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1. Introduction

Let G be a finite group, p a prime and B a p-block of G. We are interested in two different approaches to representation theory: functional and algebra theoretic.

The first deals with matrix representations and functions on groups; it includes questions about k(B), the number of irreducible complex characters of B, and about l(B), the number of irreducible Brauer characters of B.The second approach views the block B as a finite dimensional algebra. It is concerned with the module category of B including homological properties such as projective resolutions and the Auslander-Reiten quiver (see chapter 2).

A number of years ago, Brauer and Olsson studied 2-blocks B whose defect groups are dihedral or semidihedral or (generalized) quaternion from the functional point of view. They were interested in determining k(B), l(B) and to obtain information concerning the (generalized) decomposition numbers of B [4, 13].

These are also precisely the blocks which are of tame representation type [2]; and they have recently been studied from the algebra point of view. By using Auslander-Reiten theory it has been possible to classify these blocks, as algebras, by generators and relations, up to Morita equivalence (and some scalars in socle relations). In particular, this gives the Cartan matrices for all these blocks, and it allows one to calculate the decomposition numbers, hence to extend the classical results [7 to 11] of the functional approach.

The original arguments used some of the work by Brauer and Olsson from [4, 13]; however this is not necessary. The aim of this paper is to show how results on the algebras and a few general principles determine l(B), k(B) and the decomposition matrices.

We will now introduce the algebras which were studied. Let K be an algebraically closed field and Λ a finite-dimensional K-algebra.

(1.1) We say that Λ is of "dihedral" or "semidihedral" or "quaternion" type if it satisfies the following conditions:

(a) Λ is tame, symmetric and indecomposable.

(b) The Cartan matrix of Λ is non-singular.

(c) The stable Auslander-Reiten quiver of Λ has the following components:

	dihedral type	semidihedral type	quaternion type
tubes	1-tubes and	1-, 2-tubes and	1-, 2-tubes
	$\leq$ two 3-tubes	$\leq$ one 3-tube	-
others	$\mathbb{Z}A_{\infty}^{\infty}$	$\mathbb{Z}A_{\infty}^{\infty}$, $\mathbb{Z}D_{\infty}$	-
	(or $\mathbb{Z}\tilde{A}_{12}$ or $\mathbb{Z}\tilde{A}_{5}$)		

The class of these algebras contains all dihedral, semidihedral and quaternion blocks [7, 11]. To prove this, one needs, apart from general principles, the algebra structure of some local blocks.

A main step in the classification of these algebras consists in bounding the number of simple modules. We have

THEOREM *Let Λ be an algebra of dihedral or semidihedral or quaternion type. Then Λ has at most three simple modules.*

COROLLARY [Brauer, Olsson] *Suppose that B is a dihedral or semidihedral or quaternion block. Then $l(B) \leq 3$.*

In chapter 2, we shall give a proof of the Theorem for the dihedral case. The quaternion case has been done in [9]; and the semidihedral case which is somewhat longer will appear in [11].

The work to determine the algebras is more general and is independent of groups. The results may be found in [7, 8, 10, 11]; see also [6].

In the third chapter, we will calculate Cartan matrices and decomposition matrices D for all tame blocks having two simple modules.

The information we need to do this from the classification of algebras is summarized at the beginning. It is in fact convenient to study all tame blocks simultaneously, since the same Cartan matrices appear, and since the dimension of the centre of the algebra depends only on the Cartan matrix C. Given C and k(B), to calculate D one needs (almost) nothing.

We remark that the results on the heights of characters by Brauer and Olsson follow easily from the decomposition numbers, using the general fact that any block must contain an ordinary and also a Brauer character of height zero.

The dihedral case has not been published; the results for the other types are contained in [8, 10]. However, the proofs there use results by Olsson [13].

With the same techniques, one can deal with the case l(B) = 3; this is not more difficult, though it takes longer due to the number of algebras. This will appear in [11]. Blocks with l(B) = 1 create no problem.

We write $\delta(B)$ for a defect group of the block B. If M is a module then soc M is the largest semisimple submodule of M, and top M denotes its largest semisimple factor module. Any other notation should be standard. Concerning basic facts on blocks, algebras and representations we refer to [1, 12].

2. The number of simple modules for algebras of dihedral type

Let K be an algebraically closed field of arbitrary characteristic and assume that Λ is a symmetric K-algebra.

(2.1) The *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$ is the graph whose vertices are the isomorphism classes of non-projective indecomposable Λ-modules and where the number of arrows from [M] to [N] is equal to the K-dimension of the space $\mathfrak{r}(M, N)/\mathfrak{r}^2(M, N)$. Here $\mathfrak{r}(\ , \ )$ = rad $\mathrm{Hom}_\Lambda(\ , \ )$. There is a graph automorphism of $\Gamma_s(\Lambda)$, the Auslander-Reiten translation, denoted by τ.

Concerning the graph structure of $\Gamma_s(\Lambda)$ we refer to [1, 14]. Here we shall only need the following facts:

(a) For symmetric algebras, τ is induced by Ω^2, and Ω gives rise to a graph automorphism of $\Gamma_s(\Lambda)$. In particular, the tubes correspond to the Ω-periodic modules, and Ω preserves the ends of tubes.

(b) Let M = P/soc P for P indecomposable projective. Then the predecessors of [M] in $\Gamma_s(\Lambda)$ are precisely the indecomposable summands of rad P/soc P.

(c) The only algebras in (1.1) where $\Gamma_s(\Lambda)$ has multiple arrows are those which immediately lead to local algebras of length 4 and give rise only to Klein 4-group blocks with l(b) = 1 [11]. Therefore we assume here that $\Gamma_s(\Lambda)$ does not have multiple arrows.

(d) If Λ is of dihedral type then the number of arrows ending at a fixed [M] is 1 or 2, and it is 1 if and only if [M] lies at the end of a tube.

(2.2) Assume that Λ is an algebra of dihedral type, and that S is a simple Λ-module, with projective cover P. Then (rad P)/S has at most two indecomposable summands, see (2.1)(b),(d). We fix a decomposition $(\mathrm{rad}\ P)/S = U \oplus V$, and we always take $U \neq 0$.

<u>Definition of $\mathfrak{U}$ and $\underset{\sim}{V}$:</u>

The decomposition $(\mathrm{rad}\ P)/S = U \oplus V$ induces an embedding of V into P/S. We put $\mathfrak{U} := (P/S)/V$ and $\underset{\sim}{V} := \Omega\mathfrak{U}$.

Similarly, define $\mathfrak{V} = (P/S)/U$ and $\underset{\sim}{U} = \Omega\mathfrak{V}$.

The following result is crucial:

(2.3) PROPOSITION *Suppose that Λ is a symmetric algebra and that P is an indecomposable projective Λ-module.*

(a) Let $(\mathrm{rad}\ P)/(\mathrm{soc}\ P) = U \oplus V$, and let $\mathfrak{U}$ and $\underset{\sim}{V}$ be as in (2.2). Then $[\mathfrak{U}]$ and $[\underset{\sim}{V}]$ have a unique predecessor in $\Gamma_s(\Lambda)$.

(b) Let X be a submodule of rad P with a simple top with $X \not\subseteq \mathrm{rad}^2 P$. Then $[X]$ has a unique predecessor in $\Gamma_s(\Lambda)$.

For part (a), see [7, (2.8)] and part (b) is a special case of a theorem in [5].

Now, if Λ is of dihedral type then the only vertices in $\Gamma_s(\Lambda)$ with a unique predecessor are the ones at ends of tubes.

We call a module M "exceptional" if it is non-projective and [M] lies at the end of a 3-tube. Let $\mathcal{E}$ be the set of exceptional modules, thus $|\mathcal{E}| = 6$ or 3, and $\mathcal{E}$ is the union of Ω-orbits , either two of length 3, or just one Ω-orbit [see (2.1)(a)].

(2.4) LEMMA *Suppose that top $V \not\cong S$ or that soc $U \not\cong S$. Then $\mathfrak{U}$ and $\underset{\sim}{V}$ must be exceptional.*

Proof: Suppose not. Then $\Omega^2\mathfrak{U} \cong \mathfrak{U}$, and there is an exact sequence $0 \to \mathfrak{U} \to Q \to P \to \mathfrak{U} \to 0$ where Q is projective. By exactness and since the

Cartan matrix is non-singular, we have $P \cong Q$. Therefore soc $U \cong$ soc $Q \cong$ top $Q \cong$ top V.

(2.5) Proof of the Theorem for the dihedral case

Suppose there is an algebra Λ of dihedral type having at least four simple modules. Denote the simple Λ-modules by $S_0, S_1, S_2, S_3, \ldots$. and the corresponding projectives by $P_0, P_1, \ldots$, and let $(\text{rad } P_i)/S_i = U_i \oplus V_i$ be as in (2.2). Since Λ is indecomposable we have that $\text{Ext}^1(S_i, S_j) \neq 0$ for some $j \neq i$, and then S_j occurs in $\text{top}(U_i \oplus V_i)$. We may therefore assume that top $V_i \not\cong S_i$ [in addition to the convention in (2.2)].

Then the modules $\underset{\sim}{U}_i$, $\underset{\sim}{V}_i$ are exceptional, by (2.4); in particular, for each i there is a module in $\mathcal{E}$ whose top is isomorphic to S_i (and hence Λ has at most 6 simple modules.) Put top $\mathcal{E}$ = $[\text{top } M]_{M \in \mathcal{E}}$, with repetitions, similarly define soc $\mathcal{E}$. Since $|\mathcal{E}| = 6$, not all S_i can occur twice in top $\mathcal{E}$.

(1) Suppose S_0 occurs only once in top $\mathcal{E}$. Then top V_0 and soc U_0 are simple: We have soc $\mathcal{E}$ = top $\mathcal{E}$, consequently $\underset{\sim}{V}_0$ is the unique module in $\mathcal{E}$ whose socle is $\cong S_0$. Since top $\underset{\sim}{V}_0 \not\cong S_0$, there is a submodule X of $\underset{\sim}{V}_0$ such that $X \not\subseteq \text{rad}^2 P_0$, and where top X is simple but $\not\cong S_0$. Then X is exceptional, by (2.3), with socle $\cong S_0$, and it follows that $X \cong \underset{\sim}{V}_0$. Dually, soc $\underset{\sim}{U}_0$ is simple.

(2) $\mathcal{E}$ must be a union of two Ω-orbits: Suppose not. Then the minimal projective resolution of the modules in $\mathcal{E}$ is of the form

(*) $0 \to \underset{\sim}{U}_0 \to Q_5 \to Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to \underset{\sim}{U}_0 \to 0.$

Since top $\mathcal{E} = [\text{top } Q_i]$, we have that at least four of the Q_i are indecomposable. By exactness and since the Cartan matrix is non-singular, we obtain

(**) $Q_0 \oplus Q_2 \oplus Q_4 \cong Q_1 \oplus Q_3 \oplus Q_5.$

If more than four Q_i were indecomposable then they all would be, and only three isomorphism types of indecomposable projectives would occur in (**).

It follows that S_i occurs once in top $\mathcal{E}$, for $0 \leq i \leq 3$, and the remaining two modules in top $\mathcal{E}$ are not simple. Thus, in (*), there is some j such that $Q_j \cong P_i$ but Q_{j+1} is decomposable. Now, top $Q_{j+1} \cong$ top $\Omega\mathfrak{U}_i \cong$ top V_i, so top V_i is not simple. This is a contradiction to (1).

Suppose S_0 occurs only once in top $\mathcal{E}$. By (1) and (2), $\mathfrak{U}_0$ has a projective resolution of the form

(*) $0 \to \mathfrak{U}_0 \to P_j \to P_1 \to P_0 \to \mathfrak{U}_0 \to 0$ where $S_j = \operatorname{soc} U_0$ and $S_1 = \operatorname{top} V_0$.

(3) <u>$\operatorname{rad} P_0/\operatorname{rad}^2 P_0 \cong S_1 \oplus S_0$:</u> The module $\mathfrak{V}_0$ is not exceptional, for otherwise we would have $\mathfrak{U}_0 \cong \mathfrak{V}_0$ and there would be two distinct quotients of P_0 of length 2 which are isomorphic. This is not possible. Then (3) follows from (2.4) and (*).

(4) <u>We may assume that $\widetilde{U}_0 \cong \underset{\sim}{V}_j$ and $\underset{\sim}{V}_0 \cong \mathfrak{U}_1$; and then $\underset{\sim}{V}_1 \cong \mathfrak{U}_j$:</u>
By (*), S_j occurs in $\operatorname{soc}_2 P_0/S_0$, consequently S_0 occurs in rad $P_j/\operatorname{rad}^2 P_j$ (see for example [7, (2.2)]. We may assume $S_0 \subseteq \operatorname{top} V_j$; then there is a submodule X of V_j which is not contained in $\operatorname{rad}^2 P_j$, whose top is isomorphic to S_0. We deduce that X is exceptional and hence isomorphic to $\mathfrak{U}_0$. By (*) we know that $\operatorname{soc} \Omega^{-1}\mathfrak{U}_0$ is simple, so that P_j/X has a simple socle. On the other hand, $X \subseteq \underset{\sim}{V}_j$, therefore $\operatorname{rad}(P_j/X) \cong \underset{\sim}{V}_j/X \oplus U_j$, and it follows that $\underset{\sim}{V}_j/X = 0$. The second part is dual, and the last statement follows since $\mathfrak{U}_j \cong \Omega^{-1}\mathfrak{U}_0 \cong \Omega\underset{\sim}{V}_0 \cong \underset{\sim}{V}_1$.

Since we assumed that top $\underset{\sim}{V}_1 \not\cong S_1$ we have $S_j \not\cong S_1$; say $j = 2$.
We deduce from (4) that $\operatorname{rad} P_1/\operatorname{rad}^2 P_1 \cong S_2 \oplus \operatorname{top} U_1$ and $\operatorname{rad} P_2/\operatorname{rad}^2 P_2 \cong S_0 \oplus \operatorname{top} U_2$.

(5) <u>One of $\mathfrak{V}_1$, $\mathfrak{V}_2$ must be exceptional:</u> Otherwise, we would have by (2.4) that $\operatorname{rad} P_1/\operatorname{rad}^2 P_1 \cong S_1 \oplus S_2$ and $\operatorname{rad} P_2/\operatorname{rad}^2 P_2 \cong S_0 \oplus S_2$, and then using also (3), $\operatorname{Ext}^1(S_i, S_3) = 0$ for $0 \leq i \leq 2$, and Λ would be decomposable.

Say $\mathfrak{V}_1$ is exceptional, and let its projective resolution be

(**) $0 \to \mathfrak{V}_1 \to Q_2 \to Q_1 \to P_1 \to \mathfrak{V}_1 \to 0$.

(6) <u>One of Q_1, Q_2 is isomorphic to P_3:</u> We know that $\mathfrak{V}_1$ lies in $\mathcal{E}$ but not

in the Ω-orbit of $\mathfrak{U}_1$. Therefore top $\mathcal{E}$ is given by the tops of the projectives in (*) and (**).

Taking now the direct sum of (*) and (**) gives an exact sequence $0 \to \mathfrak{U}_1 \oplus \mathfrak{V}_1 \to P_0 \oplus Q_2 \to P_2 \oplus Q_1 \to P_1 \oplus P_1 \to \mathfrak{U}_1 \oplus \mathfrak{V}_1 \to 0$. By exactness and since the Cartan matrix is non-singular, we deduce $P_0 \oplus Q_2 \cong P_2 \oplus Q_1$. This is the direct sum of two indecomposables, by (6), and then there should be three isomorphism types occuring, a contradiction to the Krull-Schmidt Theorem. This completes the proof.

3. Cartan matrices and decomposition numbers for tame blocks B with l(B) = 2

From now we assume that char K = 2.

There are seven families of basic algebras Λ of dihedral, semidihedral, quaternion type with two simple modules, up to scalars in the relations. The scalars do not affect the Cartan matrix and the dimension of the centre Z(Λ) of the algebra. There are four different Cartan matrices. Moreover, the dimension of Z(Λ) depends only on the Cartan matrix. So we should study the following cases:

Type	Cartan matrix	dim Z(Λ)	Reference
$(\mathcal{A})$, $(\mathcal{A}_1)$	$\begin{bmatrix} 4k & 2k \\ 2k & k+1 \end{bmatrix}$ $(k \geq 1)$	k + 3	$(\mathcal{A})$ in [11] III in [8]
$(\mathcal{A}_2)$	$\begin{bmatrix} 4k & 2k \\ 2k & k+2 \end{bmatrix}$ $(k \geq 2)$	k + 4	II in [8] I in [10]
$(\mathcal{B})$ $(\mathcal{B}_1)$, $(\mathcal{B}_2)$	$\begin{bmatrix} 4k & 2k \\ 2k & k+s \end{bmatrix}$ $(k \geq 1,\ s \geq 2)$	k+s+2	$(\mathcal{B})$ in [11] I and IV in [8] II in [10]
$(\mathcal{B}_3)$	$\begin{bmatrix} s+2 & s \\ s & s+2 \end{bmatrix}$ $(s \geq 1)$.		V in [8]

We remark that the notation for the types indicates the ordinary quiver of the algebra. Type $(\mathcal{A})$ has three arrows, including one loop; and type $(\mathcal{B})$

has another loop which gives rise to the parameter s in the Cartan matrix.

In some cases, we will use the following facts from the classification of the algebras:

[**] Let P_0 be the first indecomposable projective. Then
(1) P_0 does not have a submodule whose composition factors are rS_0 with $r \geq 3$. Therefore the decomposition matrix does not have rows whose sum is [r 0].
(2) P_0 does not have a submodule whose composition factors are $5S_0 + S_1$. Hence the decomposition matrix does not have two rows [1 0] and a row [3 1].

Let B be a tame block. One can now determine k(B), similarly as in [4, 13], with the help of Brauer's general formula

$$k(B) = \Sigma\ l(b)$$

where the sum is taken over the conjugacy classes of subsections (see [3, 11]). Representatives for the subsections can be determined by standard methods. Now, for the non-cyclic blocks b occuring, l(b) is known, from our work on algebras. [There are details available relating l(b) to the structure of appropriate dihedral blocks.] Using this it is straightforward to calculate k(B); and we will therefore not give details here.

We remind of the results for blocks B with l(B) = 2; here 2^n is the order of a defect group:

Defect group	k(B)	
dihedral	$2^{n-2}+3$	$(n \geq 3)$
semidihedral	$2^{n-2}+3$ or $2^{n-2}+4$	$(n \geq 4)$
quaternion	$2^{n-2}+4$	$(n \geq 4)$.

For calculating the decomposition numbers we will use the following general facts: Let C be a Cartan matrix of an arbitrary p-block and D the corresponding decomposition matrix.

(F.1) The determinant of C is a power of p. The highest elementay divisor of C is the order of the defect group, it occurs with multiplicity 1.

(F.2) The entries of D are non-negative integers; also, D does not have a row consisting of zeros only.

(F.3) The matrix D has l(B) columns and k(B) rows, and $D^tD = C$.

(F.4) k(B) = dim Z(B)

(3.1) LEMMA *Assume that B is a block whose Cartan matrix is of the form*

$$\begin{bmatrix} 4k & 2k \\ 2k & k+r \end{bmatrix} \qquad (k \geq 1,\ r \geq 1),$$

and assume that dim Z(B) = k+r+2 is either odd or ≡ 0 (mod 4). Then one of the following holds:

(i) $r = 2^{n-2}$ *and* $k = 1$ *(iii)* $r = 1$ *and* $k = 2^{n-2}$

(ii) $r = 2^{n-2}$ *and* $k = 2$ *(iv)* $r = 2$ *and* $k = 2^{n-2}$.

Proof: The Cartan matrix C has determinant 4kr. For a block, we deduce that kr is a power of 2. Suppose first that dim Z(B) is odd, then one of k, r is odd and hence = 1. It follows that the first elementary divisor of C is 1, and then $4kr = |\delta(B)| = 2^n$, and one of (i) or (iii) holds.

Now assume that dim Z(B) is even. Then, by the hypothesis, k + r = 2u for u odd. So the first elementary divisor of C is 2, and then $4kr = 2|\delta(B)|$, and we get that one of (ii) or (iv) holds.

(3.2) PROPOSITION *Let B be any block with Cartan matrix C where either*

(i)
$$C = \begin{bmatrix} 4 & 2 \\ 2 & 2^{n-2}+1 \end{bmatrix} \quad \text{and } k(B) = 2^{n-2}+3, \text{ or}$$

(ii) $C = \begin{bmatrix} 8 & 4 \\ 4 & 2^{n-2}+2 \end{bmatrix}$ *and* $k(B) = 2^{n-2}+4$, *and*

[**](1) holds.

Then the decomposition matrix of B is given by

(i) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $(2^{n-2}-1)$*-times* *or* *(ii)* $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ $(2^{n-2}-1)$*-times*

respectively.

Proof: Let $[a_i]^t$ and $[b_i]^t$ be the first and second column of D, and denote by c_{ij} the Cartan numbers. At least two of the b_i must be 0 since $c_{11} = k(B) - 2$ (see (F.3)). We may assume that $b_1 = b_2 = 0$, and then a_1 and a_2 are $\neq 0$, by (F.2). Moreover we know from (F.3) that $\Sigma\,(a_i - b_i)^2 = c_{00} - 2c_{01} + c_{11} = k(B) - 2$. Hence $a_i = b_i$ for at least two values of i, say for i = 3 and 4; then b_3 and b_4 are $\neq 0$.

Assume first that (i) holds, then $\Sigma\, a_i^2 = 4$. We have just agreed that $a_i \neq 0$ for $i \leq 4$. So we deduce that $a_i = 1$ for $i \leq 4$ and $a_i = 0$ otherwise. It follows that $b_i \neq 0$ for $i \geq 5$, and in particular there are c_{11} values of i for which $b_i \neq 0$. We deduce that $b_i = 1$ for these i, and we are done.

Now suppose that (ii) holds. Then it follows from [**](1) that $a_1 = a_2 = 1$ and $b_i \neq 0$ for $i \geq 3$. Hence there are c_{11} values of i for which b_i is non-zero, and we deduce $b_i = 1$ for these i. Now we have $8 = c_{00} = 4 + \sum_{i\geq 5} a_i^2$ and $4 = c_{01} = 2 + \sum_{i\geq 5} a_i = 2$, and the rest is clear.

(3.3) PROPOSITION *Let B be a block with Cartan matrix C such that either*

(iii) $$C = \begin{bmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 \end{bmatrix} \quad \text{and } k(B) = 2^{n-2} + 3, \text{ or}$$

(iv) $$C = \begin{bmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 \end{bmatrix} \quad \text{and } k(B) = 2^{n-2}+4, \text{ and [**] holds.}$$

Then the decomposition matrix of B is given by

(iii) $$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} (2^{n-2}-1)\text{-times}$$ or (iv) $$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} (2^{n-2}-1)\text{-times}.$$

respectively.

Proof: Let $[a_i]^t$ and $[b_i]^t$ be the first and second column of D, and denote by c_{ij} the Cartan numbers. At least two of the b_i must be 0 since $c_{11} = k(B) - 2$. We may assume that $b_1 = b_2 = 0$, and then a_1 and a_2 are $\neq 0$, see (F.2). Moreover $\Sigma\ (a_i-b_i)^2 = c_{00} - 2c_{01} + c_{11} = k(B) - 2$. Hence for at least two values of i, we have $a_i = b_i$. Without loss of generality $a_i = b_i$ for i = 3 and 4, and then $b_i \neq 0$ for these i. Now observe that

(*) $\Sigma\ (a_i-2b_i)^2 = c_{00} - 4c_{01} + 4c_{11}$.

Assume first that (iii) holds; then the number in (*) is 4. We have just agreed that $a_i-2b_i \neq 0$ for $i \leq 4$, therefore we must have that $a_i = 2b_i$ for $i \geq 5$. Consequently $b_i \neq 0$ for $i \geq 5$ and then there are c_{11} values of i for which $b_i \neq 0$. We deduce that $b_i = 1$ for these i, and the rest is clear.

Now suppose that (iv) holds. Then it follows from [**](1) that $a_1 = a_2 = 1$ and $b_i \neq 0$ for $i \geq 3$. Hence there are c_{11} values of i for which $b_i \neq 0$, and we deduce $b_i = 1$ for these i. Then by [**](2) we have that $a_i \neq$

3. The number in (*) is 8 here, hence $8 = 4 + \sum_{i \geq 5} (a_i - 2)^2$. Suppose $a_i - 2 = \pm 1$ four times for $5 \leq i$. Then since $a_i \neq 3$ we have $a_i = 1$ eight times, and $a_i = 2$ otherwise. But then $\sum a_i^2 \neq 2^n$, a contradiction to (F.3). Hence we must have that $a_i = 0$ once for some $i \geq 5$ and $= 2$ otherwise, and we are done.

(3.4) **REMARK** The question arises which decomposition matrices can occur in (3.2)(ii) and (3.3)(iv) but without assuming [**]. An elementary calculation shows that there are one or two more solutions for D. In each case one finds that $k_0(B) = 8$. To exclude these for tame blocks, one could alternatively proceed by showing that $k_0(B) = 4$.

(3.5) **LEMMA** *Let B be a block of type* (B_3). *Then* $s = 2^{n-2} - 1$.

Proof: The Cartan matrix of an algebra belonging to family (B_3) has determinant $4(s+1)$. Hence for a block we deduce that s+1 is a 2-power. Then s is odd, and the lowest elementary divisor of C is 1. Consequently $s = 2^{n-2} - 1$.

(3.6) **PROPOSITION** *Let B be any block whose Cartan matrix is of the form*

$$\begin{bmatrix} 2^{n-2}+1 & 2^{n-2}-1 \\ 2^{n-2}-1 & 2^{n-2}+1 \end{bmatrix}$$

with $k(B) \geq 2^{n-2}+3$. *Then the decomposition matrix of B is given by*

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad (2^{n-2}-1)\text{-times}.$$

In particular, $k(B) = 2^{n-2}+3$.

Proof: Let $[a_i]^t$ and $[b_i]^t$ be the columns of D. We have

(*) $\Sigma\ (a_i - b_i)^2 = c_{00} - 2c_{01} + c_{11} = 4.$

Suppose that $k(B) \geq 2^{n-2} + 4$, then $k(B) - c_{ii} \geq 3$, and at least three a_i and three b_i are = 0. Then by (F.2), $a_i - b_i \neq 0$ for at least six values of i. This is not possible, by (*).

Now, $k(B) - c_{ii} = 2$, and we see that at least two of the b_i and two of the a_i are 0. We may assume that $b_1 = b_2 = 0$, then a_1 and a_2 are $\neq 0$. Then without loss of generality, $a_3 = a_4 = 0$, and then b_3 and b_4 are $\neq 0$. We deduce from (*) that $a_i = b_i$ for $i \geq 5$, and then $a_i \neq 0$ for these values of i, and the statement is now evident.

We remark that (3.2), (3.3) and (3.6) are more general and do not depend on the defect groups.

It is clear that the decomposition numbers give directly the relations satisfied by the characters on elements of odd order, as obtained in [4, 13]. Moreover, the general fact that at least one ordinary character and also at least one Brauer character in the block is of height zero, implies also the results in [4, 13] on the heights of the characters. The answer is that in each of (3.2), (3.3) and (3.6) the first four characters are of height zero, and the last row corresponds to the character of height 1.

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