\overline{\varphi})}^{\overline{G}} \in \mathcal{M}_{\overline{G}}/\overline{G}} \alpha_{\overline{(\overline{H},\overline{\varphi})}^{\overline{G}}}(\overline{\chi}) \overline{(H,\varphi)}^G.$$

This means $\alpha_{\overline{(H,\varphi)}^G}(\chi) = 0$ unless $N \leq H \leq G$ and $\varphi|_N = 1_N$, and in this case we have $\alpha_{\overline{(H,\varphi)}^G}(\chi) = \alpha_{\overline{(\overline{H},\overline{\varphi})}^{\overline{G}}}(\overline{\chi})$.

Proof. Clear with (1.12) and its subsequent remark concerning the injectivity of $\mathrm{res}_{+(G\to\overline{G})}$ on the basis $\mathcal{M}_{\overline{G}}/\overline{G}$. $\square$

Proof of k) (i): We argue by descending induction on $|H|$ using the equation (2.3)

$$(\varphi,\chi|_H)=\sum_{\overline{(H,\varphi)}^G\le\overline{(H',\varphi')}^G\in\mathcal{M}_G/G}\gamma^G_{(H,\varphi),(H',\varphi')}\alpha_{\overline{(H',\varphi')}^G}(\chi).$$

For $H=G$ equation (2.3) becomes $(\varphi,\chi|_H)=\alpha_{\overline{(G,\varphi)}^G}(\chi)$ which implies our statement in this case. Let $H<G$, then $(\varphi,\chi|_H)=0$ implies $(\varphi',\chi|_{H'})=0$ for all $(H',\varphi')\in\mathcal{M}_G$ with $(H,\varphi)\le(H',\varphi')$. By the induction hypothesis we know that $\alpha_{\overline{(H',\varphi')}^G}(\chi)=0$ for all $\overline{(H',\varphi')}^G\in\mathcal{M}_G/G$ with $\overline{(H,\varphi)}^G<\overline{(H',\varphi')}^G$. So equation (2.3) becomes $(\varphi,\chi|_H)=\gamma^G_{\overline{(H,\varphi)}^G(H,\varphi)}\alpha_{\overline{(H,\varphi)}^G}(\chi)$. $\square$

The proof of the remaining parts of k) requires further terminology. We fix a finite dimensional $\mathbf{C}G$-module V with character χ. In this fixed situation we establish something like a Galois correspondence between certain subspaces of V and certain pairs in $\mathcal{M}_G$.

(2.17) Definition. a) With V and χ as above we define for each $(H,\varphi)\in\mathcal{M}_G$ the subspace $F(H,\varphi)$ of V by

$$F(H,\varphi)=\{v\in V\mid hv=\varphi(h)v \text{ for all } h\in H\},$$

i.e. $F(H,\varphi)$ is the φ-homogeneous component of the $\mathbf{C}H$-module V and therefore we have $\dim_{\mathbf{C}}F(H,\varphi)=(\varphi,\chi|_H)$.
Conversely, we define for each $\mathbf{C}$-linear subspace $W\ne 0$ of V the pair $P(W)\in\mathcal{M}_G$ by

$$P(W)=\sup\{(H,\varphi)\in\mathcal{M}_G\mid hw=\varphi(h)w \text{ for all } h\in H,\ w\in W\}.$$

We show that $P(W)$ is well-defined. First of all the trivial pair $(E,1)$ satisfies the condition in the definition. Secondly, if two pairs (H,φ) and (K,ψ) satisfy the condition, then also does the pair consisting of the subgroup U generated by H and K and the extension μ of φ and ψ. Such an extension exists, since U acts on W by scalar multiplication as H and K do.
b) We call a pair $(H,\varphi)\in\mathcal{M}_G$ *admissible* for V, or for χ, if $F(H,\varphi)\ne 0$, i.e. $(\varphi,\chi|_H)\ne 0$. We denote by $A(V)$ or $A(\chi)$ the set of admissible pairs for V and by $S(V)$ the set of non-zero $\mathbf{C}$-linear subspaces of V. Then the maps

$$F: A(V)\longrightarrow S(V) \quad\text{and}\quad P: S(V)\longrightarrow A(V)$$

are well-defined.

(2.18) Remark. There is an obvious poset structure and an obvious G-action on $A(V)$ and on $S(V)$. For $W, W' \in S(V)$, $(K,\psi), (H,\varphi) \in A(V)$ and $s \in G$ we have the following immediate consequences of the preceding definitions:

a) $A(V)$ is closed under taking subpairs and conjugated pairs.

b) $F({}^s(H,\varphi)) = sF(H,\varphi)$ and $P(sW) = {}^sP(W)$.

c) $(K,\psi) \leq (H,\varphi) \Rightarrow F(K,\psi) \supseteq F(H,\varphi)$ and $W \subseteq W' \Rightarrow P(W) \geq P(W')$.

d) $(H,\varphi) \leq PF(H,\varphi)$ and $W \subseteq FP(W)$.

e) $F(H,\varphi) = FPF(H,\varphi)$ and $P(W) = PFP(W)$. □

(2.19) Definition. For $(H,\varphi) \in A(V)$ and $W \in S(V)$ we define the *closures* $\mathrm{cl}(H,\varphi)$ and $\mathrm{cl}(W)$ of (H,φ) and W by

$$\mathrm{cl}(H,\varphi) = PF(H,\varphi) \quad \text{and} \quad \mathrm{cl}(W) = FP(W).$$

We call $(H,\varphi) \in A(V)$, resp. $W \in S(V)$, *closed*, if it coincides with its closure.

(2.20) Lemma.

a) For all $(H,\varphi), (H',\varphi') \in A(V)$ and $W, W' \in S(V)$ we have

(i) $\mathrm{cl}(H,\varphi)$ and $\mathrm{cl}(W)$ are closed.

(ii) $F(\mathrm{cl}(H,\varphi)) = F(H,\varphi)$ and $P(\mathrm{cl}(W)) = P(W)$.

(iii) $(H,\varphi) \leq (H',\varphi') \Rightarrow \mathrm{cl}(H,\varphi) \leq \mathrm{cl}(H',\varphi')$ and $W \subseteq W' \Rightarrow \mathrm{cl}(W) \subseteq \mathrm{cl}(W')$.

b) For $(H,\varphi) \in A(V)$, $W \in S(V)$ and $s \in G$ the following statements are equivalent:

(i) (H,φ) (resp. W) is closed,

(ii) ${}^s(H,\varphi)$ (resp. sW) is closed,

(iii) (H,φ) (resp. W) is contained in the image of P (resp. F).

c) F and P are inverse bejections if restricted to the closed pairs in $A(V)$ and the closed subspaces in $S(V)$.

d) For $W, W' \in S(V)$ we have

$$\inf\{P(W), P(W')\} = P(W + W'),$$

in particular the infimum of closed pairs is again closed.

Proof. a) and b) are trivial consequences of remark (2.18) above.

c) Images under F and P are always closed by b), and FP and PF are the identity maps on closed objects by (2.18) e).

d) For $(H,\varphi) \in A(V)$ we have

$$\begin{aligned} & (H,\varphi) \le P(W) \text{ and } (H,\varphi) \le P(W') \\ \iff & H \text{ acts on } W \text{ and } W' \text{ by } \varphi \\ \iff & H \text{ acts on } W+W' \text{ by } \varphi. \end{aligned}$$

$\inf\{P(W), P(W')\}$ is the greatest pair (H,φ) having the first property and $P(W+W')$ is the greatest one having the last property. □

(2.21) Proposition. *If $(U,\psi) \le (H,\varphi) \in A(V)$ and if (H,φ) is closed, then*

$$\gamma^G_{(U,\psi),(H,\varphi)} = \gamma^G_{\mathrm{cl}(U,\psi),(H,\varphi)}.$$

Proof. Define $(U',\psi') = \mathrm{cl}(U,\psi)$, then $(U',\psi') \le (H,\varphi)$, since (H,φ) is closed. From (1.11) and (1.14) we see that

$$\mathrm{res}_{+U'}^{G}\overline{(H,\varphi)}^G = \gamma^G_{(U',\psi'),(H,\varphi)}\overline{(U',\psi')}^{U'} + \sum_s \overline{(U' \cap {}^sH, {}^s\varphi)}^{U'},$$

where s runs through those double cosets in $U'\backslash G/H$ with $\overline{(U' \cap {}^sH, {}^s\varphi)}^{U'} \ne \overline{(U',\psi')}^{U'}$. Therefore we get

$$\begin{aligned} \mathrm{res}_{+U}^{G}\overline{(H,\varphi)}^G &= \mathrm{res}_{+U}^{U'}\mathrm{res}_{+U'}^{G}\overline{(H,\varphi)}^G \\ &= \gamma^G_{(U',\psi'),(H,\varphi)}\underbrace{\mathrm{res}_{+U}^{U'}\overline{(U',\psi')}^{U'}}_{=\overline{(U,\psi)}^U} + \sum_s \mathrm{res}_{+U}^{U'}\overline{(U' \cap {}^sH, {}^s\varphi)}^{U'}, \end{aligned}$$

s as above. Since $\gamma^G_{(U,\psi),(H,\varphi)}$ is the coefficient of $\mathrm{res}_{+U}^{G}\overline{(H,\varphi)}^G$ at $\overline{(U,\psi)}^U$, it is enough to show that in the last sum, running over s, no element $\overline{(U,\psi)}^U$ occurs. Furthermore looking at (1.20) it is enough to show that for all these s in the last sum we have $\overline{(U,\psi)}^{U'} \not\le \overline{(U' \cap {}^sH, {}^s\varphi)}^{U'}$. So we complete the proof by showing

$$\overline{(U,\psi)}^{U'} \le \overline{(U' \cap {}^sH, {}^s\varphi)}^{U'} \Rightarrow \overline{(U' \cap {}^sH, {}^s\varphi)}^{U'} = \overline{(U',\psi')}^{U'}.$$

$\overline{(U,\psi)}^{U'} \le \overline{(U' \cap {}^sH, {}^s\varphi)}^{U'}$ implies that there is some $u \in U'$ such that $(U,\psi) \le {}^u(U' \cap {}^sH, {}^s\varphi) = (U' \cap {}^{us}H, {}^{us}\varphi)$. And this implies $(U',\psi') = \mathrm{cl}(U,\psi) \le \mathrm{cl}(U' \cap {}^{us}H, {}^{us}\varphi)$. Defining $(H',\varphi') = \mathrm{cl}(U' \cap {}^{us}H, {}^{us}\varphi)$ we get $\psi'|_{U' \cap {}^{us}H} = \varphi'|_{U' \cap {}^{us}H} = {}^{us}\varphi$ which implies $(U,\psi) \le (U' \cap {}^{us}H, {}^{us}\varphi) \le (U',\psi') = \mathrm{cl}(U,\psi)$. So we get $\mathrm{cl}(U' \cap {}^{us}H, {}^{us}\varphi) = (U',\psi')$. However, $(U' \cap {}^{us}H, {}^{us}\varphi) = \inf\{(U',\psi'), ({}^{us}H, {}^{us}\varphi)\}$ and both (U',ψ') and $({}^{us}H, {}^{us}\varphi) = {}^{us}(H,\varphi)$ are closed. So by (2.20) d),

$(U' \cap {}^{us}H, {}^{us}\varphi)$ is closed as well, i.e. $(U' \cap {}^{us}H, {}^{us}\varphi) = (U', \psi')$ which implies $\overline{(U' \cap {}^{s}H, {}^{s}\varphi)}^{U'} = \overline{(U', \psi')}^{U'}$. □

Proof of k) (ii): Assume that $\alpha_{\overline{(H,\varphi)}^G}(\chi) \neq 0$. Then we know from proposition k) (i) that (H, φ) is admissible for χ. We show that (H, φ) is closed for χ by descending induction on $|H|$. If $H = G$ then (G, φ) is closed for χ since it is maximal in $A(V)$. Now let $H < G$ and assume that (H, φ) is not closed for χ. Define $(U, \psi) = \mathrm{cl}(H, \varphi)$ and consider equation (2.3) for (H, φ) and (U, ψ):

$$(\varphi, \chi|_H) = \gamma^G_{(H,\varphi),(H,\varphi)} \alpha_{\overline{(H,\varphi)}^G}(\chi) + \sum_{\overline{(H,\varphi)}^G < \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \gamma^G_{(H,\varphi),(H',\varphi')} \alpha_{\overline{(H',\varphi')}^G}(\chi),$$

$$(\psi, \chi|_U) = \sum_{\overline{(U,\psi)}^G \leq \overline{(U',\psi')}^G \in \mathcal{M}_G/G} \gamma^G_{(U,\psi),(U',\psi')} \alpha_{\overline{(U',\psi')}^G}(\chi).$$

In both equations we merely have to sum up over admissible and closed pairs (H', φ') and (U', ψ') by our induction hypothesis. Since $(U, \psi) = \mathrm{cl}(H, \varphi)$ and (H, φ) is not closed, the closed pairs which are greater than (H, φ) are exactly the closed pairs which are greater or equal to (U, ψ). This means that the sums on the right side of the two equations have the same value, since by proposition (2.21) the corresponding γ-factors coincide. On the other side we have $(\psi, \chi|_U) = \dim_{\mathbb{C}} F(U, \psi) = \dim_{\mathbb{C}} F(\mathrm{cl}(H, \varphi)) = \dim_{\mathbb{C}} F(H, \varphi) = (\varphi, \chi|_H)$. So subtraction of the two equations yields $\alpha_{\overline{(H,\varphi)}^G}(\chi) = 0$ which contradicts our assumption. Thus (H, φ) is closed. $(\tilde{\varphi}, \chi|_{\tilde{H}}) = (\varphi, \chi|_H)$ implies $\dim_{\mathbb{C}} F(\tilde{H}, \tilde{\varphi}) = \dim_{\mathbb{C}} F(H, \varphi)$. Hence we have $F(\tilde{H}, \tilde{\varphi}) = F(H, \varphi)$ and $\mathrm{cl}(H, \varphi) = \mathrm{cl}(\tilde{H}, \tilde{\varphi}) \geq (\tilde{H}, \tilde{\varphi}) > (H, \varphi)$. This is a contradiction, since (H, φ) is closed. □

Proof of k) (iii): Let V be a $\mathbb{C}G$-module affording the character χ. If $\alpha_{\overline{(H,\varphi)}^G}(\chi) \neq 0$ then (H, φ) is admissible (by k) (i)) and closed for χ (by k) (ii)), and we have $F(Z(\chi), \psi) = V \supseteq F(H, \varphi)$ implying $(H, \varphi) = PF(H, \varphi) \geq PF(Z(\chi), \psi) \geq (Z(\chi), \psi)$. □

Proof of k) (iv): It is enough to show the assertion for $\chi \in \mathrm{Irr}(G)$. In this case we have $Z(G) \leq Z(\chi)$ and the proof is completed by k) (iii). □

Proof of f): Let k be arbitrary. We have to show that $b_G \Psi^k_+ a_G(\chi)$ and $\Psi^k(\chi)$ coincide after being restricted to all cyclic subgroups of G. However, this is evident, since Ψ^k_+ and Ψ^k commute with restrictions to subgroups and since (2.8) holds obviously for all cyclic groups, cf. c). Now let $(k, |G|) = 1$. We have to show that

$$\pi_H \mathrm{res}^G_{+H}(\Psi^k_+ a_G(\chi) - a_G \Psi^k(\chi)) = 0$$

for each $H \leq G$. Since Ψ^k_+, Ψ^k and a_G commute with restrictions, it is enough to show that $\pi_H(\Psi^k_+ a_H(\chi) - a_H\Psi^k(\chi)) = 0$ for $\chi \in \mathrm{Irr}H$. However, this is trivial if χ is linear, and if χ is nonlinear, then the above equation follows from the fact that $\Psi^k(\chi)$ is again irreducible and has the same dimension as χ. In fact, Ψ^k preserves dimensions by its definition and using this together with $(\Psi^k(\chi), \Psi^k(\chi)) = (\chi, \chi)$ the irreducibility of $\Psi^k(\chi)$ follows. $\square$

Proof of e): This is a consequence of part f), since $\check{\chi} = \Psi^{-1}(\chi)$ for all $\chi \in R(G)$ and since $\sigma_G\overline{(H,\varphi)}^G = \Psi_+^{-1}\overline{(H,\varphi)}^G$ for all $(H,\varphi) \in \mathcal{M}_G$. $\square$

For $k = 0$ we obtain from (2.8):

(2.22) Corollary. *For all $\chi \in R(G)$ we have*

$$\chi(1)\cdot 1_G = \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi) \mathrm{ind}_H^G 1_H. \tag{2.23}$$ $\square$

Proof of h): This follows immediately from corollary (2.22). $\square$

Proof of j): (2.11) b) follows from counting dimensions on both sides of (2.6). d) follows from taking scalar product with 1_G on both sides of (2.8), and applying Frobenius reciprocity. a) and c) are special cases of d), namely for $k = 0$ and $k = 1$. $\square$

Next we will prove the integrality of the coefficients. This proof is due to G.-M. Cram:

(2.24) Proposition. *For all $\chi \in R(G)$ and for all $(H,\varphi) \in \mathcal{M}_G$ we have $\alpha_{\overline{(H,\varphi)}^G}(\chi) \in \mathbf{Z}$.*

Proof. Assume that G is a group with minimal order such that there is some $\chi \in R(G)$ with $a_G(\chi) \notin R_+(G)$. We may assume that $\chi \in \mathrm{Irr}G$. Let furthermore $(H,\varphi) \in \mathcal{M}_G$ be maximal with $\alpha_{\overline{(H,\varphi)}^G}(\chi) \notin \mathbf{Z}$.

First we show that under the above assumptions we have $N_G(H,\varphi) = G$: We define $U = N_G(H,\varphi)$ and consider the coefficient of $a_U(\chi|_U) = \mathrm{res}_{+U}^{\;G} a_G(\chi)$ at $\overline{(H,\varphi)}^U$ which gives the equation (cf. (1.20)-(1.22))

$$\alpha_{\overline{(H,\varphi)}^U}(\chi|_U) = \frac{|U \cdot U|}{|U|}\alpha_{\overline{(H,\varphi)}^G}(\chi) + \sum_{\overline{(H,\varphi)}^G < \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} \delta_{\overline{(H,\varphi)}^U}^{\overline{(H',\varphi')}^G} \alpha_{\overline{(H',\varphi')}^G}(\chi).$$

From the maximality of (H,φ) it follows that the sum on the right side is an integer, i.e. $\alpha_{\overline{(H,\varphi)}^U}(\chi|_U) - \alpha_{\overline{(H,\varphi)}^G}(\chi) \in \mathbf{Z}$. But since $\alpha_{\overline{(H,\varphi)}^G}(\chi) \notin \mathbf{Z}$, we deduce from the minimality of $|G|$ that $U = G$.

Next we show that $H = Z(G)$: Since $H \triangleleft G$, $\chi|_H$ is the sum of G-conjugates of some $\psi \in \mathrm{Irr}H$ and since $\alpha_{\overline{(H,\varphi)}^G}(\chi) \neq 0$, we see from k) (i) that φ is one of these conjugates. But φ is G-invariant and hence we have $\chi|_H = \chi(1)\cdot\varphi$. χ and φ are inflations of $\overline{\chi} \in R(G/\ker\varphi)$ and $\overline{\varphi} \in \widehat{H/\ker\varphi}$ and corollary (2.16) implies $\alpha_{\overline{(H,\varphi)}^G}(\chi) = \alpha_{\overline{(H/\ker\varphi,\overline{\varphi})}^{G/\ker\varphi}}(\overline{\chi})$. Now the minimality of $|G|$ implies $\ker\varphi = 1$, and the existence of a faithful and G-invariant linear character of H implies $H \leq Z(G)$. k) (iv) shows that $H = Z(G)$.

By the maximality of (H,φ), by k) (i) together with $\chi|_H = \chi(1)\cdot\varphi$ and by k) (iv) we see that $\overline{(H,\varphi)}^G \in \mathcal{M}_G/G$ is the only element of $\mathcal{M}_G/G$ whose corresponding coefficient in $a_G(\chi)$ is not integral. This is a contradiction to j) a). □

Proposition (2.24) completes the proof of the existence statement of theorem (2.1). □

Proof of g): In general we have for all $(H,\varphi) \in \mathcal{M}_G$ the equation

$$\det(\mathrm{ind}_H^G\varphi) = \varepsilon_{G/H}\cdot(\varphi\mathrm{o}\mathrm{tr}_H^G)$$

where $\varepsilon_{G/H} \in \hat{G}$ is the sign character of the permutation action of G on G/H and tr_H^G : $G^{ab} \longrightarrow \mathbf{H}^{ab}$ is the transfer map, cf. [M] prop. 3.2. We have to show that for all $\chi \in R(G)$ we have $\prod_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} (\varepsilon_{G/H})^{\alpha_{\overline{(H,\varphi)}^G}(\chi)} = 1_G$ for each $\chi \in R(G)$. This follows from (2.23):

$$\begin{aligned} 1_G = (\det 1_G)^{\chi(1)} = \det(\chi(1)\cdot 1_G) &= \det\Big(\sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\mathrm{ind}_H^G 1_H\Big) \\ = \prod_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} (\varepsilon_{G/H}\cdot(1_H o tr_H^G))^{\alpha_{\overline{(H,\varphi)}^G}(\chi)} &= \prod_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} (\varepsilon_{G/H})^{\alpha_{\overline{(H,\varphi)}^G}(\chi)}. \end{aligned}$$ □

Proof of a) (iv): In order to get the explicit formula (2.5) a) for $\alpha_{\overline{(H,\varphi)}^G}(\chi)$ we will invert the matrix $\Gamma(G)$ explicitly. From (1.17) d) we obtain the decomposition

$$\Gamma(G) = \Delta(G)\cdot D(G)$$

where $\Delta(G)$ is the resulting matrix when we divide the columns of $\Gamma(G)$ by its own diagonal entries $(N_G(H,\varphi) : H)$ and where $D(G)$ is the diagonal matrix with these entries in the diagonal. We have to concentrate on $\Delta(G)$ only. By the combinatorial interpretation of $\gamma^G_{(H,\varphi),(H',\varphi')}/\gamma^G_{(H',\varphi'),(H',\varphi')}$ (cf. the note preceding (1.18)) we have for the entries $d_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}$ of $\Delta(G)$ the equation

$$d_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G} = \#\{(K,\psi) \in \overline{(H',\varphi')}^G \mid (H,\varphi) \leq (K,\psi)\}.$$

We may decompose the unipotent matrix $\Delta(G) = 1+N$ where N is an upper triangular nilpotent matrix with entries

$$n_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G} = \#\{(K,\psi) \in \overline{(H',\varphi')}^G \mid (H,\varphi) < (K,\psi)\}.$$

Since N is nilpotent we have

$$\Delta(G)^{-1} = 1 - N + N^2 - + \cdots \pm N^r$$

for sufficiently large r. Let $n^i_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}$ denote the entries of N^i, then it is easy to show by induction on i the equation

$$n^i_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G} = \#\{\mathcal{M}_G - \text{chains } (H,\varphi) = (H_0,\varphi_0) < \ldots < (H_i,\varphi_i) \mid (H_i,\varphi_i) \in \overline{(H',\varphi')}^G\}.$$

This shows equation (2.5) a). As a consequence we obtain equation (2.5) b):

$$\begin{aligned} a_G(\chi) &= \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \left(\frac{|H|}{|N_G(H,\varphi)|} \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}(\varphi', \chi|_{H'}) \right) \overline{(H,\varphi)}^G \\ &= \sum_{(H,\varphi) \in \mathcal{M}_G} \left(\frac{|H|}{|G|} \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G/G} c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}(\varphi', \chi|_{H'}) \right) \overline{(H,\varphi)}^G \\ &= \sum_{(H,\varphi) \in \mathcal{M}_G} \left(\frac{|H|}{|G|} \sum_{(H,\varphi) \leq (H',\varphi') \in \mathcal{M}_G} \mu_{(H,\varphi),(H',\varphi')}(\varphi', \chi|_{H'}) \right) \overline{(H,\varphi)}^G. \end{aligned}$$ □

Let $\mu_{H,H'}$ be the Möbius function of the poset $\mathcal{S}_G$ of subgroups of G, then for all $(H,\varphi) \leq (H',\varphi') \in \mathcal{M}_G$ we have $\mu_{(H,\varphi),(H',\varphi')} = \mu_{H,H'}$ by counting chains. Note that Möbius functions can be expressed as alternating sum of numbers of connecting chains of fixed length, cf. [R]. This yields another formula for $a_G(\chi)$ in terms of $\mu_{H,H'}$. For $\chi \in R(G)$ and $K \leq H \leq G$ we consider $p_H(\chi|_H)|_K \in \mathbf{Z}\hat{K}$ as an element in $R_+(K)$ by the decomposition given in the paragraph preceding (1.6).

(2.25) Corollary. *For all $\chi \in R(G)$ we have*

$$a_G(\chi) = \frac{1}{|G|} \sum_{K \leq H} |K| \, \mu_{K,H} \, \mathrm{ind}_{+K}^G(p_H(\chi|_H)|_K) \quad \in R_+(G)$$

and

$$\chi = \frac{1}{|G|} \sum_{K \leq H} |K| \, \mu_{K,H} \, \mathrm{ind}_K^G(p_H(\chi|_H)|_K) \quad \in R(G).$$

Proof. From (2.5) b) we obtain

$$\begin{aligned} a_G(\chi) &= \frac{1}{|G|} \sum_{(H,\varphi)\le(H',\varphi')\in\mathcal{M}_G} |H|\,\mu_{H,H'}\,(\varphi',\chi|_{H'})\mathrm{ind}_{+H}^{\,G}\varphi \\ &= \frac{1}{|G|} \sum_{H\le H'\in S_G} |H|\,\mu_{H,H'}\,\mathrm{ind}_{+H}^{\,G}\Big(\sum_{\varphi'\in\hat{H}'}(\varphi',\chi|_{H'})\varphi|_H\Big) \\ &= \frac{1}{|G|} \sum_{H\le H'\in S_G} |H|\,\mu_{H,H'}\,\mathrm{ind}_{+H}^{\,G}(p_{H'}(\chi|_{H'})|_H). \end{aligned}$$

Since $b_G\mathrm{ind}_{+H}^{\,G} = \mathrm{ind}_H^G b_H$, and since $b_H|_{\mathbf{Z}\hat{H}}$ is the identity, the second equation follows. □

3. Canonical induction formulae inducing from subgroups of special types and the Artin exponent

Throughout this section let $\mathcal{T}$ be a class of finite groups which is closed under taking isomorphic groups and subgroups and which contains all the cyclic groups. Let $S_G^{\mathcal{T}}$ be the poset of all subgroups of G of type $\mathcal{T}$, and let $\mathcal{M}_G^{\mathcal{T}}$ be the poset of all pairs $(H,\varphi)\in\mathcal{M}_G$ with $H\in S_G^{\mathcal{T}}$. G acts by conjugation from the left on $\mathcal{M}_G^{\mathcal{T}}$ and $S_G^{\mathcal{T}}$. $\mathcal{M}_G^{\mathcal{T}}/G$ and $S_G^{\mathcal{T}}/G$ are again posets in the obvious way. We define $R_+^{\mathcal{T}}(G)$, resp. $\Omega^{\mathcal{T}}(G)$, to be the free abelian group with basis $\mathcal{M}_G^{\mathcal{T}}/G$, resp. $S_G^{\mathcal{T}}/G$. If $\mathcal{T}$ is the class of all finite groups, then $\Omega^{\mathcal{T}}(G)=\Omega(G)$ is the Burnside ring of G, i.e. $\Omega(G)$ is the free abelian group over the G-conjugacy classes $\overline{H}^G=\overline{H}$ of subgroups H of G. $\Omega(G)$ can also be defined as the Grothendieck ring of the category of finite G-sets, cf. [Dr]. Note that by the identification $\overline{H}^G\mapsto\overline{(H,1_H)}^G$, $\Omega^{\mathcal{T}}(G)$ becomes a subring of $R_+^{\mathcal{T}}(G)$. The inclusion $\Omega(G)\subseteq R_+(G)$ comes from the functor which assigns to each finite G-set S the monomial representation of G whose lines are identified with the G-set S, such that G acts on the set of lines as on S and the stabilizer of every line acts trivially on it. Looking at the multiplication formula (1.5) we see that $R_+^{\mathcal{T}}(G)$ (resp. $\Omega^{\mathcal{T}}(G)$) is an ideal in $R_+(G)$ (resp. $\Omega(G)$). Hence $R_+^{\mathcal{T}}(G)$ is again a $\mathbf{Z}\hat{G}$-module. As in section 2 we indicate by a preceding $\mathbf{Q}$ that we have tensored an abelian group with $\mathbf{Q}$ over $\mathbf{Z}$. Since $\mathcal{T}$ is closed under taking subgroups, the restriction map $\mathrm{res}_{+H}^{\,G}$ maps elements of $R_+^{\mathcal{T}}(G)$ to elements of $R_+^{\mathcal{T}}(H)$ and the map

$$\rho_G^{\mathcal{T}} = (\pi_H\circ\mathrm{res}_{+H}^{\,G})_{H\in S_G^{\mathcal{T}}} : \quad R_+^{\mathcal{T}}(G)\longrightarrow\Big(\prod_{H\in S_G^{\mathcal{T}}}\mathbf{Z}\hat{H}\Big)^G$$

is injective as well as its $\mathbf{Q}$-tensored version, since it is the restriction of the injective map ρ_G, cf. (1.19). Since $R_+^{\mathcal{T}}(G)$ and $(\prod_{H\in S_G^{\mathcal{T}}}\mathbf{Z}\hat{H})^G$ have the same $\mathbf{Z}$-rank, $\rho_G^{\mathcal{T}}$ has finite cokernel and induces an isomorphism, again denoted by $\rho_G^{\mathcal{T}}$, between the corresponding $\mathbf{Q}$-algebras.

Note that $R_+^{\mathcal{T}}$ is in general not a functor. For example let $f : G' \longrightarrow G$ be a surjective group homomorphism with G of type $\mathcal{T}$ and G' not of type $\mathcal{T}$, then $\mathrm{res}_{+,f}\overline{(G,1_G)}^G = \overline{(G',1_{G'})}^{G'} \notin R_+^{\mathcal{T}}(G')$.

Again we may define duals, induction, Adams operations and bilinear forms as in the first section. The bilinear form is again non-degenerate, since with respect to the basis $\mathcal{M}_G^{\mathcal{T}}/G$ it is represented by the matrix

$$\Gamma^{\mathcal{T}}(G) = \left(\gamma^G_{(H,\varphi),(H',\varphi')}\right)_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G \in \mathcal{M}_G^{\mathcal{T}}/G}$$

which is a submatrix of $\Gamma(G)$ and again an upper triangular matrix with the old diagonal entries $(N_G(H,\varphi) : H)$, $\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}/G$. Proposition (1.28) holds also for elements of $R_+^{\mathcal{T}}(G)$, since $R_+^{\mathcal{T}}(G)$ is contained in $R_+(G)$.

Since $\mathcal{T}$ contains all the cyclic groups, the map $b_G : \mathbf{Q}R_+(G) \longrightarrow \mathbf{Q}R(G)$ restricted to $\mathbf{Q}R_+^{\mathcal{T}}(G)$ remains a surjective (cf. [Se], 9.2), $\mathbf{Q}\hat{G}$-linear and multiplicative map

$$b_G^{\mathcal{T}} : \mathbf{Q}R_+^{\mathcal{T}}(G) \longrightarrow \mathbf{Q}R(G).$$

Immitating the definition of a_G we define the map

$$a_G^{\mathcal{T}} : \mathbf{Q}R(G) \longrightarrow \mathbf{Q}R_+^{\mathcal{T}}(G), \quad \chi \mapsto \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}/G} \alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi)\overline{(H,\varphi)}^G,$$

where the $\alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi) \in \mathbf{Q}$ are the unique solution of the linear equation system

$$\text{(3.1)} \qquad \left((\varphi, \chi|_H)\right)_{\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}/G} = \Gamma^{\mathcal{T}}(G) \cdot \left(\alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi)\right)_{\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}/G}.$$

Obviously $a_G = a_G^{\mathcal{T}}$ for all G of type $\mathcal{T}$, and using exactly the same proofs we gave for a_G in the second section, we obtain the following results for $a_G^{\mathcal{T}}$:

(3.2) Theorem. *There is one and only one family of maps $a_G^{\mathcal{T}} : \mathbf{Q}R(G) \longrightarrow \mathbf{Q}R_+^{\mathcal{T}}(G)$ satisfying the two conditions*

$(*^{\mathcal{T}})$ *$a_G^{\mathcal{T}}$ commutes with restrictions to subgroups and*

$(**^{\mathcal{T}})$ *for all G, the following diagram is commutative*

$$\begin{array}{ccc} \mathbf{Q}R(G) \xrightarrow{a_G^{\mathcal{T}}} \mathbf{Q}R_+^{\mathcal{T}}(G) \subseteq \mathbf{Q}R_+(G) \\ \qquad\qquad\qquad\qquad\; \searrow^{p_G} \qquad\qquad \downarrow \pi_G \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{Q}\hat{G} \end{array}$$

This family has the following further properties:

a) *Descriptions of* $a_G^{\mathcal{T}}$:

(i) *The coefficients* $\alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi)$ *are the unique solution of of the following equations indexed by* $\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}$:

$$(\varphi, \chi|_H) = \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G^{\mathcal{T}}} \gamma^G_{(H,\varphi),(H',\varphi')} \alpha^{\mathcal{T}}_{\overline{(H',\varphi')}^G}(\chi).$$

(ii) $a_G^{\mathcal{T}}$ *is the unique map such that the diagram*

$$\begin{array}{ccc} \mathbf{Q}R(G) & \xrightarrow{a_G^{\mathcal{T}}} & \mathbf{Q}R_+^{\mathcal{T}}(G) \\ & {\scriptstyle (p_H \mathrm{res}_G^H)_{H\in S_G^{\mathcal{T}}}} \searrow & \downarrow \rho_G^{\mathcal{T}} \\ & & \left(\prod_{H\in S_G^{\mathcal{T}}} \mathbf{Q}\hat{H}\right)^G \end{array}$$

commutes.

(iii) $a_G^{\mathcal{T}}$ *is the adjoint map of* $b_G^{\mathcal{T}}$.

(iv) *Let* $\mu^{\mathcal{T}}_{(H,\varphi),(H',\varphi')}$ *and* $c^{\mathcal{T}}_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}$ *be defined for* $(H,\varphi),(H',\varphi') \in \mathcal{M}_G^{\mathcal{T}}$ *as in (2.1) a) (iv), then we have for all* $\chi \in R(G)$:

$$a)\quad \alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi) = \frac{|H|}{|N_G(H,\varphi)|} \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G^{\mathcal{T}}/G} c_{\overline{(H,\varphi)}^G,\overline{(H',\varphi')}^G}(\varphi', \chi|_{H'}),$$

$$b)\quad a_G^{\mathcal{T}}(\chi) = \frac{1}{|G|} \sum_{\overline{(H,\varphi)}^G \leq \overline{(H',\varphi')}^G \in \mathcal{M}_G^{\mathcal{T}}/G} |H|\, \mu_{(H,\varphi),(H',\varphi')}(\varphi', \chi|_{H'})\, \overline{(H,\varphi)}^G.$$

b) $b_G^{\mathcal{T}} \circ a_G^{\mathcal{T}} = \mathrm{id}_{\mathbf{Q}R(G)}$.

c) $a_G^{\mathcal{T}}$ *is* $\mathbf{Q}\hat{G}$*-linear and* $a_G^{\mathcal{T}}(\varphi) = \overline{(G,\varphi)}^G$ *for all* G *of type* $\mathcal{T}$ *and* $\varphi \in \hat{G}$.

d) $a_G^{\mathcal{T}}$ *commutes with taking duals.*

e) *For all* $k \in \mathbf{Z}$ *we have* $b_G^{\mathcal{T}} \Psi_+^k a_G^{\mathcal{T}} = \Psi^k$, *and if* $(k, |G|) = 1$, *then we have* $\Psi_+^k a_G^{\mathcal{T}} = a_G^{\mathcal{T}} \Psi^k$.

f) *For each* $\chi \in R(G)$ *with* $\chi(1) = 0$ *we have*

$$\sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G^{\mathcal{T}}/G} \alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi) \mathrm{ind}_H^G(\varphi - 1_H) = \chi.$$

g) *The equations corresponding to (2.11) hold.*

h) *For all characters* χ *of* G *and all* $(H,\varphi) \in \mathcal{M}_G^{\mathcal{T}}$ *we have*

$$(\varphi, \chi|_H) = 0 \quad \Rightarrow \quad \alpha^{\mathcal{T}}_{\overline{(H,\varphi)}^G}(\chi) = 0. \qquad \square$$

With the same arguments as in corollary (2.25) we obtain

(3.3) Corollary. *For all* $\chi \in R(G)$ *we have*

a) $a_G^{\mathcal{T}}(\chi) = \frac{1}{|G|} \sum_{K \leq H \in \mathcal{S}_G^{\mathcal{T}}} |K| \mu_{K,H} \operatorname{ind}_{+K}^G(p_H(\chi|_H)|_K)$,

b) $\chi = \frac{1}{|G|} \sum_{K \leq H \in \mathcal{S}_G^{\mathcal{T}}} |K| \mu_{K,H} \operatorname{ind}_K^G(p_H(\chi|_H)|_K)$,

where $\mu_{K,H}$ *is the Möbius function of the poset* $\mathcal{S}_G$ *or, which is the same, of the poset* $\mathcal{S}_G^{\mathcal{T}}$. $\square$

For the class $\mathcal{C}$ of all cyclic groups (3.3) b) is Brauer's explicit form (cf. [Br2] and [L], theorem 4.1) of Artin's induction theorem:

$$\chi = \frac{1}{|G|} \sum_{K \leq H \in \mathcal{S}_G^{\mathcal{C}}} |K| \mu_{K,H} \operatorname{ind}_K^G(\chi|_K).$$

It is clear by the reciprocal of Brauer's theorem proved by J. Green (cf. [Se] 11.3) that $a_G^{\mathcal{T}}(R(G))$ can't be contained in $R_+^{\mathcal{T}}(G)$ for all G if $\mathcal{T}$ doesn't contain all elementary groups. So the questions which arise are: Which are the types $\mathcal{T}$ such that $a_G^{\mathcal{T}}$ is integral; and if $a_G^{\mathcal{T}}$ is not integral, what can be said about the denominators? For a group G and $\chi \in R(G)$ we define the natural numbers

$$\text{(3.4)} \qquad \begin{aligned} A^{\mathcal{T}}(\chi) &= \min\{d \in \mathbf{N} \mid d \cdot a_G^{\mathcal{T}}(\chi) \in R_+^{\mathcal{T}}(G)\} \quad \text{and} \\ A^{\mathcal{T}}(G) &= \min\{d \in \mathbf{N} \mid d \cdot a_G^{\mathcal{T}}(R(G)) \subseteq R_+^{\mathcal{T}}(G)\}. \end{aligned}$$

$A^{\mathcal{T}}(G)$ is an invariant of G with respect to $\mathcal{T}$ which measures how far $a_G^{\mathcal{T}}$ is from being integral, or more unprecisely, how far G is from being of type $\mathcal{T}$. For example $A^{\mathcal{T}}(G) = 1$ is equivalent to the integrality of $a_G^{\mathcal{T}}$ and for each G of type $\mathcal{T}$ we have $A^{\mathcal{T}}(G) = 1$. From the explicit formula for $a_G^{\mathcal{T}}$ in (3.2) a) (iv) we obtain

$$\text{(3.5)} \qquad A^{\mathcal{T}}(G) \,\Big|\, |G|.$$

(3.6) Lemma.

a) *For all* $x \in \mathbf{Q}R_+^{\mathcal{T}}(G)$ *we have* $x \cdot a_G^{\mathcal{T}}(1_G) = x$, *i.e.* $\mathbf{Q}R_+^{\mathcal{T}}(G)$ *is again a ring with unity* $a_G^{\mathcal{T}}(1_G)$.

b) *If the class* $\mathcal{T}$ *is contained in the class* $\mathcal{U}$, *then we have for all* $\chi \in \mathbf{Q}R(G)$

$$a_G^{\mathcal{T}}(\chi) = a_G^{\mathcal{U}}(\chi) a_G^{\mathcal{T}}(1_G) \quad \in \mathbf{Q}R_+^{\mathcal{T}}(G).$$

Proof. For all $H \in \mathcal{S}_G^{\mathcal{T}}$ we have

$$\pi_H \mathrm{res}_{+H}^{\,G} a_G^{\mathcal{T}}(1_G) = \pi_H a_H^{\mathcal{T}}(1_H) = p_H(1_H) = 1_H \text{ and}$$

$$\pi_H \mathrm{res}_{+H}^{\,G} a_G^{\mathcal{T}}(\chi) = \pi_H a_H^{\mathcal{T}}(\chi|_H) = p_H(\chi|_H) = \pi_H a_H^{\mathcal{U}}(\chi|_H) = \pi_H \mathrm{res}_{+H}^{\,G} a_G^{\mathcal{U}}(\chi).$$

Now the equalities in a) and b) are shown by applying the injective map $\rho_G^{\mathcal{T}}$ to both sides. □

(3.7) Corollary.

a) $a_G^{\mathcal{T}}(\chi) = a_G(\chi) a_G^{\mathcal{T}}(1_G)$ *for all* $\chi \in \mathbf{Q}R(G)$.
b) $A^{\mathcal{T}}(G) = A^{\mathcal{T}}(1_G)$.
c) $A^{\mathcal{T}}(H) \mid A^{\mathcal{T}}(G)$ *for all* $H \leq G$. □

From the above Lemma we know that $a_G^{\mathcal{T}}(1_G)$ is an idempotent in $\mathbf{Q}R_+^{\mathcal{T}}(G)$ and theorem (3.2) h) implies that $a_G^{\mathcal{T}}(1_G)$ even lies in $\mathbf{Q}\Omega(G)$. First of all we want to obtain a criterion for $A^{\mathcal{T}}(G) = 1$ which is equivalent to $a_G^{\mathcal{T}}(1_G) \in \Omega(G)$. So we use the examinations of idempotents of $\Omega(G)$ and $\mathbf{Q}\Omega(G)$ by Dress and Yoshida (cf. [Dr], [Y]). Dress defines the injective ring homomorphism

$$\rho_G : \Omega(G) \longrightarrow \Big(\prod_{H \in \mathcal{S}_G} \mathbf{Z} \Big)^G, \quad S \mapsto (|S^H|)_{H \in \mathcal{S}_G}$$

where S is a finite G-set and S^H denotes the set of H-invariant elements of S. G acts on the product of copies of $\mathbf{Z}$ by permutation along the conjugation action of G on $\mathcal{S}_G$. ρ_G becomes an isomorphism if tensored with $\mathbf{Q}$ and it is just the restriction of the old map ρ_G of the first section, i.e. we have the commutative diagram

$$\text{(3.8)} \qquad \begin{array}{ccc} \mathbf{Q}\Omega(G) & \xrightarrow{\eta_G} & \mathbf{Q}R_+(G) \\ {\scriptstyle \rho_G}\big\downarrow & & {\scriptstyle \rho_G}\big\downarrow \\ \Big(\prod_{H \in \mathcal{S}_G} \mathbf{Z} \Big)^G & \xrightarrow{\eta_G} & \Big(\prod_{H \in \mathcal{S}_G} \mathbf{Q}\hat{H} \Big)^G \end{array}$$

where the lower map η_G embeds componentwise $z \mapsto z \cdot 1_H$ and the upper map η_G maps $\overline{H}^G$ to $\overline{(H, 1_H)}^G$. In fact, (3.8) is commutative, since for a transitive G-set $S \cong G/U$ the result of $\eta_G \rho_G$ in the H-component equals $|S^H| \cdot 1_H$ and the result of $\rho_G \eta_G$ in this component is given by

$$\pi_H \mathrm{res}_{+H}^{\,G} \overline{(U, 1_U)}^G = \pi_H \Big(\sum_{s \in H \backslash G/U} \overline{(H \cap {}^s U, 1)}^H \Big) = \sum_{s \in H \backslash G/U, H \leq {}^s U} 1_H \quad \text{and}$$

$$\{ s \in H \backslash G/U \mid H \leq {}^s U \} = \{ s \in G/U \mid H \leq {}^s U \} = (G/U)^H.$$

Let $e \in \mathbf{Q}\Omega(G)$ be the idempotent which is mapped to $a_G^{\mathcal{T}}(1_G)$ under ρ_G. Then we have

$$\text{(3.9)} \qquad \rho_G(e) = (\delta_{H \in \mathcal{T}})_{H \leq G}.$$

This follows from theorem (3.2) a) (ii), which implies $\rho_G(a_G^{\mathcal{T}}(1_G)) = (\delta_{H\in\mathcal{T}})_{H\leq G}$, and from the commutativity of diagram (3.8).

For $\overline{H}^G \in \mathcal{S}_G/G$ let $e_{\overline{H}} \in \mathbf{Q}\Omega(G)$ be the preimage by ρ_G of the primitive idempotent in $(\prod_{H\leq G} \mathbf{Q})^G$ being 1 in the components indexed by conjugates of H and 0 in the others. Since ρ_G is a ring isomorphism the idempotents of $\mathbf{Q}\Omega(G)$ are exactly the sums

$$\sum_{\overline{H}^G\in T} e_{\overline{H}} \tag{3.10}$$

where T is an arbitrary subset of $\mathcal{S}_G/G$. (3.9) shows that $a_G^{\mathcal{T}}(1_G)$ equals the sum in (3.10) for $T = \mathcal{S}_G^{\mathcal{T}}/G$. In the proof of theorem 3.1 in [Y], Yoshida characterized those subsets T which lead to idempotents in $\Omega(G)$. Using Dress' studies in [Dr] on the prime spectrum of $\Omega(G)$ Yoshida observed that the primitive idempotents in $\Omega(G)$ are exactly those sums (3.10) where T is an equivalence class of the *perfect equivalence* relation on $\mathcal{S}_G/G$:

$$\overline{K}^G \sim \overline{H}^G \iff K^p \text{ and } H^p \text{ are conjugate in } G \tag{3.11}$$

where H^p denotes the smallest perfect normal subgroup of H, i.e. the smallest $N \triangleleft H$ such that H/N is solvable. So we have

(3.12) Proposition. *$A^{\mathcal{T}}(G) = 1$ if, and only if, the set of conjugacy classes of subgroups of G of type $\mathcal{T}$ is a union of perfect equivalence classes.* □

(3.13) Corollary.

a) *$a_G^{\mathcal{T}}$ is integral for all G if and only if $\mathcal{T}$ is closed under taking extensions by cyclic groups of prime order, i.e. if for each exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow 1 \tag{3.14}$$

we have $H \in \mathcal{T} \iff G \in \mathcal{T}$.

b) *If $\mathcal{T}$ is the class of solvable groups then $a_G^{\mathcal{T}}$ is integral for all G. On the other hand, if $a_G^{\mathcal{T}}$ is integral for all G, then $\mathcal{T}$ contains the class of solvable groups.*

Proof. a) is an immediate consequence of the facts that for each exact sequence (3.14) we have $H \sim G$ and that G can be built up by such exact sequences from G^p.

b) follows from part a). □

Unfortunately we don't get a canonical and explicit version of Brauer's theorem for the class $\mathcal{T}$ of elementary groups. To obtain an integral canonical formula we have to admit at least all solvable subgroups.

For the remaining part of this section we specialize $\mathcal{T}$ to be the class $\mathcal{C}$ of all cyclic groups and we study $A^{\mathcal{C}}(G)$. We recall the definition, cf. [L] or [CR] (76.1), of the *Artin exponent* $A(G)$ of a group G:

$$A(G) = \min\{d \in \mathbf{N} \mid d{\cdot}R_{\mathbf{Q}}(G) \subseteq \sum_{H \in \mathcal{S}_G^{\mathcal{C}}} \operatorname{ind}_H^G R_{\mathbf{Q}}(H)\}$$

where $R_{\mathbf{Q}}(G)$ is the character ring of the rational representations of G. For $\chi \in R_{\mathbf{Q}}(G)$ we define

$$A(\chi) = \min\{d \in \mathbf{N} \mid d{\cdot}\chi \in \sum_{H \in \mathcal{S}_G^{\mathcal{C}}} \mathbf{Z}{\cdot}\operatorname{ind}_H^G R_{\mathbf{Q}}(H)\}.$$

Since for a cyclic group H we have $R_{\mathbf{Q}}(H) = \sum_{K \leq H} \mathbf{Z}{\cdot}\operatorname{ind}_K^H 1_K$, cf. prop.(76.6) in [CR], we obtain

$$\text{(3.15)} \qquad \begin{aligned} A(G) &= \min\{d \in \mathbf{N} \mid d{\cdot}R_{\mathbf{Q}}(G) \subseteq \sum_{H \in \mathcal{S}_G^{\mathcal{C}}} \mathbf{Z}{\cdot}\operatorname{ind}_H^G 1_H\}, \\ A(\chi) &= \min\{d \in \mathbf{N} \mid d{\cdot}\chi \subseteq \sum_{H \in \mathcal{S}_G^{\mathcal{C}}} \mathbf{Z}{\cdot}\operatorname{ind}_H^G 1_H\}, \end{aligned}$$

$A(G)$ is well-defined, since each $\chi \in R_{\mathbf{Q}}(G)$ can be expressed as a $\mathbf{Q}$-linear combination of $\operatorname{ind}_H^G 1_H$, H cyclic, cf. theorem 30 in [Se]. This means that the map

$$\text{(3.16)} \qquad b_G^{\mathcal{C}} : \mathbf{Q}\Omega^{\mathcal{C}}(G) \longrightarrow \mathbf{Q}R_{\mathbf{Q}}(G)$$

which is the restriction of $b_G : \mathbf{Q}R_+(G) \longrightarrow \mathbf{Q}R(G)$ is surjective. But since $\Omega^{\mathcal{C}}(G)$ and $R_{\mathbf{Q}}(G)$ have the same $\mathbf{Z}$-rank, cf. cor. 1 of theorem 29 in [Se], the map (3.16) is an isomorphism. Since $a_G^{\mathcal{C}}(1_G) \in \mathbf{Q}\Omega^{\mathcal{C}}(G)$, cf. (3.2) h), we know that $a_G^{\mathcal{T}}(1_G)$ is the preimage of 1_G under this isomorphism.

(3.17) Theorem.

a) $A(G) = A(1_G) = A^{\mathcal{C}}(1_G) = A^{\mathcal{C}}(G)$.

b) Let $e = \sum_{\overline{H} \in \mathcal{S}_G^{\mathcal{C}}/G} e_{\overline{H}} \in \Omega^{\mathcal{C}}(G)$, *then* $e = a_G^{\mathcal{C}}(1_G)$ *and we have*

$$A(G) = \min\{d \in \mathbf{N} \mid d{\cdot}e \in \Omega(G)\}.$$

Proof. a) $A(G) = A(1_G)$, since we have by (3.15) an expression

$$\text{(3.18)} \qquad A(1_G){\cdot}1_G = \sum_{H \in \mathcal{S}_G^{\mathcal{C}}} z_H \operatorname{ind}_H^G 1_H,$$

and multiplication of (3.18) with an arbitrary $\chi \in R_{\mathbf{Q}}(G)$ yields

$$A(1_G)\cdot\chi = \sum_{H\in S_G^c} z_H \mathrm{ind}_H^G \chi|_H$$

with $\chi|_H \in R_{\mathbf{Q}}(H)$.

$A^{\mathcal{C}}(1_G) = A^{\mathcal{C}}(G)$ by (3.7) b).

Obviously $A(1_G) \le A^{\mathcal{C}}(1_G)$, since $a_G^c(1_G)$ provides an expression (3.18) where $A(1_G)$ is replaced by $A^{\mathcal{C}}(1_G)$.

So it is enough to show $A^{\mathcal{C}}(1_G) \le A(1_G)$. We consider an expression (3.18) for $A(1_G)$. This yields an element $x \in \Omega^{\mathcal{C}}(G)$ with $b_G(x) = A(1_G)\cdot 1_G$. Since the map in (3.16) is an isomorphism, we obtain $x = A(1_G)\cdot a_G^c(1_G)$, hence $A^{\mathcal{C}}(1_G) \le A(1_G)$.

b) $e = a_G^c(1_G)$ follows from the subsequent consideration of (3.10). The equation for $A(G)$ follows now from $A(G) = A^{\mathcal{C}}(1_G)$. □

(3.19) Remark. The equation for $A(G)$ in (3.17) b) shows that the Artin exponent is just a matter of calculations in $\Omega(G)$, namely to find the unity $e = \sum_{\overline{H}\in S_G^c/G} z_{\overline{H}}\overline{H}$ of $\mathbf{Q}\Omega^{\mathcal{C}}(G)$ and to determine the least common multiple of the denominators of the coefficients $z_{\overline{H}}$.

Theorem (3.17) enables us to apply all the results about the Artin exponent to $A^{\mathcal{C}}(G)$. In particular, for odd primes p, the p-part of $A^{\mathcal{C}}(G)$ can be calculated explicitly by the theorems 7.4 and 7.12 in [L]. Lam also determined the Artin exponent for 2-groups (theorem 6.3 in [L]). Conversely we can determine the Artin exponent of G by computing $a_G^c(1_G)$. Since $A(G) = 1$ if, and only if, G is cyclic ([L], theorem (2.9)), we obtain

(3.20) Corollary. *Let $\mathcal{C}$ be the class of all cyclic groups then a_G^c is integral if, and only if, G is cyclic.* □

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