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On the p-part of character degrees of solvable groups

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by
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1 Introduction

The Fong-Swan Theorem shows a relation between irreducible Brauer characters and ordinary irreducible characters by the following: Let φ be an irreducible Brauer character of a p-solvable group G. There exists a p-rational irreducible character χ of G, such that $\chi = \varphi$ as a Brauer character. Especially: Every condition on ordinary characters is valid for Brauer characters (in a p-solvable group). We now ask for a kind of inversion of this relation and consider the character degrees. Let q be a prime, such that $q^2 \nmid \beta(1)$ for all $\beta \in IBr_p(G)$. Do we get a bound $n \in \mathbb{N}$, such that $q^n \nmid \chi(1)$ for all $\chi \in Irr(G)$? In general this is impossible.

2 Example

Let $\Gamma(8)$ be the group of all semilinear maps on $GF(8)$. It is easy to prove that the set of degrees of all irreducible Brauer charcters in characteristic 7 is $cd_7(\Gamma(8)) := \{1, 7\}$. The set of ordinary character degrees is $cd(\Gamma(8)) := \{1, 7, 3\}$. We put

$$H_n := \Gamma(8) \times \ldots_n \ldots \times \Gamma(8).$$

Clearly: $3 \nmid \beta(1)$ for all $\beta \in IBr_7(H_n)$, but there exists a $\chi \in Irr(H_n)$, such that $\chi(1) = 3^n$. (There exists many other examples for primes $p \neq 7$ too.)

For the prime p it is possible to show the following:

3 Theorem

Let G be solvable with $O_p(G) = E$ and $p^2 \nmid \beta(1)$ for all $\beta \in IBr_p(G)$. It follows:

a) G has elementary abelian Sylow-p-subgroups.

b) $p^2 \nmid \chi(1)$ for all $\chi \in Irr(G)$.

Before we start to prove this theorem, we need some short lemma:

[1] This is a small part of a Dissertation at Mainz.

4 Lemma

Let $N \trianglelefteq G$, V an irreducible KN-modul and K a field. Further let $T := T_G(V)$ be the inertiagroup of V and W an irreducible KT-modul, such that $W_N = e\,V$ with $e \in I\!N$. Then W^G is an irreducible KG-modul and

$$|G : T_G(V)| \cdot \dim V \;\Big|\; \dim W.$$

Proof: Manz[4]; Lemma 1 .

5 Lemma

Let N be an abelian normal subgroup of G, such that $(|G/C_G(N)|, |N|) = 1$ and $G/C_G(N)$ is abelian.

a) $G/C_G(N)$ has a regular orbit on N, $Irr(N)$ and $IBr_p(N)$ if $p \nmid |N|$.

b) There exists a $\chi \in Irr(G)$ resp. $\beta \in IBr_p(G)$ if $p \nmid |N|$, such that

$$|G/C_G(N)| \;\Big|\; \chi(1) \;\; \text{resp.} \;\; |G/C_G(N)| \;\Big|\; \beta(1).$$

Proof:

a) Isaacs[3], 13.24 yields that $Irr(N)$ and N are isomorphic as permutation modules. Then the result follows by Passman[5],lemma 2.2 .

b) This follows by a) and Lemma 4.

6 Proof of theorem 3

We denote by $F(G)$ the Fitting subgroup of G and define $F_j/F_{j-1} := F(G/F_{j-1})$, $(F_0 := E)$.

a) Let G be a minimal counterexample to statement a).

(i) $\Phi(G) = E$, where $\Phi(G)$ is the Frattini subgroup of G :
Proof: Clear, since $p \nmid |F_1|$ and $F(G/\Phi(G)) = F_1/\Phi(G)$.

(ii) $p \;\Big|\; |F_2/F_1|$, but $p^2 \nmid |F_2/F_1|$:
Proof: Assume that $p \nmid |F_2/F_1|$, hence $O_p(G/F_1) = E$. Since G is a minimal counterexample, G/F_1 has elementary abelian Sylow-p-subgroups and out of $p \nmid |F_1|$ it follows, that G has elementary abelian Sylow-p-subgroups too; a contradiction. Therefore $p \;\Big|\; |F_2/F_1|$.
Let $P_0/F_1 \in Syl_p(F_2/F_1)$ and $A \leq G$, such that A/F_1 is a maximal abelian normal subgroup of P_0/F_1, esp. $A \trianglelefteq\trianglelefteq G$. Hence $p^2 \nmid \alpha(1)$ for all $\alpha \in IBr_p(A)$ (Clifford theory). Since we have $C_A(F_1) = F_1$, Lemma 5 yields an $\alpha \in IBr_p(A)$, such that $|A/F_1| = \alpha(1)$. Hence $|A/F_1| \;\Big|\; p$ and therefore $A = P_0$.

(iii) Conclusion: Let $P_0 \trianglelefteq G$ be as defined in (ii). Lemma 5a) yields a $\lambda_0 \in IBr_p(F_1)$, such that

$$T_G(\lambda_0) \cap P_0 = F_1.$$

In particular $p \;\Big|\; |G : T_G(\lambda_0)|$. But by Lemma 4 we now have, that $p^2 \nmid |G : T_G(\lambda_0)|$ since $p^2 \nmid \beta(1)$ for all $\beta \in IBr_p(G)$. Hence

$$|T_G(\lambda_0)|_p = \frac{1}{p}|G|_p.$$

Now let $P_1/F_1 \in Syl_p(T_G(\lambda_0)/F_1)$. Obviously

$$P_0/F_1 \cap P_1/F_1 = E$$

and because of the order of $T_G(\lambda_0)$ we get

$$(P_0P_1)/F_1 \in Syl_p(G/F_1).$$

We claim, that P_1/F_1 is elementary abelian:
Since $P_0/F_1 = O_p(G/F_1)$ it is obviously, that $O_p(G/P_0) = E$. Furthermore G is a minimal counterexample and therefore G/P_0 has elementary abelian Sylow-p-subgroups. Hence $P_1/F_1 \cong (P_1P_0)/P_0$ is elementary abelian.
Step (ii) yields $|P_0/F_1| = p$ and therefore P_1/F_1 is operating trivial on P_0/F_1. So $(P_0P_1)/F_1$ is elementary abelian and for any $P \in Syl_p(G)$ we have:

$$P = P/(P \cap F_1) \cong (PF_1)/F_1 \cong (P_0P_1)/F_1 \quad \text{is elementary abelian.}$$

b) Assume $p^2 \mid \chi(1)$ for a $\chi \in Irr(G)$. Let D be the defect group of the block of χ. Then it follows by a) that D is elementary abelian. If the p-part of the order of G is $|G|_p = p^a$ and $|D| = p^d$, Brauer's Theorem about the height of characters yields, that $p^{(a-d)} \mid \chi(1)$ and $p^{a-d+1} \nmid \chi(1)$ (Fong[2];Thm.3C). Since we have the assumption $p^2 \mid \chi(1)$ it follows, that $a - d \geq 2$. Let now $\beta \in IBr_p(G)$ belong to the same block as χ. Then $p^{a-d} \mid \beta(1)$ (Feit[1];IV,4.5) and therefore $p^2 \mid \beta(1)$; a contradiction.

7 Remark

Tsushima proved in [6]; Theorem 5 the following:
If G is solvable and $p^2 \nmid \chi(1)$ for all $\chi \in Irr(G)$, then $G/O_p(G)$ has elementary abelian Sylow-p-subgroups.
With the theorem of Fong-Swan (Feit[1];X,2.1) this is a corollary of theorem 1. (Or use the proof of theorem 1 directly).

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