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INTRODUCTION

Let $G$ be a finite group, $k$ an algebraically closed field whose characteristic $p$ divides the order of $G$, and denote by $A$ a symmetric interior $G$-algebra (that is a symmetric $k$-algebra $A$ together with a homomorphism $\phi: G \to A^\ell$). In [2], we introduce the notion of Auslander-Reiten system of $G$ over $A$ (cf. §1), as a “generalization” in terms of idempotents of the usual notion of almost split sequence of $kG$-modules, which then corresponds to the case where $A$ is the algebra of $k$-endomorphisms of a $kG$-module. Furthermore we show in [2] that every primitive idempotent $i$ of $A^G$ such that $i \not\in A^G_0$ is the right extremity of a unique Auslander-Reiten system, up to embedding of $A$ into a symmetric interior $G$-algebra and to conjugacy by invertible $G$-fixed elements of that algebra (we will be talking abusively of “the Auslander-Reiten system ending with $i$”); our construction of that system proceeds mainly like the construction of the almost split sequence terminating in a given $kG$-module $M$ (that is we take the pullback of a projective cover of $\Omega M$ over a generator of the $\text{End}_{kG}(M)$-socle of $\text{Ext}^1(M, \Omega^2 M)$).

Let $k$ be the trivial $kG$-module and denote by $R_k$ the almost split sequence terminating in $k$, and by $L_k$ the corresponding Auslander-Reiten system of $G$. In [1], Auslander and Carlson show that the tensor product of any indecomposable $kG$-module $M$ with the sequence $R_k$ is either split or almost split up to an injective factor, and they give various criteria describing the second case (3.6., 4.7.). Using a different approach, we investigate a similar question for Auslander-Reiten systems: we give here sufficient conditions for certain pullbacks from the tensor product $i \otimes L_k$ to be either split, or else equal up to a trivial system to the Auslander-Reiten system ending with $i$. In other words, denoting by $u_G$ a representative of a generator of the socle of $\text{Ext}^1(k, \Omega^2 k)$ (see §1), we look for cases when the equivalence class of some tensor product with $u_G$ lies in the socle of the bimodule which corresponds here to $\text{Ext}^1(M, \Omega^2 M)$. Since our sufficient conditions are not satisfied for all $kG$-modules, we obtain the related results of [1] (3.6, 4.7.) for very specific modules only. Yet our approach applies when $i$ is a source idempotent of a block, and in that significant case (see [2], VII) we obtain an explicit generator of the socle. This last result, which was the starting point of this study, was suggested to me by Lluis Puig.
Section 1 presents our notations, the main definitions of [2] which are of use here, as well as two preliminary lemmas. We give our results in section 2.

§1 NOTATIONS AND PRELIMINARIES

We write $A^\times$ for the group of units of $A$ and denote by $a^x$ the element $\phi(x^{-1})a\phi(x)$ of $A$, where $a \in A$ and $x \in G$. The corresponding action of $G$ makes $A$ into a $G$-algebra. If $H$ and $K$ are two subgroups of $G$ with $K \subset H$, we denote by $A^H$ the algebra of $H$-fixed elements of $A$, and by $\text{Tr}^H_ K : A^K \to A^H$ the relative trace map, defined by $\text{Tr}^H_ K(a) = \sum a^x$, where $x$ runs over a right transversal of $H$ modulo $K$; the image of this map is the two-sided ideal $A^H_ K$ of $A^H$. If $P$ is any $p$-subgroup of $G$, we denote by $B_{BrP} : A^P \to A(P)$ the Brauer morphism, that is the homomorphism corresponding to the quotient by the ideal $\sum_{Q \in P} A^P_ Q$ of $A^P$. Furthermore we set $\overline{A^H} = A^H/A^H_ K$, and for any $a$ in $A^H$ we denote by $\overline{a}$ the image of $a$ in $\overline{A^H}$.

All modules and algebras are finite dimensional $k$-spaces, and the modules are left modules. We denote by $A^{opp}$ the opposite algebra of $A$, by $\text{J}(A)$ the Jacobson radical of $A$, and by $H^M$ (resp. $Q^M$) a projective cover (resp. a Heller translate) of the module $M$. Our tensor products are taken over $k$; note that the tensor product of two (symmetric) interior $G$-algebras is again a (symmetric) interior $G$-algebra.

Suppose we are given three mutually orthogonal idempotents $i, i^o, i'$ of $A^G$, together with two elements $d \in iA^G i^o$ and $d' \in i^o A^G i'$: we say that $S = (i, i^o, i', d, d')$ is a system of $G$ over $A$ if we have $dd' = 0$ and if there exists $(s, s')$ in $i^oAi \times i'Ai^o$ satisfying the conditions:

$$i = ds, \\ i^o = sd + d's' \quad \text{and} \quad i' = s'd'.$$

In case $i^o \in A^G_ i$, we call $S$ a Heller system; we say $S$ is trivial if $i = 0$, and split if $i \in dA^G$. Set $i^+ = i + i^o + i'$. The commuting algebra of the system $S$ is by definition the interior $G$-subalgebra $A_S$ of $i^oAi^+$ whose elements commute with $i, i^o, i', d$ and $d'$ simultaneously. We call $S$ an Auslander-Reiten system if it is a non trivial system, if the algebra $A_S$ is local, and if for every symmetric interior $G$-algebra $B$ and every embedding of interior $G$-algebras $f : A \to B$, we have $f(d)B = f(i)J(B^G)$ (by definition an embedding $f : A \to B$ is a homomorphism of interior $G$-algebras that is one-to-one and satisfies $\text{Im } f = f(1)Bf(1)$). The idempotent $i$ is then primitive in $A^G$ (cf. [2]).

The following additional notations are fixed throughout this note. Considering projective covers of the $kG$-modules $k$ and $\Omega k$, we denote by $E$ the interior $G$-algebra $\text{End}_k(k \oplus \Omega k \oplus \Omega \Omega k \oplus \Omega^2 k)$ and write $e, e^*, e', e''$ and $e^r$ for the orthogonal idempotents of $E$ corresponding to the projections on $k, \Omega k, \Omega \Omega k$ and $\Omega^2 k$ respectively. We consider Heller systems $H_k = (e, e^*, e', h, h')$ and $H_{\Omega k} = (e', e'^*, e'', m, m')$ of $G$ over $e$ (cf. [2]), and denote by $u_G$ an element of $e' E^G e$ whose class $\overline{u_G}$ generates the 1-dimensional $k$-space $e' E^G e$ (cf. [2], III 3).

On the other hand, we denote by $P$ a Sylow $p$-subgroup of $G$, and we write $A(G)$ for the quotient of $A^G$ by its two-sided ideal $\sum_{Q \in P} A^G_ Q = \text{Tr}^G_ P(\text{ker } BrP)$ (if $G$ is a $p$-group, then $P = G$ and we have $A(G) = Br_G(A^G)$, so the notation is consistent.) We begin with two lemmas which do not require $A$ to be symmetric:

**Lemma 1.** For any element $a$ in $\sum_{Q \in P} A^G_ Q$, we have $\overline{a} \otimes \overline{u_G} = 0$. 

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Proof: Let $Q$ be a proper subgroup of $P$. We have $u_G \in E_1^Q \cap E^Q$ (cf. [2], IV 2.1.), so every $a' \in A^Q$ satisfies $\text{Tr}_{G}(a') \otimes u_G \in (A \otimes E)_1^Q$.

The converse of lemma 1 is true under certain conditions:

**Lemma 2.** Suppose that $G = P$ and that the interior $P$-algebra $A$ has a $P$-stable basis. Take $u$ in $E^P$ such that $u \neq 0$ and let $a$ be an element in $A^P$ such that $Br_{P}(a) \neq 0$. Then $a \otimes u \neq 0$.

**Proof:** Set $v = a \otimes u$ and let $B$ be a $P$-stable basis of $A$. The condition $Br_{P}(a) \neq 0$ ensures the existence of $u$ in $E^P$ such that $u \neq 0$. We consider the projection of $v$ onto $u \otimes u$, in the decomposition $u \otimes u = (A \otimes E)_1^P$. Since $u$ is not in $E_1^P$, we get $v \notin (A \otimes E)_1^P$.

§2

Fix an idempotent $i$ in $A^G$, and set $B = iAi \otimes E$, $I = \sum_{Q \subseteq P} iA^Q_i$. The tensor product with $e$ defines an embedding of interior $G$-algebras from $iAi$ to $B$, and $B$ is symmetric. Denote by $\mathcal{H} = (j_j, j^*_j, c, c')$ the Heller system $i \otimes \mathcal{H}_i$ of $G$ over $B$. We recall from [2] that the $(j^*_jB^Gj; j^*B^Gj)$-bimodule $j^*_jB^Gj$ has same socles as a left and as a right module, and that if $i$ is primitive the socles have dimension 1 (cf. [2], III); furthermore in this case, if $u$ is any element in $j^*_jB^Gj$ whose class $\bar{u}$ generates that socle, the Auslander-Reiten system ending with $i$ is equal, up to a trivial system and to embedding, to the pullback of the Heller system $i \otimes \mathcal{H}_i$ over $(j, u)$ (cf. [2], VI).

Lemma 1 shows in particular that the condition $i \in I$ (which in case $i$ is primitive means that the Sylow subgroup $P$ is not a defect group of $i$), implies that the tensor product with $i$ or with any idempotent of $iA^G_i$, of the Auslander-Reiten system ending with $i$ does not belong to the ideal $I$.

**Proposition 1.** For all $a$ in $iA^G_i$ whose module $I$ is the $(iAi)(G)^{op}$-socle of $(iAi)(G)$, the element $a \otimes u_G$ is in the socle of $j^*_jB^Gj$.

**Proof:** Let $a$ be such an element and set $u = a \otimes u_G$. Let $\hat{B}_i$ denote the Heller algebra $B_i \otimes j^*_jB_j$ of $\mathcal{H}$ (cf. [2]). In [2], III 3. we show that every "symmetrising" form $\tau$ on $B$ determines a central form $\tau_{H,G}$ on $B^G_{i}$, which annihilates $(B^G_{i})^G$ and induces a symmetrising form on $B_{i}^G$; moreover the socle of $B_{i}^G$ coincides with the orthogonal of the radical of $B_{i}^G$. Thus it is sufficient to prove that $\tau_{H,G}(u \cdot J(\hat{B}^G_{i})) = 0$. Since we have $u \cdot J(\hat{B}^G_{i}) = uJ((iAi \otimes E)e)^G$ and $eE = (eEe)^G \simeq k$, all we need to show is that the restriction of the form $\tau_{H,G}$ to the space $aJ(iA^G_i) \otimes u_G$ is zero. But the hypothesis yields $aJ(iA^G_i) \in I$, so we conclude by lemma 1.

**Corollary.** Suppose we have $J(iA^G_i) = I$. Then for all $a$ in $iA^G_i$ the element $a \otimes u_G$ lies in the socle of $j^*_jB^Gj$.

**Proof:** In this case the radical of $(iAi)(G)$ is $\{0\}$.

Let us assume next that $G = P$ and that $i$ is primitive. Thus the algebra $(iAi)(P)$ is local, and it has a right socle of dimension 1. In certain cases we obtain a generator of the socle of $j^*_jB^Pj$:
Proposition 2. Suppose that the $P$-algebra $iAi$ has a $P$-stable basis. Let $a$ be an element of $iAi^P$ whose image under $Br_P$ generates the $(iAi)(P)^{op}$-socle of $(iAi)(P)$. The socle of $j^1B^Dj$ is the $k$-space generated by $a \otimes u_P$.

Proof: Lemma 2 shows that $a \otimes u_P \neq 0$. The result then follows from proposition 1.

Remark: (Application to $kG$-modules) Let us consider the special case where $A$ is the algebra of $k$-endomorphisms of an indecomposable $kG$-module $M$ with vertex $P$, and where $i = \text{id}_M$. If $M$ is simple, or if $G = P$ and $M$ is an endo-permutation $kP$-module, then $(iAi)(G) \cong k$ (cf. [5], 5.8.), so the corollary applies: for $a = i$, our statement says that if $M$ is simple, the tensor product of the almost split sequence $1 \otimes k$ with $M$ is either split, or almost split up to a projective direct summand. On the other hand if $M$ is an endo-permutation $kP$-module, proposition 2 shows that this same tensor product is, up to a projective direct summand, the almost split sequence terminating in $M$. These are two special cases of the results of Auslander and Carlson on the tensor product of the sequence $1 \otimes k$ with a $kG$-module, cf. [1], 3.6., 4.7. (indeed if $M$ is an endo-permutation module, we have $p | \dim M$, because every $P$-stable basis of $iAi = \text{End}_k(M)$ contains a unique fixed point, cf. [4], 2.8.4.).

Application to the source algebra of a block. Let $b$ be a primitive idempotent of the center $ZkG$ of $kG$, and set $A = kB$. Assume $b$ has a non trivial defect group, say $D$, and let $i$ be a $D$-source of $b$, that is a primitive idempotent of $A^D$ such that $b \in \text{Tr}_D^G(AD_iA^B)$. The hypotheses of proposition 2 are satisfied for the source algebra $iAi$ and $P = D$. Let $SZ(D)$ denote the element $\sum_{x \in Z(D)} x$ of $kD$. We obtain an explicit generator of the socle of $j^1B^Dj$:

Theorem. Set $a = SZ(D) \cdot i$. The socle of $j^1B^Dj$ is the $k$-vector space generated by $a \otimes u_D$.

Proof: Viewing the isomorphism of interior $Z(D)$-algebras $(iAi)(D) \cong kZ(D)$ (cf. [4], 14.5.) as an identification, the Brauer morphism $Br_D : iA^Di \to (iAi)(D)$ maps $a$ to the element $SZ(D)$ of $kZ(D)$. Thus the $kZ(D)^{op}$-module generated by $Br_D(a)$ is trivial, that is isomorphic to $k$ and equal to the $kZ(D)^{op}$-socle of $kZ(D)$. The conclusion now follows from proposition 2.

References