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A SPECIAL DECOMPOSITION OF THE NILPOTENT CONE OF A CLASSICAL LIE ALGEBRA

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1. Introduction

1.1. The *special unipotent classes* of a simple group were introduced by Lusztig in [Lu1] (last remark of the paper), in relation with *special representations* of Weyl groups. At the same time Spaltenstein remarked in [Sp1] that there is a natural involution on a certain subset of the unipotent classes of a simple group (cf. [Sp2, chap. II.1]). In case of a classical group it is induced by the standard involution of the unipotent classes in GL_n (defined via the duality of the corresponding partitions) and has a purely combinatorial description [loc.cit]. It is not hard to see that this subset is exactly the set of special classes of Lusztig. Spaltenstein also notes that the sets

$$\widehat{C} := \overline{C} \setminus \bigcup_{\substack{C' \text{ special} \\ \overline{C'} \subset \overline{C}}} \overline{C'}$$

where C runs through the special classes, form a partition of the unipotent variety, i.e. any unipotent class is contained in a $\widehat{C}$ for a unique special class C. For completness we will prove this in section 4 (Proposition 4.2).

1.2. Lusztig conjectures in [Lu2, Conjecture 3] that the varieties $\widehat{C}$ for special classes C are *rational homology manifolds*. This has been verified for "minimal" special classes in [loc.cit], and for E_6, E_7, E_8 in a remark in [BS]. The case F_4 can be handled using [Sh], and for G_2 it is quite easy.

The purpose of this paper is to prove it for the classical groups.

Theorem. *Let C be a special unipotent conjugacy class of a classical group. Define $\widehat{C}$ as above. Then $\widehat{C}$ consists of 2^d conjugacy classes, where d is the number of irreducible components of $\widehat{C} \setminus C$. There is a smooth variety Y with an action of the group $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, d copies, and an isomorphism*

$$Y/\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 \xrightarrow{\sim} \widehat{C}$$

which identifies the stratification of C by conjugacy classes with the stratification of the quotient by isotropy groups. (These are the 2^d subproducts of ${\mathbf{Z}_2}^d$.) In particular $\widehat{C}$ is a rational homology manifold.

*Partially supported by Schweizerischer Nationalfonds

1.3. The partition of the unipotent variety into subsets $\widehat{C}$ can also be justified by the following result of Lusztig.

Proposition. *Let C_1, C_2 be two unipotent classes and let ρ_1, ρ_2 be the Springer representation of the Weyl group W corresponding to $(C_1, \mathbf{1})$, $(C_2, \mathbf{1})$, where $\mathbf{1}$ is the constant sheaf. Then C_1 and C_2 belong to the same $\widehat{C}$ if and only if ρ_1 and ρ_2 belong to the same two sided cell of W.*

1.4. The paper depends in an essential way from the results and methods of [KP2] of which it should be considered as a continuation. We try to help the reader to extract the necessary information from there by recalling as we proceed the main definitions.

Acknowledgement. We like to thank George Lusztig for pointing out to us his conjecture and for his helpful remarks concerning this introduction.

2. Conjugacy Classes and Partitions

2.1. Let ε be $+1$ or -1. A finite dimensional vector space V with a non-degenerate form $(\ ,\ )$ such that $(u, v) = \varepsilon(v, u)$ for all $u, v \in V$ will be called a *quadratic space of type* ε (shortly an *orthogonal space* in case $\varepsilon = 1$, a *symplectic space* in case $\varepsilon = -1$). We denote by $G(V)$ the subgroup of $\mathrm{GL}(V)$ leaving the form invariant. So we have $G(V) = \mathrm{O}_n$ or $G(V) = \mathrm{Sp}_n$, $n = \dim V$, according to $\varepsilon = 1$ or $\varepsilon = -1$. Similarly $\mathbf{g}(V)$ denotes the Lie algebra of $G(V)$.

2.2. For any quadratic space V of type ε the conjugacy class C_D of a nilpotent element $D \in \mathbf{g}(V)$ is completely determined by its conjugacy class in $\mathrm{End}(V)$, hence by its associated partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0, \quad |\lambda| := \sum_{i=1}^{s} \lambda_i = \dim V,$$

given by the sizes of the blocks of the Jordan normal form of D in $\mathrm{End}(V)$ (cf.[KP2, 2.1]. If we denote by $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_s)$ the dual partition (i.e. $\widehat{\lambda}_i := \#\{j \mid \lambda_j \geq i\}$) we have for all j

$$\dim \ker D^j = \sum_{i=1}^{j} \widehat{\lambda}_i.$$

2.3. The partition λ associated to a nilpotent $D \in \mathbf{g}(V)$ satisfies the following condition (cf. [KP2, 2.2]):

(Y_ε) *If V is orthogonal (V is symplectic) every row of even length (of odd length) occurs an even number of times.*

Such partitions will be called *ε-partitions*. We denote by $C_{\varepsilon,\lambda}$ the associated nilpotent conjugacy class in $\mathbf{g}(V)$, where V is a quadratic space of type ε of dimension $|\lambda|$.

2.4. It is convenient to represent the partitions geometrically as Young-diagrams with rows consisting of $\lambda_1, \lambda_2, \ldots, \lambda_s$ boxes respectively. Then the dual partition $\widehat{\lambda}$ is defined by setting $\widehat{\lambda}_j$ equal to the length of the jth column of the diagram of λ.

Also the inclusion behavior $C_{\varepsilon,\sigma} \subset \overline{C}_{\varepsilon,\lambda}$ has a purely combinatorial description: A necessary and sufficient condition is that the diagram σ is obtained from λ by moving down a number of boxes (cf. [KP2, 2.5]). In this case we write $\sigma \leq \lambda$ and call σ an *ε-degeneration* of λ.

3. Special Conjugacy Classes

3.1. Let $C = C_{\varepsilon,\lambda}$ be a nilpotent conjugacy class. It is not difficult to determine in terms of the partition λ whether the class C is *special* in the sense of Lusztig [Lu1] (cf. Introduction). We use the following definition which is easily seen to be equivalent to the one given by Spaltenstein [Sp2] (see also [Ke, §6]).

Definition. Let λ be an ε-partition. Define the sequence $(s_1, s_2, s_3, \ldots)$ by

$$s_i = \begin{cases} \lambda_1 + \lambda_2 + \ldots + \lambda_i, & \text{if } |\lambda| \text{ is even;} \\ 1 + \lambda_1 + \lambda_2 + \ldots + \lambda_{i-1}, & \text{if } |\lambda| \text{ is odd.} \end{cases}$$

Then λ is called *special* if the numbers $s_2, s_4, s_6, \ldots$ are all even.

3.2. Remark. The following two assertions are easily deduced from the definitions.

(a) Let σ be an orthogonal partition with $|\sigma|$ even and let σ' be the symplectic partition obtained from σ by removing the first column. Then $|\sigma'|$ is even, and σ' is special if and only if σ is special.

(b) Let σ be a orthogonal partition with $|\sigma|$ odd and assume that the first row σ_1 is odd. Denote by σ° the orthogonal partition obtained from σ by removing the first row. Then $|\sigma^\circ|$ is even and σ° is special if and only if σ is special.

3.3. Lemma. *Let σ be a symplectic partition. Then σ is special if and only if $\widehat{\sigma}$ is symplectic and special.*

PROOF: For a symplectic partition σ it is clear that $\widehat{\sigma}$ is special, i.e. that all sums $\widehat{\sigma}_1 + \widehat{\sigma}_2 + \ldots + \widehat{\sigma}_{2i}$ are even, since since every odd row has to occur an even number of times. To prove that for a special partition σ the dual $\widehat{\sigma}$ is symplectic, it suffices to show that either $\widehat{\sigma}_1 = \widehat{\sigma}_2$ or that $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ both are even. In fact this implies that the partition σ'' obtained from σ by removing the first two columns, is again special (and symplectic), and by induction we may assume that its dual $\widehat{\sigma''}$ is symplectic. Since $\widehat{\sigma}$ is obtained from $\widehat{\sigma''}$ by adding two rows of either the same length or both of even length the partition $\widehat{\sigma}$ is symplectic too. It remains to show that for $\widehat{\sigma}_1 \neq \widehat{\sigma}_2$ both $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are even. Now $l := \widehat{\sigma}_1 > \widehat{\sigma}_2$ and $s_l = |\sigma|$ is even. Hence $s_{l-1} = |\sigma| - 1$ is odd and therefore l has to be even. Since σ is symplectic $\widehat{\sigma}_1 - \widehat{\sigma}_2 = \#$rows of length one is even and the claim follows. ∎

3.4. Lemma. *Let σ be an orthogonal partition with $|\sigma|$ odd. Then σ is special if and only if $\widehat{\sigma}$ is orthogonal and special.*

PROOF: Clearly $\widehat{\sigma}_1$ is odd. Now add a new row of odd length $> \sigma_1$ and remove the first column, which is now even, to obtain a symplectic partition ν. It follows from (a) and (b) that ν is special in case σ is special, and then $\widehat{\nu}$ is special and symplectic by (c). Now $\widehat{\sigma}$ is obtained from $\widehat{\nu}$ by adding first an even column and then an odd row, hence $\widehat{\sigma}$ is special and orthogonal. ∎

4. Small Degenerations

4.1. A degeneration $\sigma \leq \lambda$ is called *minimal* if $\sigma \leq \mu \leq \lambda$ implies $\mu = \sigma$ or $\mu = \lambda$. In [KP2] we have established an equivalence relation between degenerations. Under this relation the minimal degenerations fall into 8 classes a, b, c,..., h which are presented in [KP2, 3.4 table I]. In the following we will use degenerations of type a and g which are equivalent to the pairs $(1^{2n}) \leq (2, 1^{2n-2})$ of symplectic partitions.

In addition we have introduced in [KP2, 13.6] the concept of a degeneration $\sigma \leq \lambda$ being *decomposed into independent degeneration.* This means that we can decompose λ and σ into blocks consisting of consecutive rows

$$\lambda = \begin{matrix} \lambda^{(1)} \\ \lambda^{(2)} \\ \vdots \\ \lambda^{(s)} \end{matrix} \qquad \sigma = \begin{matrix} \sigma^{(1)} \\ \sigma^{(2)} \\ \vdots \\ \sigma^{(s)} \end{matrix}$$

such that all $\lambda^{(i)}$, $\sigma^{(i)}$ are ε-partitions, that $\lambda^{(i)}$ has the same number of rows as $\sigma^{(i)}$ except perhaps the last $\lambda^{(s)}$, and that $\sigma^{(i)}$ is a degeneration of $\lambda^{(i)}$.

Definition. An ε-degeneration $\sigma \leq \lambda$ is called *small* if σ is obtained from λ by independent minimal degenerations of type a or g.

Similarly we say that a degeneration $C' \subset \overline{C}$ of nilpotent conjugacy classes in $\mathbf{g}(V)$ is *small* if the degeneration of the corresponding ε-partitions is small.

4.2. Proposition (Spaltenstein). *Given an ε-partition σ there is a uniquely determined minimal special ε-partition λ such that $\sigma \leq \lambda$. In addition the ε-degeneration $\sigma \leq \lambda$ is small.*

PROOF: (1) We first consider the case where σ is symplectic. Assume that s_{2i} is even for $i = 1, \ldots, r-1$ and that s_{2r} is odd. (See 3.1 for the definition of s_i.) Then σ_{2r-1} is even and σ_{2r} is odd, and there is an $r' > r$ such that $\sigma_{2r} = \sigma_{2r+1} = \ldots = \sigma_{2r'-1} > \sigma_{2r'}$. Now define

$$\sigma'_i = \begin{cases} \sigma_{2r} + 1, & \text{for } i = 2r; \\ \sigma_{2r'-1} - 1, & \text{for } i = 2r' - 1; \\ \sigma_i, & \text{in all other cases.} \end{cases}$$

It is easy to see that σ' is again symplectic and that $\sigma' \leq \sigma$ is a minimal degeneration of type a (if $r' = r+1$) or g (if $r' > r+1$) (see [KP2, 3.4 table I]). In particular it is a small degeneration. Furthermore the sequence $(s'_1, s'_2, s'_3, \ldots)$ for σ' (i.e. $s'_i := \sigma'_1 + \sigma'_2 + \ldots + \sigma'_i$) has the property that s'_{2i} is even for $i = 1, 2, \ldots, r'-1$. To finish the proof in the symplectic case it suffices by induction to show that for any special symplectic partition $\lambda \geq \sigma$ we also have $\lambda \geq \sigma'$. Let $t_i := \lambda_1 + \lambda_2 + \ldots + \lambda_i$. By assumption $\lambda_i \geq \sigma_i$ for all i and λ_{2j} is even for all j. By definition we have

$$s'_i = \begin{cases} s_i, & \text{for } i \leq 2r-1 \text{ and } i > 2r'-1; \\ s_i + 1, & \text{for } 2r \leq i \leq 2r'-1. \end{cases}$$

Hence we have to show that

$$(*) \qquad t_j > s_j \text{ for } 2r \le j < 2r' - 1.$$

By assumption $s_{2r}, s_{2r+2}, \ldots, s_{2r'-2}$ are all odd and so $(*)$ is satisfied for the even j's. Now assume that $t_{2k+1} = s_{2k+1}$ for some k with $r \le k < r'-1$. Then $\lambda_{2k+1} = t_{2k+1} - t_{2k} < s_{2k+1} - s_{2k}$, hence $\lambda_{2k+2} \le \lambda_{2k+1} < \sigma_{2k+1} = \sigma_{2k+2}$, and so $t_{2k+2} = t_{2k+1} + \lambda_{2k+2} < s_{2k+1} + \sigma_{2k+2} = s_{2k+2}$ which is a contradiction.

(2) Now let σ be orthogonal and $|\sigma|$ even and denote by σ° the symplectic partition obtained from σ by adding a column of length $\hat{\sigma}_1$. We have seen in (1) that there is a unique minimal special symplectic partition $\lambda^\circ \ge \sigma^\circ$ and that the degeneration $\sigma^\circ \le \lambda^\circ$ is small. Since the first two columns of σ° have the same length it follows that $\widehat{\lambda^\circ}_1 = \widehat{\sigma^\circ}_1 = \hat{\sigma}_1$. Removing the first column we obtain a special orthogonal partition λ (Remark (a)) such that σ is a small degeneration of λ, and clearly λ is also minimal under these conditions.

(3) Finally assume that σ is orthogonal with $|\sigma|$ odd. This time we add a first row of odd length $\ge \sigma_1 + 2$ to obtain an orthogonal partition σ° with $|\sigma^\circ|$ even, and proceed like in (2), using Remark (b). ∎

4.3. Proposition. *The map $\sigma \mapsto \hat{\sigma}$ induces a duality on the set $\mathbf{P}_{2n}^{symp}$ of special symplectic partitions of $2n$ and on the set $\mathbf{P}_{2n+1}^{orth}$ of special orthogonal partitions of $2n+1$. In addition there is a bijection*

$$\mathbf{P}_{2n+1}^{orth} \xrightarrow{\sim} \mathbf{P}_{2n}^{symp}$$

respecting the partial ordering and the duality.

PROOF: The first part is a reformulation of Lemma 3.3 and 3.4. Furthermore we define a map $\mathbf{P}_{2n+1}^{orth} \longrightarrow \mathbf{P}_{2n}^{symp}$ in the following way: Remove the last box from the special orthogonal partition σ to get a partition σ' of $2n$. Now there is a uniquely determined maximal symplectic partition $\lambda \le \sigma'$ and one shows that this λ is special. We leave the details to the reader. ∎

5. Geometry of Small Degenerations

5.1. Let us start with a quadratic space V of type ε and a nilpotent element $D \in \mathfrak{g}(V)$ with conjugacy class $C_D = C_{\varepsilon,\lambda}$. The image $U := D(V)$ is in a natural way a quadratic space of type $-\varepsilon$ and $D' := D|_U$ is a nilpotent element of $\mathfrak{g}(U)$ with conjugacy class $C_{D'} = C_{-\varepsilon,\lambda'}$, where λ' is obtained from λ by removing the first column [KP2, 4.1]. Furthermore we get the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(V,U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \downarrow\rho & & \\ \mathfrak{g}(V) & & \end{array}$$

The two maps are defined by $\pi(X) := XX^*$ and $\rho(X) := X^*X$, where $X^* : U \to V$ is the dual map of X. (We identify U^* with U and V^* with V using the non-degenerate form $(\ ,\ )$.) π and ρ are the quotient maps under $G(V)$ and $G(U)$ respectively [KP2, 1.2].

We embed $\mathrm{Hom}(V,U)$ in $\mathrm{Hom}(V,U) \times \mathrm{Hom}(U,V)$ by $X \mapsto (X, X^*)$. The $G(V) \times G(U)$-orbits in $\mathrm{Hom}(V,U)$ are completely determined by the corresponding $\mathrm{GL}(V) \times \mathrm{GL}(U)$-orbits in

$\mathrm{Hom}(V,U)\times\mathrm{Hom}(U,V)$ which are classified by their ab-diagrams (cf. [KP2, §6] and [KP1, §4]). Given the ab-diagram of X, e.g.

$$\begin{array}{l} bababab \\ babab \\ babab \\ bab \\ ba \\ ab \end{array}$$

then the Young-diagrams of $\pi(X)$ and $\rho(X)$ are obtained from it by removing all a's (resp. all b's), e.g.

$$\begin{array}{lcl} bbbb & & aaa \\ bbb & & aa \\ bbb & \text{and} & aa \\ bb & & a \\ b & & a \\ b & & a \end{array}$$

representing the partitions $(4,3,3,2,1,1)$ and $(3,2,2,1,1,1)$. This fact allows to discuss in detail the decomposition into orbits of $\pi^{-1}(C)$ for a given conjugacy class C; it will be used in the proof of Lemma 5.3 below.

Now put $N_{\varepsilon,\lambda} := \pi^{-1}(\overline{C}_{-\varepsilon,\lambda'})$.

5.2. Lemma. *Let $C_{\varepsilon,\sigma}\subset\overline{C}_{\varepsilon,\lambda}$ with $\widehat{\sigma}_1=\widehat{\lambda}_1$. Then*

(a) $O_\sigma := \rho^{-1}(C_{\varepsilon,\sigma})$ is an orbit under $G(U)\times G(V)$ contained in $N_{\varepsilon,\lambda}$.

(b) $\pi(O_\sigma) = C_{-\varepsilon,\sigma'}$.

(c) $G(U)$ acts freely on O_σ.

(d) π is smooth on O_σ.

PROOF: (a) and (b) follow from [KP2, Lemma 4.3]. Since $\widehat{\sigma}_1=\widehat{\lambda}_1$ the elements of $C_{\varepsilon,\sigma}$ have the same rank as those of $C_{\varepsilon,\lambda}$, namely $\dim U$, hence $\rho^{-1}(C_{\varepsilon,\sigma})\subset\{X\in\mathrm{Hom}(V,U)\mid X \text{ surjective}\}$. Now (c) and (d) follow from [KP2, Proposition 11.1]. ∎

5.3. Lemma. *Assume V symplectic, hence U orthogonal, and let $C_{-1,\sigma}\subset\overline{C_{-1,\lambda}}$ with $\widehat{\lambda}_1=\widehat{\sigma}_1-1$ and $\widehat{\lambda}_2=\widehat{\sigma}_2+1$. Then*

(a) $O_\sigma := \rho^{-1}(C_{-1,\sigma})$ is an orbit under $G(U)\times G(V)$ contained in $N_{-1,\lambda}$.

(b) $\pi(O_\sigma) = C_{1,\sigma''}$, where σ'' is obtained from σ by removing the first row and then adding one box.

(c) The connected component $G(U)^0 = SO(U)$ acts freely on O_σ, and the stabilizer of $X\in O_\sigma$ is $\mathbf{Z}_2$.

(d) π is smooth on O_σ.

PROOF: Since $\dim U = \dim V - \widehat{\lambda}_1$ there is a unique ab-diagram with $\dim U$ a's lying on top of λ, i.e. which by removing the a's gives λ. (It is obtained by inserting between all consecutive b's of λ an a.) Hence $O_\lambda := \rho^{-1}(C_{-1,\lambda})$ is the $G(U)\times G(V)$-orbit with this ab-diagram. By assumption the last parts of λ and σ have the form

$$(\ldots,\underbrace{2,1,\ldots,1}_{2s-1}) \quad\text{and}\quad (\ldots,\underbrace{1,1,\ldots,1,1}_{2s}),$$

where $2s = \widehat{\sigma}_1 - \widehat{\sigma}_2$. In particular $\operatorname{rk} D = \dim U - 1$ for $D \in C_{-1,\sigma}$. As a consequence the ab-diagram of O_λ has the following last part:

$$\text{last } 2s-1 \text{ rows} \quad \left\{ \begin{array}{l} bab \\ b \\ b \\ \vdots \\ b \end{array} \right.$$

But then there is only one possibility for an ab-diagram on top of $C_{-1,\sigma}$: The first $\widehat{\sigma}_2$ rows are completely determined by σ and λ and are the same as for O_λ, and the last part must have the following form (see [KP2, 6.3 table III]):

$$\text{last } 2s+1 \text{ rows} \quad \left\{ \begin{array}{l} a \\ b \\ b \\ b \\ \vdots \\ b \end{array} \right.$$

Hence $\rho^{-1}(C_{\varepsilon,\sigma})$ is an orbit and $\pi(\rho^{-1}(C_{\varepsilon,\sigma})) = C_{-\varepsilon,\sigma''}$, where σ'' is obtained from σ by removing the first row and then adding one box. This gives (a) and (b).

Now (c) follows from [KP2, Proposition 11.5] since

$$\rho^{-1}(C_{\varepsilon,\sigma}) \subset L'' := \{X \in \operatorname{Hom}(V,U) \mid \operatorname{rk} \rho(X) \geq \dim U - 1\} \setminus L'$$

where

$$L' := \{X \in \operatorname{Hom}(V,U) \mid X \text{ surjective}\},$$

and (d) follows from [KP2, Proposition 11.4], since

$$L^0 := \{X \mid \operatorname{codim} X(V) \leq 1\} \supset L''$$

because $\operatorname{rk} X^*X \leq \operatorname{rk} X$. ∎

5.4. We now recall the fundamental construction of [KP2] associated to a nilpotent element $D \in \mathfrak{g}(V)$ with conjugacy class $C_{\varepsilon,\lambda}$ [KP2, §5]. It is obtained by iterating the construction above. The spaces $V_i := D^i(V)$ are quadratic of type $(-1)^i\varepsilon$ and $D_i := D|_V \in \mathfrak{g}(V_i)$ is nilpotent with Young-diagram λ_i obtained from λ by removing the first i rows. By forming an iterated fiber product we get the following diagram:

$$\begin{array}{ccccccc}
Z & \xrightarrow{\phi} & Z_1 & \longrightarrow & \dots & \longrightarrow & \overline{C}_{D_t} = 0 \\
\downarrow & & \downarrow & & & & \\
\bullet & \longrightarrow & \bullet & & \dots & & \\
\downarrow & & \vdots & & & & \\
\vdots & & \vdots & & & & \\
\bullet & \longrightarrow & \bullet & & & & \\
\downarrow & & \downarrow & & & & \\
N_D & \xrightarrow{\pi} & \overline{C}_{D_1} & & & & \\
\downarrow\rho & & & & & & \\
\overline{C}_D & & & & & &
\end{array}$$

By construction all the squares in this diagram are fibre products. As a subdiagram we obtain

$$(**)\qquad \begin{array}{ccc} Z & \xrightarrow{\phi} & Z_1 \\ \downarrow\theta' & & \downarrow\theta_1 \\ N_D & \xrightarrow{\pi} & \overline{C}_{D_1} \\ \downarrow\rho & & \\ \overline{C}_D & & \end{array}$$

where the composition $\theta := \rho \circ \theta' : Z \to \overline{C}_D$ is the quotient map under the group

$$G := G(V_1) \times G(V_2) \times \ldots \times G(V_t).$$

5.5. Proposition. *Let $C' \subset \overline{C}_D$ be a small degeneration. Then Z is smooth in $Z' := \theta^{-1}(C')$ and the connected component G^0 of the group G acts freely on Z'. Furthermore the stabilizer in G of any point of Z' is isomorphic to $\mathbf{Z}_2^d$, where d is the number of minimal degenerations needed to obtain the ε-diagram of C' from that of C_D.*

PROOF: Let $C' = C_{\varepsilon,\sigma}$ and $C_D = C_{\varepsilon,\lambda}$. Since $\sigma \leq \lambda$ is a small degeneration we have either $\widehat{\sigma}_1 = \widehat{\lambda}_1$ or V is symplectic (and V_1 orthogonal) and $\widehat{\lambda}_1 = \widehat{\sigma}_1 - 1 \geq \widehat{\lambda}_2 = \widehat{\sigma}_2 + 1$ (see Definition). It follows from Lemma 5.2 and 5.3 that π is smooth on $N' := \rho^{-1}(C')$ and that $G(V_1)^0$ acts freely on N'. The fibre product diagram $(**)$ implies that ϕ is smooth on Z' and that $G(V_1)^0$ acts freely on Z'. Furthermore $\pi(\rho^{-1}(C'))$ is a conjugacy class $C_1' \subset \overline{C}_{D_1}$ with ε-diagram σ' obtained from σ by cancelling the first row (and adding one box in case $\widehat{\lambda}_1 = \widehat{\sigma}_1 - 1$). It is clear from this that $C_1' \subset \overline{C}_{D_1}$ is again a small degeneration. From the following diagram

$$\begin{array}{ccccccc} Z' & \subset & Z & \xrightarrow{\phi} & Z_1 & \supset & Z_1' \\ \downarrow & & \downarrow\theta' & & \downarrow\theta_1 & & \downarrow \\ M & \subset & N_D & \xrightarrow{\pi} & \overline{C}_{D_1} & \supset & C_1' \\ \downarrow & & \downarrow\rho & & & & \\ C' & \subset & \overline{C}_D & & & & \end{array}$$

we see that the outer square

$$\begin{array}{ccc} Z' & \xrightarrow{\phi} & Z_1' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi} & C_1' \end{array}$$

is cartesian too, hence the claim follows by induction. ∎

6. A Special Decomposition

6.1. Definition. For a special ε-partition λ we define

$$S_\lambda := \bigcup_{\sigma \leq \lambda} C_{\varepsilon,\sigma},$$

where the union is taken over all σ such that λ is the minimal special ε-partition $\geq \sigma$.

It follows from Proposition 2.1 that S_λ is a locally closed subvariety of $\mathbf{g}(V)$, and that

$$S_\lambda = \bigcup_{\substack{\lambda \text{ special} \\ \lambda' < \lambda}} C_{\varepsilon,\lambda'}.$$

6.2. Theorem. *Let λ be a special ε-degeneration and let d be the number of different minimal degenerations of λ of type a or g. Then S_λ consists of 2^d conjugacy classes, and there is a smooth variety Y_λ with an action of the group $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, d copies, and an isomorphism*

$$S_\lambda \simeq Y_\lambda / \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$$

which identifies the stratification of S_λ by conjugacy classes with the stratification by isotropy groups.

PROOF: It follows from Proposition 4.2 that every conjugacy class $C' \subset S_\lambda$ is a small degeneration of C_λ. Hence $\theta^{-1}(S_\lambda)$ is a smooth open subvariety of Z with a free action of G° by Proposition 5.5. As a consequence $Y_\lambda := Z_\lambda / G^\circ$ is smooth too and $S_\lambda \simeq Z_\lambda / F$ with $F = G/G^\circ \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, d copies. The rest is clear. ■

6.3. Corollary. *The variety S_λ is a rational homology manifold.*

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