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# DEFECT GROUPS AND THE ISOMORPHISM PROBLEM

by Leonard L. Scott[1]

**ABSTRACT** As a ring, a block may arise from more than one finite group. The resulting conjugacy issue for defect groups is important for the group ring isomorphism problem and for understanding block theory in general. There are even indirect structural consequences for finite groups, through G. Robinson's work on the "Z–star theorem" for odd primes. A positive answer to the defect group conjugacy problem is given here for the principal block in the case of cyclic, T.I., Sylow p–subgroups.

In a talk at Arcata [S] I raised the following question regarding defect groups: Let B be a block of group rings $\mathbb{Z}_p G$ and $\mathbb{Z}_p H$, where $\mathbb{Z}_p$ denotes the ring of p–adic integers and G, H are (possibly nonisomorphic) finite groups. Suppose B has defect group D in G and E in H. Identify D and E with their projections on B. Is it then true that, after applying some suitable normalization process to D and E (preserving their isomorphism types), these groups must then be conjugate by a unit of B? In case B is a principal block, normalization should just be the familiar and innocuous projection onto the units of augmentation 1, using an augmentation $B \to S$ obtained, say, from the sum–of–coefficients map $SG \to S$. For other blocks the correct formulation of normalization remains part of the problem.

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As Alperin has observed, a weaker version of the question is simply to ask if the block determines the defect group up to isomorphism. This allows the question to be meaningfully investigated over modular fields (and also, we note here, for Morita or derived equivalent blocks, though it remains reasonable in all these cases to hope also for a version describing more explicitly how the isomorphism takes place). C. Bessenrodt has obtained a number of interesting results in the modular context, as discussed at this conference. However, for intended later applications to the ismorphism problem — Does $\mathbb{Z}G \simeq \mathbb{Z}H$ imply G is isomorphic to H? — I have focused on the version stated above. Let me also mention that both Roggenkamp and Weiss have expressed the view that the question might be formulated even more strongly, in terms of p–groups that should be conjugate to a *subgroup* of the defect group.

In this note I wish to present the following theorem, including a somewhat condensed proof. The statement is an improvement over the result presented at this Luminy conference, in that I do not need to assume E is the image of D under an automorphism of D. The argument, however, is essentially the same.

*Theorem* *Let* S *be an unramified* (and integrally closed) *finite extension of* $\mathbb{Z}_p$. *Let* B *be the principal block of group algebras* SG, SH *over* S, *for finite groups* G *and* H. *Let* D *and* E *be Sylow* p–*subgroups of* G *and* H, *respectively, and identify them with their projections onto* B. *Assume that* E *is normalized in the sense of mapping to* 1 *under the augmentation* $B \to S$ *induced by* $SG \to S$ (which certainly also sends D to 1).

*Assume also that* D *is cyclic* (which implies that E is cyclic), *and that* D *and* E *are* T.I. *sets in* G *and* H, *respectively*. *Then* D *is conjugate to* E *in* B.

Since every finite simple group appears to have a cyclic T.I. set Sylow subgroup for some prime p, I am hopeful that even this basic case might have an impact on the isomorphism problem for nonsolvable groups. In the solvable (or just p–constrained) case, Roggenkamp and I have proved that, if the $\mathbb{Z}$–forms $\mathbb{Z}G/O_{p'}(G)$ and $\mathbb{Z}H/O_{p'}(H)$ for a principal block B agree, then any defect (Sylow) groups D for G and E for H, normalized in B as above, are conjugate by a unit of B. Indeed, the normalized projections of G and H on B are conjugate in this case [S]. The equality over $\mathbb{Z}$ (rather than $\mathbb{Z}_p$) is only needed to insure, in the case $O_{p'}(G) = 1$, that $O_p(H)$ maps to 1 in $\mathbb{Z}_pG/O_p(G)$. Roggenkamp and I would very much like to know <u>if this already follows</u>, if H is a group of augmentation 1 units which is a $\mathbb{Z}_p$–basis in $\mathbb{Z}_pG$. Perhaps someone reading this article may be able to answer this question.

I would like to mention that the first version of the above theorem, the defect 1 case with E the image of D under an automorphism, was obtained in collaboration with Roggenkamp, using his explicit determinations [Ro] of the blocks involved as orders. Of course, the first positive answer to any kind of defect group conjugacy question was the p–group case Roggenkamp and I treated in [RS].

One ingredient in the proof of the above theorem is the following lemma, which I found while thinking about work of G. Robinson on the $Z^*$ theorem for primes $p \geq 5$; the latter would follow from results of Robinson and a positive answer to the defect group conjugacy question for principal blocks [R].

<u>Lemma</u> <u>Let</u> S <u>be a complete</u> p–<u>adic domain or its residue field</u>. <u>Suppose</u> B <u>is a block of both</u> SG <u>and</u> SH <u>for finite groups</u> G <u>and</u> H, <u>for which</u> B <u>has defect groups</u> D <u>and</u> E, <u>respectively</u>. <u>Assume that</u> B <u>has trivial source as an</u> $SG \times H$–<u>module</u>. <u>Let</u> $e \in C_B(D)$ <u>be a source idempotent</u> (in the sense of Puig [P]) <u>for</u> B <u>with respect to</u> D <u>and</u> G, <u>and</u> $f \in C_B(E)$ <u>a source idempotent for</u> B <u>with respect to</u> E <u>and</u> H.

<u>Then</u> eD <u>is conjugate by a unit of</u> B <u>to</u> fE. <u>In particular</u>, e <u>is conjugate to</u> f, <u>and</u> D <u>is isomorphic to</u> E.

Briefly, the proof is as follows: First one shows that Be is an indecomposable $SH \times D$ module with vertex V a diagonal copy of D in $E \times D$. (Note Be is projective on both sides, while its restriction to V contains a trivial summand. Thus V is diagonal. If W is its projection onto D, then Be is projective relative to $H \times W$, thus projective relative to $B \otimes SW$ and $G \times W$. It follows that $W = D$, since e is a source idempotent.) Symmetry now gives an isomorphism of E with D, which may be used to make Be an $SH \times E$–module. As such, it has trivial source and vertex the standard diagonal of E. Hence it is a direct summand of the bimodule SH, so must be isomorphic to $Bf'$ for some primitive idempotent $f'$ of the centralizer in B of E. Examination of this isomorphism gives a conjugation of $f'E$ to eD. It can now be concluded that the idempotent $f'$ has defect group E, and is thus conjugate to f by a unit of B normalizing E, completing the proof.

Broué noted at the conference that he has encountered similar hypotheses in his work on fusion–compatible isometries, except that he uses the "endo–permutation", rather than the "trivial source" condition (and a "diagonal" hypothesis on the vertex of his Morita equivalence module, the analogue of B). The "endo–permutation" condition would at least suffice in the beginning of the argument above, replacing Be by $\mathrm{End}_S(Be)$.

Intellectual predecessors of the above argument include Ward [Wa] and Coleman [C], which concern fusion and automorphisms of p–subgroups of unit groups of group rings, and more modern observations of Puig dealing similarly with source algebras. I am grateful to R. Sandling for pointing out to me the results in Ward's paper.

When D and E are T.I. sets, as in the hypothesis of the theorem, one can fairly easily conclude from the lemma that D is conjugate to E in B: Use the isomorphism in the lemma to make B and Bf into $G \times D$ modules. The lemma implies that Bf is isomorphic to $Be_\beta$, a twist of Be by some automorphism $\beta$ of D. To prove D is conjugate to E, it is enough to show B is isomorphic to $B_\beta$. However, these modules have the same character on $G \times D$, and they differ from the isomorphic modules Bf and $Be_\beta$ by projectives, which are determined by their characters. This proves the desired isomorphism, and thus D is conjugate to E in B.

To prove the theorem, then, one just needs to verify the trivial source hypothesis of the lemma. That is, one needs to check that B is a permutation module for $D \times E$. Using Weiss's results [W], it suffices to check that B/BI(SE) is a permutation module for SD, where I(SE) is the augmentation ideal of SE. Thus, it suffices to check that each nonprojective indecomposable component M of B/BI(SE) has trivial source as an SG–module. Certainly M has trivial source and vertex E as an SH–module, since it is a nonprojective indecomposable direct summand of $SH/SH \cdot I(SE) \cong SH/E$. If S contains sufficient $p'$–roots of unity, which we may assume, the indecomposable S–free trivial source modules with vertex E for the normalizer $N_H(E)$ are just obtained from evident rank 1 actions of $N_H(E)/C_H(E)$, and their reductions modulo p are the simple $N_H(E)$–modules over the residue field. All these indecomposables may be viewed as syzygies of certain degrees of the trivial module. Since E is cyclic and a T.I. set, it is easy to see, from Green's results on walking around the Brauer tree [G], that their Green correspondents, such as M, are the corresponding syzygies of the trivial module for SH. Conversely, every syzygy of the trivial module of appropriate degree (even, unless E has order 2) has trivial source. Note at this point the case where E has order 2 is trivial, since it implies H has a normal 2–complement, and consequently B has rank 2 as an S–module. Hence we may assume E has order greater tham 2, and, similarly, that D has order greater than 2. Next, modifying the augmentation of SH, if necessary (without disturbing its restriction to SE), we may assume the augmentation $B \to S$ induced by $SG \to S$ agrees with that induced by $SH \to S$. Thus "syzygy of the trivial module" of any given degree now has the same meaning with respect to SG as SH, and so M also has trivial source as an SG–module. Q.E.D.

In closing let me thank Puig and his students for their early and continuing interest in the theorem and the defect group conjugacy question. From Puig's point of view, a source algebra is an S–algebra equipped with an embedding of a defect group (or, more precisely, a conjugacy class of such embeddings), and a block might be regarded similarly. The question that we are raising here is the extent to which such an embedding is essentially determined by the S–algebra. The above results show this is true at least in some significant cases.

**Late note:** Results of H. Blau, using the classification, make the "T.I. set" assumption in our Theorem unnecessary. Details will appear elsewhere, with a version for source algebras (of principal blocks).

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