EDUARDO CATTANI
AROLDO KAPLAN

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DEGENERATING VARIATIONS OF HODGE STRUCTURE

Eduardo Cattani[1] and Aroldo Kaplan[1]

This article is an expanded version of the authors' lectures given at the CIRM Conference in Luminy. As such, it is a survey of some techniques and results on degenerations of pure polarized Hodge structures and their consequences, notably the L_2 realization of related Intersection Cohomology groups. It tries to complement and update the existing expository literature ([6], [9], [15], [16], [18]) and to present the results in a manner suited to other applications.

For a unipotent variation of (pure) polarized Hodge structures over the complement of a normal-crossings divisor, Schmid's Nilpotent Orbit Theorem asserts that, relative to the flat structure, the Hodge bundles have only logarithmic singularities along the divisor; in fact, they extend holomorphically as subbundles of the canonical extension. In terms of the latter, the "approximating nilpotent orbit" is just the corresponding constant term, which by itself defines a variation of polarized Hodge structure. In turn, the SL_2-Orbit Theorems reduce the analysis of nilpotent orbits to the case of certain $SL(2,\mathbf{R})^n$-equivariant ones. The notion of nilpotent orbit turns out to be equivalent to one of polarized mixed Hodge structure; SL_2-orbits correspond then to those that split over $\mathbf{R}$ in a sense to be specified.

These ideas, together with the asymptotic representations and the properties of the local monodromy implied by them, are discussed in sections 1 to 4. The

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remaining two sections contain applications to estimates for the Hodge metric and to some global questions, including a brief sketch of the proof of an isomorphism between the natural L_2 and the Intersection Cohomologies in this context.

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1. Preliminaries.

Let X be a connected complex manifold. We view a real variation of Hodge structure (VHS) over X as given by the data $(V,\nabla,V_{\mathbf{R}},F)$, where $V \to X$ is a holomorphic vector bundle, ∇ a flat connection on V, $V_{\mathbf{R}}$ a flat real form and F a finite decreasing filtration of V by holomorphic subbundles - the Hodge filtration - satisfying

(1.1)
(i) $\nabla F^p \subset \Omega^1_X \otimes F^{p-1}$ (Griffiths' transversality)
(ii) $V = F^p \oplus \bar{F}^{k-p+1}$ ($\bar{F}$ = conjugate of F relative to $V_{\mathbf{R}}$)

for some integer k - the weight of the variation. As a C^∞-bundle, V may then be written as a direct sum

$$(1.2)\qquad V = \bigoplus_{p+q=k} V^{p,q}, \qquad V^{p,q} = F^p \cap \bar{F}^q \;;$$

the integers $h^{p,q} = \dim V^{p,q}$ are the Hodge numbers. A polarization of the VHS is a flat non-degenerate bilinear form S on V, defined over $\mathbf{R}$, of parity $(-1)^k$, whose associated flat Hermitian form $S^h(\cdot,\cdot) = i^{-k}S(\cdot,\bar{\cdot})$ satisfies

(1.3)
(i) the decomposition (1.2) is S^h-orthogonal
(ii) $(-1)^p S^h$ is positive-definite on $V^{p,k-p}$.

Such a polarization determines then a positive-definite Hermitian metric on V:

$$(1.4)\qquad \mathcal{H} = \sum_p (-1)^p S^h\Big|_{V^{p,k-p}} ,$$

the Hodge metric.

Specialization to a fiber defines the notion of polarized Hodge structure on a $\mathbf{C}$-vector space V. Fixing the latter together with the real structure $V_{\mathbf{R}}$, the polarizing form S, the weight and the Hodge numbers and allowing the Hodge filtration F to vary, describes the corresponding <u>classifying space</u> D of polarized Hodge structures. Its Zariski closure $\check{D}$ in the appropriate variety of flags consists of all filtrations F in V with $\dim F^p = \sum_{r \geq p} h^{r,k-r}$ satisfying

$$S(F^p, F^{k-p+1}) = 0. \tag{1.3.i'}$$

The complex Lie group $G_{\mathbf{C}}$ of all automorphisms of (V,S) acts transitively on $\check{D}$ - therefore $\check{D}$ is smooth - and the group of real points $G_{\mathbf{R}}$ has D as an open dense orbit. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ denote the Lie algebra of $G_{\mathbf{C}}$, $\mathfrak{g}_{\mathbf{R}} \subset \mathfrak{g}$ that of $G_{\mathbf{R}}$. The choice of a base point $F \in \check{D}$ defines a filtration in $\mathfrak{g}$

$$F^a\mathfrak{g} = \{T \in \mathfrak{g} \mid TF^p \subset F^{p+a}\} . \tag{1.5}$$

If $F \in D$, then (1.5) determines a Hodge structure of weight 0 on $\mathfrak{g}$, with

$$\mathfrak{g}^{a,-a} = \{T \in \mathfrak{g} \mid TV^{p,q} \subset V^{p+a,q-a}\}. \tag{1.6}$$

The Lie algebra of the isotropy subgroup $B \subset G_{\mathbf{C}}$ at F is $F^0\mathfrak{g}$ and $F^{-1}\mathfrak{g}/F^0\mathfrak{g}$ is an Ad(B)-invariant subspace of $\mathfrak{g}/F^0\mathfrak{g}$. The corresponding $G_{\mathbf{C}}$-invariant subbundle of the holomorphic tangent bundle of $\check{D}$ is the <u>horizontal tangent bundle</u>, denoted ty $T_h(\check{D})$. A polarized VHS over a manifold X determines - via parallel translation to a typical fiber - a holomorphic map $f: X \to D/\Gamma$ where Γ is the monodromy group (Griffiths' period map). By definition, it has local liftings into D whose differentials take values on the horizontal tangent bundle.

In the sequel we shall be interested in the asymptotic behavior of a polarized VHS, relative to some smooth compactification $\bar{X}$ of the base where $\bar{X} - X$ is a normal-crossings hypersurface. Such $\bar{X}$ exists, for example, if X is quasi-projective. Locally at infinity we may then replace X by a product $\Delta^{*n} \times \Delta^m$ of punctured disks and disks. We let U denote the upper half plane, covering Δ^* via the map $z \to s = e^{2\pi i z}$, and let

(1.7) $$\phi: U^n \times \Delta^m \to D$$

be a lifting of the corresponding period map $\Delta^{*n} \times \Delta^m \to D/\Gamma$ to the universal covering of $\Delta^{*n} \times \Delta^m$ (we shall also refer to ϕ as the period map). If $\{e_j\}$ is the standard basis of $\mathbf{C}^n$, then

(1.8) $$\phi(\underline{z} + e_j, \underline{w}) = \gamma_j \phi(\underline{z}, \underline{w}); \qquad (\underline{z}, \underline{w}) \in U^n \times \Delta^m$$

where $\{\gamma_j\} \subset G_{\mathbf{R}}$ are the images under the monodromy representation of the standard generators of $\pi_1(\Delta^{*n})$ taken in a clockwise direction. In particular, the γ_j's commute. We now assume

(1.9) the γ_j's are quasi-unipotent.

(1.10) REMARKS. (i) The definition of a real, polarized VHS given above is due to Griffiths [15] and is abstracted from his fundamental result on the geometric situation: given an algebraic family of smooth projective varieties $f: M \to X$, the Hodge structures on the local system $\mathbf{V} = R^k f_* \mathbf{C}$ determine a VHS of weight k whose restriction to the primitive cohomology is polarizable. In this case, the local system of $\mathbf{Z}$-modules $R^k f_* \mathbf{Z}$ determines an additional flat, integral structure. If the latter is incorporated in the definition of VHS, the condition (1.9) is automatically satisfied, as asserted by the Monodromy Theorem [23], [25], [28].

(ii) It is clear that any holomorphic, horizontal map $\phi: U^n \times \Delta^m \to D$ satisfying (1.8) is a lifting of a period map of a VHS over $\Delta^{*n} \times \Delta^m$, with monodromy generated by the γ_i's.

The transformations $N_j = \frac{1}{\ell_j} \log \gamma_j^{\ell_j}$, where ℓ_j = unipotency index of γ_j, lie in $\mathfrak{g}_{\mathbf{R}}$, are nilpotent and mutually commute. To any nilpotent $N \in \mathfrak{g}(V)$ one associates its weight filtration: it is the unique increasing filtration $W = W(N)$ of V satisfying

(i) $NW_\ell \subset W_{\ell-2}$

(ii) $N^\ell: Gr^W_\ell \to Gr^W_{-\ell}$ is an isomorphism.

One has the "Lefschetz decomposition"

(1.11) $$Gr^W_\ell = \bigoplus_{j\geq 0} N^j(P_{\ell+2j}(N))$$

in terms of the "primitive parts" $P_\ell(N) = \mathrm{Ker}(N^{\ell+1}:Gr^W_\ell \to Gr^W_{-\ell-2})$. The weight filtrations associated to various linear combinations of the N_j's play a crucial role in the analysis to follow, as weight filtrations of the limiting mixed Hodge structures determined by a VHS.

Recall that a (real) mixed Hodge structure (MHS) on $(V,V_{\mathbf{R}})$ is a pair (W,F) of filtrations of V such that the weight filtration W is increasing and defined over $\mathbf{R}$ and the Hodge filtration F is decreasing and induces in Gr^W_ℓ a Hodge structure of weight ℓ. Given a MHS (W,F), the subspaces $I^{p,q} = I^{p,q}(W,F)$ defined by

(1.12) $$I^{p,q} = F^p \cap W_{p+q} \cap (\bar{F}^q \cap W_{p+q} + \sum_{j\geq 1} \bar{F}^{q-j} \cap W_{p+q-j-1})$$

are a bigrading of (W,F), in the sense that

(1.13) $$W_\ell = \bigoplus_{a+b\leq\ell} I^{a,b}, \qquad F^p = \bigoplus_{a\geq p} I^{a,b},$$

which is uniquely characterized by the property

(1.14) $$I^{p,q} \equiv \overline{I^{q,p}} \qquad (\mathrm{mod.} \bigoplus_{\substack{a<p\\ b<q}} I^{a,b})$$

(cf. [13], [2], [4]). In terms of this bigrading, a MHS (W,F) splits over $\mathbf{R}$ if $I^{p,q}(W,F) = \overline{I^{q,p}(W,F)}$, in which case $I^{p,q}(W,F) = F^p \cap \bar{F}^q \cap W_{p+q}$. The following result of Deligne [13], [4], associates to any MHS one that splits over $\mathbf{R}$, in a natural manner. Define

$$L^{-1,-1}(W,F) = \{T \in \mathrm{End}(V) \mid T(I^{p,q}) \subset \bigoplus \{I^{a,b} \mid a < p,\ b < q\}\}$$

$$L^{-1,-1}_{\mathbf{R}}(W,F) = L^{-1,-1}(W,F) \cap \mathrm{End}(V_{\mathbf{R}}) .$$

(1.15) THEOREM. Given a MHS (W,F) there exists a unique $\delta \in L^{-1,-1}_{\mathbf{R}}(W,F)$ such that $(W, \exp(-i\delta)\cdot F)$ is a MHS that splits over $\mathbb{R}$. Moreover, δ commutes with every morphism of (W,F) and $L^{-1,-1}(W,F) = L^{-1,-1}(W, \exp(-i\delta)\cdot F)$.

The limiting MHS determined by a polarized VHS carry distinguished polarizations. Given V, $V_{\mathbb{R}}$, an integer k and a real non-degenerate bilinear form S of parity $(-1)^k$, we introduce:

(1.16) DEFINITION. A MHS (W,F) on $(V,V_{\mathbb{R}})$ is said to be polarized by $N \in \mathfrak{g}_{\mathbb{R}}$ relative to the data above if

- (i) $N^{k+1} = 0$
- (ii) $W = W(N)[-k]$
- (iii) $S(F^p, F^{k-p+1}) = 0$
- (iv) $NF^p \subset F^{p-1}$
- (v) The Hodge structure of weight $k + \ell$ induced by F on $P_\ell(N)$, is polarized by the bilinear form $S_\ell(\cdot,\cdot) = S(\cdot, N^\ell\cdot)$.

Conditions (ii) and (iv) guarantee that N is a $(-1,-1)$-morphism of (W,F). By strictness of such morphisms, the Hodge structure on $Gr^W_{k+\ell}$ does indeed restrict to a Hodge structure on $P_\ell(N)$, giving sense to (v). Together with (1.11), the latter implies that the Hodge structure on Gr^W_ℓ is also polarizable - in other words, (W,F) is a graded polarizable mixed Hodge structure.

2. Approximation by Nilpotent Orbits.

We keep the notation of §1 and consider a period mapping $\phi: U^n \times \Delta^m \to D$, making two simplifying assumptions: $m = 0$ and the monodromy transformation are unipotent. The first entails no loss of generality - in fact, the statements will hold uniformly on compact subsets of Δ^m - while the second amounts to replacing Δ^n by a finite branched cover. We shall return to these points whenever necessary. The mapping

$$\phi: U^n \to D$$

is holomorphic, horizontal and satisfies $\phi(\underline{z} + e_j) = (\exp N_j)\phi(\underline{z})$. Since

$\exp(\Sigma z_j N_j)$ lies in $G_{\mathbf{C}}$, the map $\tilde{\psi}(\underline{z}) = \exp(-\Sigma z_j N_j) \cdot \phi(\underline{z})$ is $\check{D}$-valued, holomorphic and invariant under translation by the $\underline{e}_j$'s. Therefore $\tilde{\psi}(\underline{z}) = \psi(e^{2\pi i \underline{z}})$, with $\psi: \Delta^{*n} \to \check{D}$ holomorphic.

(2.1) NILPOTENT ORBIT THEOREM [28].

(i) the map ψ extends holomorphically to Δ^n;

(ii) the map $\theta: \mathbf{C}^n \to \check{D}$ given by $\theta(\underline{z}) = \exp(\Sigma z_j N_j) \cdot \psi(\underline{0})$ is horizontal and there exist $\alpha > 0$ such that $\theta(\underline{z}) \in D$ for $\operatorname{Im} z_j > \alpha$, $1 \leq j \leq n$;

(iii) for any $G_{\mathbf{R}}$-invariant distance d on D there exist positive constants β, K such that, for $\operatorname{Im} z_j > \alpha$,

$$d(\phi(\underline{z}),\theta(\underline{z})) \leq K \sum_j (\operatorname{Im} z_j)^{\beta} e^{-2\pi \operatorname{Im} z_j} .$$

Moreover, the constants α, β, K depend only on the choice of d and the weight and Hodge numbers used to define D.

The proof of the Nilpotent Orbit Theorem hinges upon the existence on D of $G_{\mathbf{R}}$-invariant Hermitian metrics whose holomorphic sectional curvatures along horizontal directions are negative and bounded away from zero [17]. We refer the reader to [9] and [18] for expository accounts and to [29] for an enlightening proof in the case when D is Hermitian symmetric; the latter is also explicitly worked out in [16] for VHS of weight one. We should remark that the distance estimate in (2.1; iii) is stronger than that in Schmid's original version ([28], 4.12) and is due to Deligne (cf. [4] for a proof).

In terms of the *canonical extension* of the bundle V to Δ^n, the theorem is viewed as follows. Recall [10] that this extension is determined by trivializing V over Δ^{*n} with frames of sections of the form

$$\tilde{v}(\underline{s}) = \exp\left(\Sigma \frac{\log s_j}{2\pi i} N_j \right) \cdot v(\underline{s}) \tag{2.2}$$

where $v(\underline{s})$ denotes the flat multivalued section of V determined by an element $v \in V_{\mathbf{R}} = (V_{\mathbf{R}})_{\underline{s}_0}$. (The difference in sign with the usual definition is caused by our convention (1.8)). If one represents the Hodge bundles F^p in terms of the $v(\underline{s})$'s, they can be viewed as a holomorphically varying filtration of V, determined up to monodromy. This is given by the period map $\Delta^{*n} \to D/\Gamma$, whose lifting is ϕ. If, instead, one represents them in terms of the single-valued $\tilde{v}(\underline{s})$'s, one obtains the mapping $\psi:\Delta^{*n} \to \check{D}$; the fact that ψ takes values in $\check{D}$, rather than D, reflects the fact that these sections are not real. The theorem then asserts that the Hodge bundles extend holomorphically as subbundles of the canonical extension ((i)) and that their constant part - always relative to the trivialization $(v,\underline{s}) \to \tilde{v}(\underline{s})$ - also define a polarized VHS ((ii)); θ is just the lifting of the associated period map and part (iii) estimates the proximity between the two Hodge structures.

We consider now arbitrary <u>nilpotent orbits</u>, i.e. maps $\underline{z} \to \exp(\Sigma z_j N_j) \cdot F$, where $N_1,\ldots,N_n$ are commuting nilpotent elements of $\mathfrak{g}_{\mathbb{R}}$, $F \in \check{D}$, $N_j \in F^{-1}\mathfrak{g}$ and $\exp(\Sigma z_j N_j) \cdot F \in D$ for $\operatorname{Im} z_j >> 0$.

(2.3) <u>THEOREM</u>. <u>If</u> $\{N_1,\ldots,N_n;F\}$ <u>determine a nilpotent orbit, then</u>

(i) $N_j^{k+1} = 0$, <u>where</u> k <u>is the weight of the structures in</u> D.

(ii) <u>Given</u> $I \subset \{1,\ldots,n\}$, <u>every element in the cone</u>

$$C_I = \{ \sum_{j\in I} \lambda_j N_j \mid \lambda_j \in \mathbb{R},\ \lambda_j > 0\}$$

<u>defines the same weight filtration</u> W^I.

(iii) $(W^{(n)}[-k],F)$ <u>is a mixed Hodge structure, polarized by every</u> $N \in C_{(n)}$ <u>(we write</u> (r) <u>for</u> $\{1,\ldots,r\}$).

Conversely, if commuting nilpotent elements $N_j \in (F^{-1}\mathfrak{g}) \cap \mathfrak{g}_{\mathbb{R}}$, $F \in \check{D}$, satisfy (i), (ii) for $I = (n)$ and (iii) for some $N \in C_{(n)}$, then they determine a nilpotent orbit with the filtration F.

Parts (i) and (iii) are due to Schmid [28] and follow from the SL_2-orbit theorem, to be discussed in the next section. In the geometric case, (i) is part of the monodromy theorem (cf. Landman [25] and Katz [23]) while (iii) was also obtained by Steenbrink [30] and Clemens [8] (cf. also Chapter VII of [16]). The statement (ii) was proved by the authors in [1]. The converse statement is a consequence of the several-variables SL_2-orbit theorem [4].

A period mapping $\phi: U^n \to D$ can now be written as

$$\phi(\underline{z}) = \exp(\Sigma z_j N_j) \cdot \psi(\underline{s}),$$

with $\psi: \Delta^n \to \check{D}$ holomorphic. The limiting mixed Hodge structure $(W^{(n)}[-k], F)$ given by (2.3; iii) can be used to define a distinguished lifting of ψ to $G_{\mathbb{C}}$. Let $I^{*,*}\mathfrak{g}$ denote the bigrading (1.12) of the MHS $(W^{(n)}\mathfrak{g}, F\mathfrak{g})$ induced in $\mathfrak{g}$. Then the graded subalgebra

$$(2.4) \qquad p = \bigoplus_{a \leq -1} p_a; \qquad p_a = \bigoplus_b I^{a,b}$$

is a linear complement of $F^0\mathfrak{g}$ = isotropy subalgebra at F. Thus, for $\underline{s}$ in a possibly smaller polydisk around $\underline{0}$, we can write uniquely

$$(2.5) \qquad \psi(\underline{s}) = \exp\Gamma(\underline{s}) \cdot F$$

with $\underline{s} \to \Gamma(\underline{s}) \in p$ holomorphic and $\Gamma(\underline{0}) = 0$.

(2.6) PROPOSITION. Let $D_j = \{\underline{s} \in \Delta^n | \ s_j = 0\}$ and $\Gamma_j = \Gamma|_{D_j}$. Then $[N_j, \Gamma_j] = 0$.

Proof: Since $N_j \in I^{-1,-1}\mathfrak{g} \subset p_{-1}$, we can write $\exp(\Sigma z_j N_j)\exp\Gamma(\underline{s}) = \exp X(\underline{z})$ with $X(\underline{z}) \in p$, so that $\phi(\underline{z}) = \exp X(\underline{z}) \cdot F$. The logarithmic derivatives of $\exp X(\underline{z})$ lie in p and, due to the horizontality of ϕ, in $F^{-1}\mathfrak{g}$ as well. Hence $e^{-X(z)} \frac{\partial}{\partial z_j} e^{X(\underline{z})} \in p_{-1}$. In terms of Γ, this becomes

$$(2.7) \qquad e^{-\operatorname{ad}\Gamma(\underline{s})}N_j + 2\pi i s_j e^{-\Gamma(\underline{s})} \frac{\partial}{\partial s_j} e^{\Gamma(\underline{s})} \in p_{-1}$$

and, setting $s_j = 0$, one concludes:

$$e^{-\operatorname{ad}\Gamma_j(\underline{s})} N_j \in p_{-1} .$$

Given that $N_j \in p_{-1}$ and $\Gamma \in p$, this is possible only if $[\Gamma_j, N_j] = 0$.

(2.8) THEOREM. Let $\{N_1, \ldots, N_n; F\}$ define a nilpotent orbit and let $\Gamma: \Delta^n \to p$

be holomorphic and satisfy (2.7), so that the mapping

$$\phi(\underline{z}) = \exp(\Sigma z_j N_j) \cdot \exp\Gamma(\underline{s}) \cdot F$$

is horizontal. Then $\phi(\underline{z})$ is a period mapping, i.e. $\phi(\underline{z}) \in D$, for $\mathrm{Im}\, z_j >> 0$.

The proof of (2.8) uses the several-variables SL_2-orbit theorem and will be postponed until §4. We note the following implication: given a period mapping $\phi(\underline{z}) = \exp(\Sigma z_j N_j)\gamma(\underline{s}) \cdot F$, $\gamma = \exp\Gamma$, and a subset $I \subset \{1,\ldots,n\}$, set $D_I = \bigcap_{j\in I} D_j$ and $\gamma_I(\underline{s}) = \gamma \circ$(projection of $\underline{s}$ to D_I). The map $\phi_I : U^n \to \check{D}$,

$$\phi_I(\underline{z}) = \exp(\Sigma z_j N_j) \cdot \gamma_I(\underline{s}) \cdot F$$

is clearly horizontal and therefore, by (2.8), defines a period map for $\mathrm{Im}(z_j)$ sufficiently large. Moreover, since $\gamma_I(\underline{s})$ commutes with N_i for $i \in I$, it preserves the filtration W^I, as does, of course, $\exp(\Sigma z_j N_j)$. Recall now the following result from [1].

(2.9) THEOREM. Let $I \subset \{1,\ldots,n\}$, $J = \{1,\ldots,n\} - I$ and let $\tilde{N}_j$, $j \in J$, (resp. $\tilde{F}$) denote the endomorphisms induced by the N_j's on $Gr_\ell^{W^I}$ (resp. the filtration induced by F). Then

(i) $\{\tilde{N}_j, j \in J; \tilde{F}\}$ determine a nilpotent orbit on $Gr_\ell^{W^I}$.

(ii) The weight filtration of $\tilde{N}_j$ is the projection to $Gr_\ell^{W^I}$ of $W^{I\cup\{j\}}[\ell]$.

Combining (2.8) and (2.9) we obtain

(2.10) PROPOSITION. For $\mathrm{Im}\, z_j$, $j \in J$, sufficiently large, $\left(W^I[-k], \exp\left(\sum_{j\in J} z_j N_j\right)\gamma_I(\underline{s}) \cdot F\right)$ is a mixed Hodge structure, polarized by all $N \in C_I$.

In terms of the original VHS, this shows that the filtration F_I defined on the canonical extension over D_I by $\lim_{\mathrm{Im}\, z_i \to \infty} \exp\left(- \sum_{i\in I} z_i N_i\right)\phi(\underline{z})$, together with the filtration $\mathcal{W}^I$ determined by W^I, define a graded polarizable variation of MHS over D_I. According to (2.9; ii), this variation is admissible in the sense of Kashiwara [20] and Elzein [14].

3. SL_2-Orbits in One Variable.

Schmid's SL_2-Orbit Theorem associates to each nilpotent orbit $\exp zN \cdot F$ a nearby one, $\exp zN \cdot F_0$, which is equivariant under natural actions of $SL(2,\mathbf{R})$. For these SL_2-orbits it is fairly easy to show that N, F_0, determine a polarized MHS, as well as exact statements on its asymptotic behavior. The corresponding properties for the original orbit then follow from the specific proximity between the two. Now, the MHS defined by the SL_2-orbit splits over $\mathbf{R}$ - indeed, SL_2-orbits and polarized MHS that split over $\mathbf{R}$ are equivalent notions. Therefore, the theorem can be interpreted *a posteriori* as assigning to any polarized MHS (W,F,N) another one (W,F_0,N) that splits over $\mathbf{R}$. Understanding the map $(N,F) \to F_0$ and its relation with that given by (1.15) leads to a generalization of those results to the several-variables case, to be discussed in §4.

Let (W,F_0) be a MHS on V, split over $\mathbf{R}$ and polarized by $N \in F_0^{-1}\mathfrak{g}_{\mathbf{R}}$. Since $W = W(N)[-k]$, the subspaces

$$V = \bigoplus_{p+q=k+\ell} I^{p,q}(W,F_0), \qquad -k \le \ell \le k$$

constitute a grading of $W(N)$ defined over $\mathbf{R}$. Let $Y = Y(W,F_0)$ denote the real semisimple endomorphism of V which acts on V_ℓ as multiplication by the integer ℓ. Since $NV_\ell \subset V_{\ell-2}$,

(3.1) $$[Y,N] = -2N.$$

Because N polarizes the MHS (W,F_0) one also obtains (cf. [4], 2.7):

(3.2) $$Y \in \mathfrak{g}_{\mathbf{R}}\,.$$

(3.3) There exist $N^+ \in \mathfrak{g}_{\mathbf{R}}$ such that $[Y,N^+] = 2N^+$, $[N^+,N^-] = Y$.

Therefore, there is a Lie algebra homomorphism $\rho_*: s\ell_2(\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ defined over

$\mathbf{R}$ such that, for the standard generators

(3.4) $$Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

one has

(3.5) $$\rho_*(Y) = Y, \qquad \rho_*(N) = N, \qquad \rho_*(N^+) = N^+.$$

The upper-half plane U can be regarded as the classifying space for polarized Hodge structures of weight 1 and $h^{1,0} = h^{0,1} = 1$ on $\mathbf{C}^2$, for which $G_{\mathbf{R}} = SL(2,\mathbf{R})$. According to (1.6), the choice of the base point $\sqrt{-1} \in U$ determines a Hodge structure of weight 0 on $s\ell_2(\mathbf{C})$, given in this case by

(3.6) $$\begin{aligned} (s\ell_2(\mathbf{C}))^{-1,1} &= \overline{(s\ell_2(\mathbf{C}))^{1,-1}} = \mathbf{C}(\sqrt{-1}\,Y + N + N^+) \\ (s\ell_2(\mathbf{C}))^{0,0} &= \mathbf{C}(N^+ - N). \end{aligned}$$

A homomorphism $\rho_*: s\ell(\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ is said to be <u>Hodge at</u> $F \in D$, if it is morphism of Hodge structures: that given by (3.6) on $s\ell_2(\mathbf{C})$ and the one determined by $F\mathfrak{g}_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$. The lifting $\rho: SL(2,\mathbf{C}) \to G_{\mathbf{C}}$ of such a morphism induces a horizontal, equivariant embedding

(3.7) $$\tilde{\rho}: \mathbf{P}^1 \to \check{D}$$

by $\tilde{\rho}(g \cdot \sqrt{-1}) = \rho(g) \cdot F$, $g \in SL(2,\mathbf{C})$. Moreover,

(3.8)
(i) $\rho(SL(2,\mathbf{R})) \subset G_{\mathbf{R}}$ and therefore $\tilde{\rho}(U) \subset D$.
(ii) $\tilde{\rho}(z) = (\exp z\rho_*(N))(\exp(-i\rho_*(N))) \cdot F$.
(iii) $\tilde{\rho}(z) = (\exp x\rho_*(N))(\exp(-\frac{1}{2}\log y\,\rho_*(Y))) \cdot F$

for $z = x + iy \in U$.

(3.9) <u>PROPOSITION</u>. <u>Let</u> (W,F_0) <u>be a MHS split over</u> $\mathbf{R}$ <u>and polarized by</u> $N \in F_0^{-1}\mathfrak{g}$ <u>Then</u>

(i) <u>The filtration</u> $F_{\sqrt{-1}} := \exp\sqrt{-1}\,N \cdot F_0$ <u>lies in</u> D.

(ii) <u>The homomorphism</u> $\rho_*: s\ell_2(\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ <u>defined by (3.5) is Hodge at</u> $F_{\sqrt{-1}}$.

<u>Conversely, if a homomorphism</u> $\rho_*: s\ell_2(\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ <u>is Hodge at</u> $F \in D$, <u>then</u>

(3.10) $(W(\rho_*(N))[-k], \exp(-i\rho_*(N)) \cdot F)$ <u>is a MHS, split over</u> $\mathbf{R}$ <u>and polarized by</u> $\rho_*(N)$.

The proof of (3.9) and (3.10) is implicit in ([4], 3.12): one reduces the problem to the case of the three elementary $\mathbf{R}$-split, polarized MHS:

(a) $V = \mathbf{C} = I^{1,1}$; $N = 0$.

(3.11) (b) $V = \mathbf{C}^2 = I^{k,0} \oplus \overline{I^{k,0}}$; $N = 0$

(c) $V = \mathbf{C}^2 = I^{0,0} \oplus I^{1,1}$; $NI^{1,1} = I^{0,0}$

with the obvious polarizing forms, where it becomes straightforward. Note that (3.8; ii) describes the horizontal embedding $\rho: \mathbf{P}^1 \to \check{D}$ as a nilpotent orbit, while (iii) gives a real analytic lifting of $\tilde{\rho}|_U : U \to D$ to $G_{\mathbf{R}}$.

We may now state the SL_2-orbit theorem - the following is a simpler version, suitable for many applications. As an illustration, we show how it yields the equivalence between nilpotent orbits and polarized MHS.

(3.12) <u>THEOREM</u>. <u>Let</u> $z \to \exp zN \cdot F$ <u>be a nilpotent orbit</u>. <u>There exist</u>

(a) <u>a filtration</u> $F_{\sqrt{-1}} \in D$;

(b) <u>a homomorphism</u> $\rho_*: s\ell_2(\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$, <u>Hodge at</u> $F_{\sqrt{-1}}$;

(c) <u>a real analytic</u>, $G_{\mathbf{R}}$<u>-valued function</u> $g(y)$, <u>defined for</u> $y >> 0$, <u>such that</u>

(i) $N = \rho_*(N)$;

(ii) <u>for</u> $y >> 0$, $\exp(iyN) \cdot F = g(y)\exp(iyN) \cdot F_0$, <u>where</u> $F_0 = \exp(-iN) \cdot F_{\sqrt{-1}}$;

(iii) <u>both</u> $g(y)$ <u>and</u> $g(y)^{-1}$ <u>have convergent power series expansions at</u> $y = \infty$, <u>of the form</u> $1 + \sum_{n=1}^{\infty} A_n y^{-n}$, <u>with</u>

$$A_n \in W_{n-1}(N)\mathfrak{g} \cap \ker(\operatorname{ad}N)^{n+1} .$$

We refer to Schmid's original paper [28], as well as to [2], [4], for details.

(3.13) THEOREM. $z \to \exp zN \cdot F$ is a nilpotent orbit if and only if $(W(N)[-k],F)$ is a MHS, polarized by N.

We review the proof of (3.13). Given the nilpotent orbit, let F_0, $g(y)$, be as in (3.12). Then $F = \exp(-iyN)g(y)\exp(iyN) \cdot F_0$. Writing $g(y) = 1 + \sum_{n=1}^{\infty} g_n y^{-n}$ for $y >> 0$, one finds $\exp(-iyN)g(y)\exp(iyN) = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_m((\mathrm{ad}N)^m g_n)y^{m-n}$, where the c_m's are suitable complex constants. Letting $y \to \infty$ one obtains

$$F = (1 + \sum_{n=1}^{\infty} c_n((\mathrm{ad}N)^n g_n)) \cdot F_0.$$

Since $(\mathrm{ad}N)^n g_n \in W_{-n-1}(N)\mathfrak{g} \subset W_{-2}(N)\mathfrak{g}$, the two filtrations F and F_0 define the same filtrations on $Gr^{W(n)}$. But the data N, F_0, comes from a Hodge representation, hence by (3.11) $(W(N)[-k],F_0)$ - and therefore $(W(n)[-k],F)$ - are MHS, polarized by N.

Conversely, suppose (W,F) is a MHS polarized by N and let (W,F_0) be the MHS split over $\mathbf{R}$ associated to (W,F) by (1.15). Thus (W,F_0) is still polarized by N and

$$F_0 = \exp(-i\delta) \cdot F \qquad \text{with} \qquad \delta \in L_{\mathbf{R}}^{-1,-1}(W,F).$$

Let ρ_* be the Hodge representation determined by N, F_0 and let $Y = \rho_*(Y)$. Using (3.8) and the fact that N, as a $(-1,-1)$-morphism of (W,F), commutes with δ, we can write

$$\begin{aligned} \exp zN \cdot F &= \exp zN \exp i\delta \cdot F_0 \\ &= \exp xN \exp(-\tfrac{1}{2}\log yY)\exp(i\,\mathrm{Ad}(\exp\tfrac{1}{2}\log yY)\delta) \cdot F_{\sqrt{-1}} . \end{aligned}$$

Since $\delta \in W_{-2}(N)\mathfrak{g}$, $\lim_{y\to\infty} \mathrm{Ad}(\exp\frac{1}{2}\log yY)\delta = 0$, which implies that for $y >> 0$, $\exp(i\,\mathrm{Ad}(\exp\frac{1}{2}\log yY)\delta) \cdot F_{\sqrt{-1}}$ $(\sim F_{\sqrt{-1}})$ lies in a relatively compact subset of D. Since both $\exp xN$ and $\exp(-\frac{1}{2}\log yY)$ lie in $G_{\mathbf{R}}$, it follows that $\exp zN \cdot F \in D$ for $\mathrm{Im}\, z >> 0$, so that the latter is indeed a nilpotent orbit.

The objects F_0, ρ_*, $g(y)$, are not uniquely determined by the properties (i)-(iii) but there is a distinguished choice, as in [28], whose properties are useful for extending the theorem to several variables. We emphasize that the corresponding filtration F_0 will not in general agree with that of the **R**-split MHS associated to $(W(N)[-k],F)$ by (1.15), which we denote now by $\tilde{F}_0$:

$$\tilde{F}_0 = \exp(-i\delta)\cdot F, \qquad \delta \in L_{\mathbf{R}}^{-1,-1}(W,F) .$$

(3.14) THEOREM. The filtration F_0 and the function $g(y)$ in (3.12) can be chosen so that

(i) $F_0 = \exp\xi \cdot \tilde{F}_0$, where ξ is a universal non-commutative polynomial in the components $\delta^{a,b}$, $a \le -1$, $b \le -1$, of δ relative to the bigrading $I^{*,*}(W,\tilde{F}_0\mathfrak{g})$. In particular,

$$\xi \in L^{-1,-1}(W\mathfrak{g},\tilde{F}_0\mathfrak{g}) = L^{-1,-1}(W\mathfrak{g},F_0\mathfrak{g}) = L^{-1,-1}(W\mathfrak{g},F\mathfrak{g});$$

(ii) The coefficients A_n in the series expansions of $g(y)$ and $g(y)^{-1}$ in powers of y^{-1}, are universal non-commutative polynomials in the $\delta^{a,b}$'s and the transformation $\mathrm{ad}N^+$, where N^+ is associated to N, $\tilde{F}_0$, as in (3.3).

(3.15) COROLLARY. The filtration F_0 depends only on the MHS $(W(N)[-k],F)$ and not on the particular N or polarizing form S.

4. SL_2-Orbits and the Asymptotics of Period Maps in Several Variables.

We now consider an arbitrary nilpotent orbit $(z_1,\ldots,z_n) \to \exp(\Sigma z_j N_j)\cdot F$ and recall the notation and statements of Theorem (2.3). The MHS $(W^{(n)}[-k],F)$ is polarized by every $N \in C_{(n)}$ and, according to (3.15), the **R**-split MHS associated to it by the SL_2-orbit theorem is independent of N. We make a slight notational change and denote the Hodge filtration of the latter by $F_{(n)}$ (rather than F_0).

The **R**-split MHS $(W^{(n)}[-k],F_{(n)})$ is again polarized by every $N \in C_{(n)}$; in particular, $\exp iC_{(n)} \cdot F_{(n)} \subset D$ and therefore

$$\exp\left(\sum_{j=1}^{n-1} z_j N_j\right) \cdot (\exp i N_n \cdot F_{(n)}) \in D \quad \text{for} \quad \operatorname{Im} z_j > 0 \quad .$$

In other words, $(N_1,\ldots,N_{n-1};\ \exp iN_n \cdot F_{(n)})$ determine a nilpotent orbit of $n-1$ variables and we may repeat the procedure to obtain an $\mathbb{R}$-split MHS $(W^{(n-1)}[-k],F_{(n-1)})$ polarized by every $N \in C_{(n-1)}$. Continuing in this way, one obtains filtrations $F_{(1)},\ldots,F_{(n)}$ - depending on the given ordering of the variables - with the following properties:

(4.1)
(i) $(W^{(r)}[-k],F_{(r)})$ is a MHS split over $\mathbb{R}$ and polarized by every $N \in C_{(r)}$.

(ii) $(W^{(r)}[-k],F_{(r)})$ is associated to the MHS $(W^{(r)}[-k],\ \exp i N_{r+1} \cdot F_{(r+1)})$ by (3.12).

Let now $Y_{(r)} \in \mathfrak{g}_{\mathbb{R}}$ denote the semisimple element defined by the $\mathbb{R}$-grading of $(W^{(r)}[-k],F_{(r)})$ as in §3.

(4.2) <u>THEOREM</u>. <u>The elements</u> $Y_{(1)},\ldots,Y_{(n)}$ <u>commute</u>. <u>Thus</u>, <u>there is a multi-grading</u> $V = \bigoplus_{\underline{\ell}\in\mathbb{Z}^n} V_{\underline{\ell}}$ <u>defined over</u> $\mathbb{R}$, <u>such that</u>

$$W_s^{(r)} = \bigoplus_{\ell_r \le s} V_{\underline{\ell}}, \qquad Y_{(r)}|V_{\underline{\ell}} = \ell_r 1|V_{\underline{\ell}} \quad .$$

The proof of (4.2) is given in [4] and depends on the properties of the function $g(y)$ of the one-variable SL_2-orbit theorem. A different proof of the existence of common $\mathbb{R}$-gradings of the filtrations $W^{(1)},\ldots,W^{(n)}$, was given by Kashiwara [19]; this suffices, for example, to obtain the norm-estimates (5.1), but the fact that the specific gradings in (4.2) satisfy (4.1; ii) seems to be needed for some of the other applications, in particular (5.2) and (5.3). We should also note that (4.2) does not hold if one replaces the $\mathbb{R}$-splittings $Y_{(1)},\ldots,Y_{(n-1)}$, by those given by the simpler construction (1.15); on the other hand, an explicit formula for them in terms of the bigradings $I^{p,q}$ is given in [6].

Define $\hat{N}_1,\dots,\hat{N}_n$ in $\mathfrak{g}_{\mathbf{R}}$ by: $\hat{N}_1 = N_1$ and $\hat{N}_r$ = component of N_r in $\bigcap_{j=1}^{r-1} \ker \operatorname{ad}Y_{(j)}$, relative to the decomposition of $\mathfrak{g}_{\mathbf{R}}$ in common eigenspaces of $\operatorname{ad}Y_{(1)},\dots,\operatorname{ad}Y_{(r-1)}$. Set

$$N_{(r)} = N_1 + \dots + N_r, \quad \hat{N}_{(r)} = \hat{N}_1 + \dots + \hat{N}_r, \quad Y_r = Y_{(r)} - Y_{(r-1)} ,$$

and let y_r, N_r, N_r^+ denote the standard generators (3.4) of the r-th factor of $s\ell_2(\mathbf{C})^n$.

(4.3) <u>THEOREM</u>.

(i) $W(\hat{N}_{(r)}) = W(N_{(r)}) \quad (= W^{(r)})$.

(ii) <u>The filtration</u> $F_{\sqrt{-1}} = \exp i\hat{N}_{(r)} \cdot F_{(r)}$ <u>is independent of</u> r <u>and lies in</u> D.

(iii) <u>There is a homomorphism</u> $\rho_*: s\ell_2(\mathbf{C})^n \to \mathfrak{g}_{\mathbf{C}}$, <u>Hodge at</u> $F_{\sqrt{-1}}$, <u>such that</u> $\rho_*(y_r) = Y_r$ <u>and</u> $\rho_*(N_r) = N_r$.

We refer to [4] for a proof. There, we also give an expression for the original nilpotent orbit $\exp(\Sigma z_j N_j) \cdot F$ as the translate of the SL_2^n-orbit $\exp(\Sigma z_j N_j) \cdot F_n$ by g(y)-like functions, as in (3.12). Here, we will use (4.3) directly to describe the asymptotic behavior of a period map, in a way that seems better suited for applications (cf. [6]).

Given a period map $\phi: U^n \to D$ we write it as in (2.5)

$$\phi(\underline{z}) = \exp(\Sigma z_j N_j)\gamma(\underline{s}) \cdot F,$$

where $s_j = e^{2\pi i z_j}$ and $\gamma(\underline{s}) = \exp\Gamma(\underline{s})$, $\Gamma: \Delta^n \to p$ holomorphic and $\Gamma(\underline{0}) = 0$. Let $Y_{(j)}, \hat{N}_{(j)}$, etc. be the SL_2-data associated to the nilpotent orbit $\underline{z} \to \exp(\Sigma z_j N_j) \cdot F$ and the given ordering of the variables and define, for $y_j > 0$,

$$t_j = \frac{y_j}{y_{j+1}} \quad \text{for} \quad 1 \le j \le n-1, \quad t_n = y_n,$$

$$e(\underline{t}) = \prod_{j=1}^{n} \exp(\tfrac{1}{2} \log t_j Y_{(j)}) = \prod_{j=1}^{n} \exp(\tfrac{1}{2} \log y_j Y_j). \tag{4.4}$$

Then $e(\underline{t}) \in G_{\mathbf{R}}$ and it acts on $V_{\underline{\ell}}$ as multiplication by $t_1^{\ell_1/2} \dots t_n^{\ell_n/2}$.

(4.5) LEMMA.

(i) $Ad(e(\underline{t}))(\Sigma y_j N_j)$ is a polynomial in $t_1^{-1/2},\dots,t_n^{-1/2}$ whose term of degree zero is $\hat{N}_{(n)} = \Sigma \hat{N}_j$.

(ii) As $t_j \to \infty$, $1 \le j \le n$, $e(\underline{t})\gamma(\underline{s})e(\underline{t})^{-1} \to 1$; in fact, $\|e(\underline{t})\gamma(\underline{s})e(\underline{t})^{-1} - 1\| \sim e^{-ct_n}$ for a suitable constant $c > 0$.

Proof: For $r \ge j$, N_j is a $(-1,-1)$ morphism of $(W^{(r)}[-k],F_{(r)})$, so that $[Y_{(r)},N_j] = -2N_j$. Therefore $Ad(\prod_{r\ge j} \exp(\frac{1}{2}\log t_r Y_{(r)}))(y_j N_j) = t_j^{-1}\dots t_n^{-1} y_j N_j = N_j$. On the other hand, for $r < j$, $N_j \in W_0^{(r)}\mathfrak{g}$ and so it lies in the sum of eigenspaces of $adY_{(r)}$ with eigenvalues ≤ 0. Since, by definition, $\hat{N}_j$ is the component of $\hat{N}_j$ in $\bigcap_{r<j} \ker(adY_{(r)})$, (i) follows.

To prove (ii), write $\Gamma(\underline{s}) = \sum_{\underline{\ell}\in\mathbf{Z}^n} \Gamma_{\underline{\ell}}(\underline{s})$ according to the common eigenspaces $V_{\underline{\ell}}\mathfrak{g}$. For a given $\underline{\ell}$, if $\ell_j > 0$, then $\Gamma_{\underline{\ell}}|_{D_{(j)}} = 0$ (cf. (2.6)) and therefore

$$\|Ad(\exp \frac{1}{2}\log t_j Y_{(j)})\Gamma_{\underline{\ell}}(\underline{s})\| \le t_j^{\ell_j/2} \sum_{r=1}^{j} e^{-c_r y_r} ;$$

since $y_r = t_t \dots t_n$, this norm goes to 0 with order at least $e^{-c_r y_r}$. For $\ell_j \le 0$ this estimate holds for trivial reasons.

(4.6) REMARK. In proving (4.5; ii) we have used only that the function $\Gamma(\underline{s}) = \log \gamma(\underline{s})$ satisfies (2.6). The proof also shows that $e(\underline{t})\gamma(\underline{s})e(\underline{t})^{-1} \to 1$ as $y_1 \to \infty,\dots,y_n \to \infty$, provided that the ratios $t_j = \dfrac{y_j}{y_{j+1}}$ remain bounded away from zero.

According to (3.14; i) and (1.15) we may write $F = \exp \eta \cdot F_{(n)}$, where $\eta \in L^{-1,-1}(W^{(n)}[-k],F)$ and commutes with every morphism of $(W^{(n)}[-k],F)$. In particular, $e(\underline{t})\exp\eta\, e(\underline{t})^{-1} \to 1$ as $t_1 \to \infty,\dots,t_n \to \infty$. Because of (4.3), $e(\underline{t})$ leaves invariant the filtration $F_{(n)}$. We can then write

$$\phi(\underline{z}) = \exp(\Sigma z_j N_j)\gamma(\underline{s})\cdot F$$
$$= \exp(\Sigma x_j N_j)e(\underline{t})^{-1}(e(\underline{t})\exp(i\Sigma y_j N_j)\gamma(\underline{s})\exp\eta e(\underline{t})^{-1})\cdot F_{(n)}\ .$$

But

$$e(\underline{t})\exp(\Sigma y_j N_j)\gamma(\underline{s})\exp\eta e(\underline{t})^{-1} \to \exp i\hat{N}_{(n)}\ ,$$
$$e(\underline{t})\exp(\Sigma x_j N_j)e(\underline{t})^{-1} \to 1 \quad \text{for} \quad |x_j| \quad \text{bounded}$$

as $t_1 \to \infty,\dots,t_n \to \infty$, and $\exp i\hat{N}_{(n)}\cdot F_{(n)} = F_{\sqrt{-1}}$. Hence we obtain,

(4.7) THEOREM. For any positive ε,

$$\{e(\underline{t})\phi(\underline{z})\mid t_j > \varepsilon\} \quad \text{and} \quad \{e(t)\exp(-\Sigma x_j N_j)\phi(\underline{z})\mid t_j > \varepsilon,\ |x_j| < 1\}$$

are relatively compact subsets of D.

In fact, the following more precise description of ϕ can be obtained by unraveling (4.6) and arguing as in the proof of (4.5) (cf. [6]).

(4.8) THEOREM. The period mapping can be written as

$$\phi(\underline{z}) = \exp(\Sigma x_j N_j)e(\underline{t})^{-1}p(t)q(\underline{x},\underline{y})\cdot F_{\sqrt{-1}}\ ,$$

where $p(\underline{t})$ is a $G_{\mathbb{C}}$-valued polynomial in $t_1^{-1/2},\dots,t_n^{-1/2}$ with constant term one and $q(\underline{x},\underline{y})$ is a $G_{\mathbb{C}}$-valued analytic function satisfying the following estimate: for any $\varepsilon > 0$ there are positive constants c, K such that $\|q(\underline{x},\underline{y}) - 1\| < e^{-cy_n}$ for $t_j > \varepsilon$, $1 \le j \le n-1$; $t_n > K$. The same estimate holds if one replaces $q(\underline{x},\underline{y}) - 1$ by the derivatives $\prod(y_j \frac{\partial}{\partial y_i})q$, $\prod(y_j \frac{\partial}{\partial x_j})q$.

The objects $F_{(r)}$, $Y_{(r)}$, $\hat{N}_{(r)}$, for $r < n$ and the function $e(\underline{t})$, depend on an ordering of the variables, as does the region of validity of (4.7) and (4.8). Also, the construction leading to (4.1) can be carried out for any chain of index sets $I_0 \subset I_1 \subset \dots \subset I_m = (n)$, to obtain common splittings of the weight filtrations $W^{I_1},\dots,W^{I_m}$. One obtains corresponding versions of (4.3)-(4.8), either by letting the role of $N_1,\dots,N_n$ be played by fixed elements $T_r \in C_{I_r}$, or by letting T_r be an arbitrary element in C_{J_r}, $J_r = I_r - I_{r-1}$, and viewing the resulting data as depending on parameters.

We conclude this section with the proof of Theorem (2.8). The equation (2.7) is equivalent to the horizontality of ϕ, so that one only needs to show

$$(4.9) \qquad \phi(\underline{z}) \in D \quad \text{for} \quad \operatorname{Im} z_j >> 0 .$$

For a given ordering of the variables let $e(\underline{t})$ be the corresponding function and write

$$\begin{aligned} e(\underline{t})\exp(-\Sigma x_j N_j)\phi(\underline{z}) &= \\ &= e(\underline{t})\exp i(\Sigma y_j N_j)\gamma(\underline{s}) \cdot F \\ &= (\mathrm{Ad}(e(\underline{t}))\exp i(\Sigma y_j N_j)\gamma(\underline{s}))(e(\underline{t})\exp i(\Sigma y_j N_j) \cdot F). \end{aligned}$$

By assumption, $\{N_1,\ldots,N_n;F\}$ determine a nilpotent orbit, therefore $\exp i(\Sigma y_j N_j) \cdot F \in D$ for $\operatorname{Im} z_j >> 0$. Also, according to (4.7), the filtrations $e(\underline{t})\exp i(\Sigma y_j N_j) \cdot F$ lie in a compact subset of D for $t_j \geq \varepsilon > 0$. On such a region, the element $\mathrm{Ad}(e(\underline{t}))(\Sigma y_j N_j)$ remains bounded (cf. (4.5; i)) and, according to (4.6), $e(\underline{t})\gamma(\underline{s})e(\underline{t})^{-1} \to 1$ as $y_j \to \infty$. Consequently, the filtration $e(\underline{t})\exp(-\Sigma x_j N_j) \cdot \phi(\underline{z})$, and therefore $\phi(\underline{z})$ itself, lie in D for $\operatorname{Im} z_j >> 0$ as long as the t_j's remain bounded away from zero. Permuting the variables yields (4.9).

5. Some Applications.

The analysis of period mappings given in the previous section yields good descriptions of the degeneration of the Hodge metric relative to the flat structure near a normal-crossings divisor. For example, let $V \to \Delta^{*n}$ be the vector bundle underlying a VHS with unipotent monodromy and let $W^{(1)} = W(N_1),\ldots, W^{(n)} = W(N_1 + \ldots + N_n)$ be the sequence of flat monodromy weight filtrations associated, as in (2.3), to the ordering $(s_1,\ldots,s_n)$ of the coordinates in Δ^{*n}.

(5.1) THEOREM. *For some flat, $V_{\mathbb{R}}$-compatible Hermitian metric Q on V,* defined up to monodromy, the corresponding Q-orthogonal gradings of the filtrations $W^{(1)},\ldots, W^{(n)}$ are mutually compatible. Moreover, over any region of the form

$$\{\underline{s} \in \Delta^{*n} \Big| \ |\underline{s}| < a < 1; \ \frac{\log|s_j|}{\log|s_{j+1}|} > \varepsilon, \quad 1 \leq j \leq n-1\}$$

the Hodge metric is quasi-isometric to

$$\bigoplus_{\underline{\ell}\in\mathbf{Z}^n} \left(\frac{\log|s_1|}{\log|s_2|}\right)^{\ell_1} \cdots (-\log|s_n|)^{\ell_n} Q_{\underline{\ell}} \ ,$$

where $Q = \oplus\, Q_{\underline{\ell}}$ corresponds to the multigrading $V \cong \bigoplus_{\underline{\ell}\in\mathbf{Z}^n} Gr^{W^{(n)}}_{\ell_n} \cdots Gr^{W^{(1)}}_{\ell_1} V$

These estimates were obtained by Schmid [28] in the one-variable case and by Schmid and the authors [4] and by Kashiwara [19] in the general case. We repeat the argument here to illustrate the use of Theorem (4.7). Let $V \times U^n \to U^n$ denote the pullback of $V \to \Delta^{*n}$ to the universal cover and $\phi: U^n \to D$ the corresponding period mapping. Each $F \in D$ determines a Hodge metric H_F on V as in (1.4) and, if $g \in G_{\mathbf{R}}$, then $H_{g\cdot F} = g \cdot H_F$. In these terms, the pullback of the Hodge metric to $V \times U^n$ is represented by the family of metrics $H_{\phi(\underline{z})}$, $\underline{z} \in U^n$, on V. According to (4.7), for $\underline{z} = \frac{1}{2\pi i} \log \underline{s}$ in a region of the form $\frac{y_j}{y_{j+1}} < \varepsilon$, $y_n > \varepsilon$, $|x_j| < 1$, the filtration $e(\underline{t})\phi(\underline{z})$ remains in a compact subset of D and therefore the metrics $H_{e(\underline{t})\phi(\underline{z})}$ are uniformly quasi-isometric to any given metric, say $H_{F_{\sqrt{-1}}}$. Consequently, the metrics $H_{\phi(\underline{z})} = e(\underline{t})^{-1} H_{e(\underline{t})\phi(\underline{z})}$ are uniformly quasi-isometric to $e(\underline{t})^{-1} \cdot H_{F_{\sqrt{-1}}}$. The multigrading $V = \oplus\, V_{\underline{\ell}}$ of $W^{(1)},\ldots, W^{(n)}$ is $H_{F_{\sqrt{-1}}}$-orthogonal (as can be seen by decomposing the Hodge representation into irreducibles) and $e(\underline{t})$ acts with eigenvalue $\left(\frac{y_1}{y_2}\right)^{\ell_1/2} \cdots y_n^{\ell_n/2}$ on $V_{\underline{\ell}}$. Hence, letting Q denote the multivalued flat form induced by $H_{F_{\sqrt{-1}}}$ on V, the theorem follows.

With the more detailed information of (4.8) one can obtain corresponding estimates for the curvature of the Hodge metrics on the various Hodge bundles [4]. Although these metrics are not "good" in the sense of Mumford, one still obtains:

(5.2) THEOREM. Let M be the complement of a normal-crossings hypersurface in a compact Kähler manifold $\bar{M}$, $(V, V_{\mathbf{R}}, F, S)$ a PVHS over M with unipotent monodromy. Then the Chern forms for the Hodge metric on any of the bundles $F^q \to M$ define currents on $\bar{M}$, which represent the corresponding Chern classes of the canonical extension $\bar{F}^q \to \bar{M}$.

We refer to [32], [26], for the case $\dim M = 1$ and to [4], [24], for the case $\dim M > 1$ and its applications.

The following result concerning the locii of Hodge cycles in a family, answers a question of Deligne, who also gave a proof (unpublished) in the case of one parameter. We will give a general argument, based on the results of §4, in a forthcoming paper.

(5.3) THEOREM. Let $(V, V_{\mathbf{Z}}, F, S)$ be an integral PVHS of weight $2p$ over a smooth algebraic variety M and let $K \in \mathbf{Z}$. Then the projection onto M of the set $\{v \in F^p \quad V_{\mathbf{Z}} \mid S(v,v) = K\}$, is algebraic.

6. L_2 and Intersection Cohomology. Purity.

Let $\bar{X}$ be a compact Kähler manifold, $\mathbf{V}_{\mathbf{Z}} \to X \quad \bar{X}$ a local system of $\mathbf{Z}$-modules on the complement of some normal-crossings divisor of $\bar{X}$ underlying a polarized VHS. There are natural L_2 cohomology groups $H^{\bullet}_{(2)}(\bar{X}, \mathbf{V})$ in this setting, defined as follows. We endow X with a Kähler metric g which is asymptotic - locally along $\bar{X} - X$ - to the curvature form of that divisor (such metrics exists [10], [31] and are necessarily complete). On $\mathbf{V}$, we consider the Hodge metric (1.4) associated to the polarized VHS. Define a complex of sheaves on $\bar{X}$,

$(\mathbf{A}^{\bullet}_{(2)}(\mathbf{V}),d)$, by letting, for any open $U \subset \bar{X}$

$\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})(U)$ = $\mathbf{V}$-valued forms ω on $U \cap X$ with coefficients that are locally L_2 and have locally L_2 derivatives and such that, for any compact $K \subset U$,

$$\int_{K\cap X} \|\omega\|^2 \, dV_g < \infty \qquad \int_{K\cap X} \|d\omega\|^2 \, dV_g < \infty\}$$

By definition, $H^{\bullet}_{(2)}(\bar{X},\mathbf{V}) = H(\bar{X},\mathbf{A}^{\bullet}_{(2)}(\mathbf{V}))$, the hypercohomology of $\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})$. We shall sketch a proof of the following

(6.1) THEOREM. The complex $\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})$ satisfies the axioms of the (middle) Intersection cohomology sheaf with values in $\mathbf{V}$. Thus, $H^{\bullet}_{(2)}(\bar{X},\mathbf{V}) \cong IH^{\bullet}(\bar{X},\mathbf{V})$.

(6.2) COROLLARY. A polarizable VHS of weight k on $\mathbf{V}_{\mathbb{Z}}$ determines a canonical polarizable Hodge structure of pure weight $k + p$ in $IH^p(\bar{X},\mathbf{V})$.

These statements were conjectured by Deligne. He gave a proof of (6.2) for the case $X = \bar{X}$ (cf. [31])- where (6.1) is classical - which can be adapted to the general setting once (6.1) is known to hold. For a curve $\bar{X}$, in which case $IH^{\bullet}(\bar{X},\mathbf{V}) \cong H^{\bullet}(\bar{X},i_*(\mathbf{V}))$, the proof is due to Zucker [31]. In [3], we considered the case of surfaces $\bar{X}$ with $\mathbf{V}_{\mathbb{Z}}$ underlying a VHS of weight one. The general proof is due to Schmid and us [5] and to Kashiwara and Kawai [22]. Kashiwara-Kawai also obtained an algebraic description of the resulting Hodge filtration in $IH^p(\bar{X},\mathbf{V})$. We should mention that for geometric VHS over a quasi-projective base, Saito proved (6.2) by formal reduction to the one-dimensional case, where Zucker's result applies. We refer to the articles of Kashiwara and Saito in this same volume for the details on these two points.

We make some preliminary observations about the L_2 cohomology, concerning in particular the implication (6.1) $\Longrightarrow$ (6.2). Although defined in terms of specific metrics, the sheaves $\mathbf{A}^p_{(2)}(\mathbf{V})$ are actually attached to the data $\bar{X}$, $\mathbf{V}$. Indeed, the square-integrability conditions depend only on the local quasiisometry class of the metrics chosen; in the case of g, that is determined by the specified asymptotic behavior along $\bar{X} - X$, while that of the Hodge metric is independent of the particular polarized VHS supported by $\mathbf{V}$, as implied by the norm estimates (5.1). The sheaves $\mathbf{A}^p_{(2)}(\mathbf{V})$ are fine, due to the local product structure of the metric g:

(6.3) There is a basis for $\bar{X}$ consisting of polycylindrical open sets $\mathcal{U} \approx \Delta^{r+m}$, with $\mathcal{U} \cap X \approx \Delta^{*r} \times \Delta^m$ and such that, for any compact $K \subset \mathcal{U}$, g is quasi-isometric on $K \cap X$ to the product of the Poincaré metric $\dfrac{i\, ds \wedge d\bar{s}}{|s|^2(\log|s|^2)^2}$ in the Δ^* factors and the Euclidean metric $i ds \wedge d\bar{s}$ in the Δ factors

Thus, $H^\bullet_{(2)}(\bar{X},\mathbf{V})$ can be computed from the complex $\Gamma^\bullet(\mathbf{A}^\bullet_{(2)}(\mathbf{V}))$ of global sections. Because $\bar{X}$ is compact, this can be identified with the complex of global $\mathbf{V}$-valued forms ω on X, with coefficients that are locally L_2 and have local L_2 derivatives, such that ω and $d\omega$ are globally L_2. Once we know its cohomology to be finite dimensional, we can replace the regularity condition on the coefficients by "C^∞", up to quasi-isomorphism; moreover, since the metric g is complete, this cohomology will be representable by harmonic forms. Thus

$$(6.4) \qquad H^\bullet_{(2)}(\bar{X},\mathbf{V}) = H^\bullet(\Gamma^\bullet(\mathbf{A}^\bullet_{(2)}(\mathbf{V}))) \cong \mathcal{H}^\bullet.$$

where $\mathcal{H}^p$ = space of square-integrable $\mathbf{V}$-valued harmonic p-forms. Now, a VHS in $\mathbf{V}$ together with the bigrading of the $\mathbf{C}$-valued forms determines a natural Hodge bigrading of the $\mathbf{V}$-valued forms, of weight $k + p$ in degree p. This induces a Hodge filtration in the L_2 complex, as can be deduced from the results of Sections 4-5. As mentioned before, Deligne's proof of the Kähler identities extend from the case $X = \bar{X}$ to our setting, because $\bar{X}$ is compact Kähler. The classical argument then puts - via (6.4) - pure Hodge structures of weight $k + p$ in $H^p_{(2)}(\bar{X},\mathbf{V})$, which are polarized by the natural form incorporating the metrics on the base and the polarization; the Hodge filtration per se is independent of these. In particular, (6.1) does imply (6.2) in our setting.

The $\mathbf{Z}$-structure in $\mathbf{V}$ insures that the local monodromy along $\bar{X} - X$ is quasi-unipotent, but it can be otherwise replaced throughout by an $\mathbf{R}$-structure. Moreover, for Theorem (6.1) to hold, the requirement that $\bar{X}$ be compact Kähler is unnecessary, as long as X carries a metric with suitable behavior along $\bar{X} - X$. Specifically, one proves

(6.5) THEOREM. Let $\bar{X}$ be a complex manifold, $X \subset \bar{X}$ the complement of a normal-crossings divisor carrying a metric g satisfying (6.3) and let $\mathbf{V}$ be a local system of $\mathbf{C}$-vector spaces on X underlying a real, polarized VHS. Assume that the local monodromy of $\mathbf{V}$ along $\bar{X} - X$ is quasi-unipotent. Then, the complex $\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})$ satisfies the axioms of the (middle) Intersection cohomology of $\bar{X}$ with coefficients in $\mathbf{V}$.

The hypotheses are preserved upon restriction to open subsets of $\bar{X}$. On X, $\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})$ is quasi-isomorphic to $\mathbf{V} - \mathbf{V}$ regarded as a complex concentrated in degree zero - because the Poincaré lemma applies in the setting of L_2 forms. Elementary properties of Intersection and L_2 cohomologies together with Poincaré duality, reduce then the problem to showing: for any small neighborhood U of a point in a stratum Σ of $\bar{X} - X$, U of the form (6.3), $H^p_{(2)}(U,\mathbf{V}) \cong H^p_{(2)}(U - \Sigma,\mathbf{V})$ for $p < \frac{1}{2}\operatorname{codim}\Sigma$ and is zero otherwise. We may inductively assume this to be the case for $\operatorname{codim}\Sigma < n = \dim X$, so that it will suffice to prove

$$(6.6)\qquad H^p_{(2)}(\Delta^n,\mathbf{V}) = \begin{cases} H^p_{(2)}(\Delta^n - (0),\mathbf{V}) & \text{if } p < n \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, we may assume $\Delta^n \cap X = \Delta^{*n}$ (i.e. $r = n$ in (6.3)) and that the monodromy of $\mathbf{V}$ on Δ^{*n} is unipotent. Let $N_1,\ldots,N_n$ be the monodromy logarithms acting on a typical fibre V, $W = W(N_1,\ldots,N_n)$ the associated weight filtration of V and $Y \in \operatorname{End}(V)$ a particular real splitting of W compatible with the N_j's - for example, the $Y_{(n)}$ constructed in §4. In particular, $Gr^W_\ell = V_\ell :\cong \ell$-eigenspace of Y. We define an action of Y on the $\mathbf{V}$-valued forms: if $v \in V$, then $\hat{v} = \exp(\Sigma \frac{\arg s_j}{2\pi i} N_j) \cdot v$ can be viewed as a single-valued section of $C^\infty \otimes \mathbf{V_R}$; we then let Y act on a form

$$\bigwedge_{i\in I}\left(\frac{d|s_i|}{|s_i|}\right) \wedge \left(\bigwedge_{j\in J} d\left(\frac{\arg s_j}{2\pi i}\right)\right) \otimes \hat{v} \quad \text{with } v \in V_\ell, \text{ as multiplication by } 2|J| + \ell.$$

When the action is restricted to the rotation-invariant forms, it commutes with the differential and is compatible with the L_2 conditions. The latter forms still calculate the same cohomology, so that one obtains actions of Y on the L_2-cohomologies of Δ^n and $\Delta^n - (0)$, and corresponding gradings

$$H^\bullet_{(2)}(\Delta^n,\mathbf{V}) = \bigoplus_{\ell\in\mathbf{Z}} H^\bullet_{(2)}(\Delta^n,\mathbf{V})_\ell, \qquad H^\bullet_{(2)}(\Delta^n-(0),\mathbf{V}) = \bigoplus_{\ell\in\mathbf{Z}} H^\bullet_{(2)}(\Delta^n-(0),\mathbf{V})_\ell .$$

(6.7) <u>LEMMA</u>.

$$H^\bullet_{(2)}(\Delta^n,\mathbf{V})_\ell \cong \begin{cases} H^\bullet_{(2)}(\Delta^n-(0),\mathbf{V})_\ell & \text{if } \ell < n, \\ H^{\bullet-1}_{(2)}(\Delta^n-(0),\mathbf{V})_n \otimes M & \text{if } \ell = n, \\ 0 & \text{if } \ell < n, \end{cases}$$

<u>for certain vector space</u> M.

Although the proof of (6.7) is technically involved, the idea is simple. The norm estimates are given in a useful form on partial regions around $\underline{0}$. A typical such region is the projection onto Δ^{*n} of the set $P \subset U^n$ (= product of upper-half planes) defined as follows. For $s = e^{2\pi iz} \in \Delta^*$ we write $z = x + iy$, $t = y_1$, $u_i = y_i/y_1$ for $2 \le i \le p+1$, $v_j = y_j/y_1$ for $p+1 < j \le n$ and define, for any $r > 1$,

$$P = \{\underline{z} \in U^n \mid 0 < t < \infty,\ r^{-1} < u_i < r,\ r^{-1} < v_j < \infty,\ u_i < r\min\{u_k, v_j\}\}.$$

In terms of these coordinates, the Hodge and Poincaré metrics satisfy:

$$\|\hat{v}\|_{(\underline{x},t,\underline{u},\underline{v})} \sim t^{\ell/2}\|v\|_{(\underline{x},1,\underline{u},\underline{v})} \quad \text{for } v \in V_\ell ,$$

$$g \sim t^{-2}(dt^2 + dx_1^2) + \sum_i (du_i^2 + t^{-2}dx_i^2) + \sum_j v_j^{-2}(dv_j^2 + t^{-2}dx_j^2)$$

This and Hardy's inequality is all one needs to calculate the L_2 cohomology over P. The contribution to cohomology from the t-directions is controlled by the weights of Y while that from the complementary u,v-directions can be interpreted as coming from the cohomology of $\Delta^n - (0)$. This "explains" the isomorphisms in (6.7), which come about by incorporating the calculation on the regions P into a Mayer-Vietoris spectral sequence adapted to them. M is an infinite-dimensional space of functions related to the failure of Hardy's inequality in a critical weight; it already appears in Zucker's calculation for $n = 1$. We refer to [5] for details.

It is easy to see that (6.6) follows from (6.7), together with

$$H^p_{(2)}(\Delta^n - (0),\mathbf{V})_\ell = 0 \quad \text{for} \quad p < n \leq \ell \quad \text{or} \quad \ell \leq n \leq p.$$

By our inductive hypothesis, this statement can be replaced by the analogous one for Intersection cohomology, once a compatible notion of weight is defined there. Because of the isomorphisms

$$IH^p(\Delta^n - (0),\mathbf{V}) = \begin{cases} IH^p(\Delta^n,\mathbf{V}) & \text{if} \quad p < n \\ IH^{2n-p-1}(\Delta^n,\mathbf{V})^* & \text{otherwise,} \end{cases}$$

the required statement follows from

$$\text{(6.8)} \qquad IH^p(\Delta^n,\mathbf{V})_\ell = 0 \quad \text{for} \quad \ell > p.$$

This "semipurity" statement - for the corresponding notion of weight - had been conjectured by Deligne in an unpublished letter, based on an analogous result of Gabber for the ℓ-adic situation. As he observed, it amounts to the following property of the monodromy: For $J = (j_1,\ldots,j_r)$, $1 \leq j_1 < \cdots < j_r \leq n$, write $N_J = N_{j_1} \ldots N_{j_r}$ and let $(B^\bullet,\delta)$ be the simple complex associated to the multiple complex with

$$B_J = N_J V \quad \text{and} \quad sg(j,J)N_j\colon B_J \to B_{J\cup\{j\}}\,, \quad j \notin J,$$

as components and differentials, respectively. Then $(B^\bullet,\delta)$ computes $IH\,(\Delta^n,\mathbf{V})$.

$W = W(N_1,\ldots,N_n)$ determines a filtration $WB^\bullet$ of $B^\bullet$, with $W_\ell B_J = N_J W_{\ell+|J|}$, relative to which δ is a (-2)-morphism. The statement (6.8) amounts to:

(6.9) THEOREM. $Gr_\ell^{WH^\bullet(B^\bullet)} = 0$ for $\ell > 0$.

In [5], we proved the following somewhat stronger version. Write $N = N_1 + \ldots + N_n$ and let $(NB)^\bullet$ denote the analogous complex defined on NV. Then, the map induced by N:

(6.10) $N:H^\bullet(B^\bullet) \to H^\bullet((NB)^\bullet)$ is the zero map.

This, in turn, was deduced from the existence of a MHS on $(B^\bullet,d)$. Specifically, let F be a filtration on $B^0 = V$ and S a bilinear form such that (W,F) is a MHS split over $\mathbb{R}$ and polarized by S and every element in the cone C spanned by $N_1,\ldots,N_n$; for example, that defined in §3, with $F = F_{(n)}$, by the given variation. Again, F and S determine corresponding objects on each B^p, by

$$F^r B_J = N_J F^{r+|J|}, \qquad S_J(N_J u, N_J v) = S(u, N_J v),$$

and one has (cf. (3.5) in [5])

(6.11) THEOREM. (WB^p, FB^p) is a MHS on B^p, polarized by S and every $T \in C$, and d is a $(-1,-1)$ morphism.

The statement (6.8) shows the strong restriction that a polarizable VHS imposes on the monodromy of the underlying local system. It also has the following implication, of rather different character ([4], (1.17)). Let X be a compact Kähler manifold, c_1 and c_2 be cohomology classes of X such that $\lambda_1 c_1 + \lambda_2 c_2$ is a Kähler class for all $\lambda_1,\lambda_2 > 0$ and let L_1, L_2, be the corresponding Kahler operators. Then,

(6.12) THEOREM. $H^{\ell-\ell_1-\ell_2}(X,\mathbf{C}) \cap \ker(L_1^{\ell_1} L_2^{\ell_2}) \subset \ker L_1^{\ell_1} + \ker L_2^{\ell_2}$ for $\ell \leq \dim X$ $\ell_1,\ell_2 > 0$.

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Eduardo Cattani

Aroldo Kaplan

Department of Mathematics
University of Massachusetts
Amherst, MA 01003
U.S.A.