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ON THE HOMOLOGY CLASSES FOR THE COMPONENTS OF SOME FIBRES OF SPRINGER'S RESOLUTION

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ABSTRACT: We compute the homology classes of the components of the fibres of Springer's resolution in terms of Schubert classes when the unipotent element is of "one hook" type.

0. Introduction

Let $G$ be a connected reductive group over $\mathbb{C}$. Denote by $\mathcal{B}$ the variety of all Borel subgroups of $G$. If $u$ is a unipotent element of $G$, the fibre of Springer's resolution $\mathcal{B}_u$ is the variety of Borel subgroups containing $u$. The inclusion $\mathcal{B}_u \hookrightarrow \mathcal{B}$ induces a homomorphism of homology groups $H_*(\mathcal{B}_u;\mathbb{Z}) \rightarrow H_*(\mathcal{B};\mathbb{Z})$, which is injective if $G = GL_n(\mathbb{C})$ [8]. When $w$ runs over the elements in the Weyl group $W$ of $G$, the Schubert classes $[X_w]$ form a basis of $H_*(\mathcal{B};\mathbb{Z})$ [2]. If $C$ is a component of $\mathcal{B}_u$, it defines a homology class in $H_*(\mathcal{B}_u;\mathbb{Z})$, whose image in $H_*(\mathcal{B};\mathbb{Z})$ is denoted by $[C]$. We can then write

$$[C] = \sum_{w \in W} n_C(w)[X_w] \quad \text{with} \quad n_C(w) \in \mathbb{Z}$$

In this paper we shall consider the case with $G = GL_n(\mathbb{C})$ and with $u$ a unipotent element whose Jordan decomposition is of "one hook" type, i.e. such that there is at most one Jordan block of size greater than one. The result in that case is that $n_C(w)$ is the cardinal of a set of reduced expressions of $w$, depending on $C$. We believe that a similar result could be true in general, at least for $GL_n(\mathbb{C})$. For example, we have obtained such a result in the case that the Jordan decomposition of $u$ has only two blocks.

We want to express our deep gratitude to Professor Springer. He proposed the problem [13], and inspired all our work. He also read the paper and implemented it considerably. The clarity the reader can find comes from him.
1. Combinatorial results about tableaux and permutations

1.1. A good reference for some terminology about tableaux is Macdonald's book [10], for links between tableaux and reduced decompositions the reader is referred to [5], [9], [14].

Consider "strict standard staircase tableaux" with entries in the set \(\{1, \ldots, n-1\}\), i.e. tableaux \(T\) for a Dartition \((m, m-1, \ldots, 1)\) such that the integers \(a_{p,q}\) in the place \((p,q)\) satisfy \(a_{p,q} < a_{p+1,q} < a_{p+1,q+1}\) (the columns are strictly increasing and the diagonals are increasing, it follows that the rows are strictly increasing).

In the symmetric group \(S_n\), let \(s_{i,j}\) denote the transposition \((i,j)\), let \(s_i\) be the fundamental transposition \(s_{i,i+1}\), \(1 < i < n-1\). Let \(l(w)\) denote the length of an element \(w \in S_n\) and \(w < w'\) the Bruhat order relative to the set of generators \((s_i)\) \(1 \leq i \leq n-1\) [3].

Denote by \(|X|\) the cardinality of a set \(X\).

Associate to such a tableau \(T\) a permutation \(w = w_T \in S_n\), namely \(w = c_1 \cdots c_m\) where \(c_p = s_{a_{m-p+1},p} \cdots s_{a_1,p}\).

We say \(T\) is reduced if \(l(w_T) = \frac{1}{2}m(m+1)\).

1.2. We list a number of properties.

1.2.1. Write \(r = s_{a_{m-p+1},p} \cdots s_{a_1,p}\). Then \(w = r_m \cdots r_1\).

1.2.2. If \(T\) is reduced then \(a_{p+1,q} = a_{p,q+1}\) implies \(a_{p,q+1} = a_{p,q}\).

1.2.3. Let \(i\) defined by \(w_i = 1\). Then \(a_{p,q} = p+q-1\) for \(p+q < i\) and \(a_{i,i} > i\).

Define the number \(\tau = \tau_p = \tau(T,p)\) as follows: in the \(p\)th column of \(T\) we have \(a_{1,p} = a_{2,p-1} = \cdots = a_{\tau,p} = a_{\tau+1,p-1} = \tau\).

1.2.4. For \(p < q < i-1\) we have \(\tau_p > \tau_q\). If \(p < i\) then \(w_p = \tau_p + 1\). Hence if \(p < q < i\) then \(w_p > w_q\).

1.2.5. If \(T\) is reduced and \(a_{i,i} = i+1\) then \(\tau_i < \tau_{i-1}\). Moreover \(\tau_{i-1} = \tau_i + 1\) if and only if \(\tau_i + 2 = w(i-1) < w(i+1)\) and \(\tau_{i-1} > \tau_i + 1\) if and only if \(w(i-1) > w(i+1) = \tau_i + 2\).
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1.2.1.* If \( p > u, q > v \) then \( a_{p,q} \cdot a_{u,v} \geq 2 \). It follows that \( a_{p,q} \) \( a_{u,v} \) commute. The proof follows from this observation.

1.2.2.* Let \( T \) be not necessarily reduced. We show by induction on \( m-i \) that \( l(r_m \ldots r_i) > l(r_m \ldots r_i s_{a_{i,j}}) \) \( (1 \leq i \leq m, 1 \leq j \leq m-i+1) \). This is clear if \( j = m-i+1 \) or \( a_{i,j+1} > a_{i,j} \), otherwise \( s_{a_{i,j}} s_{a_{i,j+1}} s_{a_{i,j}} = s_{a_{i,j}} s_{a_{i,j+1}} s_{a_{i,j}} = a_{i+1,j} \) (diagonals increase) so that \( r_i a_{i,j} \) \( s_{a_{i+1,j}} = s_{r_i} \). By the induction hypothesis we have \( l(r_m \ldots r_{i+1} a_{i,j}) < l(r_m \ldots r_{i+1}), \) whence the asserted inequality.

It follows that if \( a_{p+1,q} > a_{p,q+1} \) and \( a_{p,q+1} > a_{p,q+1} \), we have

\[ l(r_m \ldots r_{p+1} a_{p+1,q}) < l(r_m \ldots r_{p+1}), \]

showing that \( T \) is not reduced.

1.2.3.* We have \( w = r_m \ldots r_1 \) and \( r_i \) fixes 1 if \( p \geq 2 \), so \( i = w^{-1}(1) = r_1^{-1}(1) = a_{1,m} a_{1,m} \ldots a_{1,1} \). Since \( a_{1,p} \) \( p \) it follows that \( a_{1,p} = p \) for \( p \leq i-1, \)

\[ a_{1,p} \geq p \text{ for } p > i. \]

1.2.4.* That \( \tau > \tau_q \) if \( p < q < i \) follows from the definitions (\( T \) is a strict tableau). Now \( c_j \) fixes \( p \) if \( j > p \), \( c_p = p + \tau_p \) for \( p < i \) and \( c_i = t-1 \) if \( j < t < j + \tau_j \); thus because \( p < i \) we have \( w = c_1 \ldots c_{p-1}(p) = c_1 \ldots c_{p-1}(p+\tau_p) = \tau_p + 1. \)

1.2.5.* That \( \tau_i > \tau_{i-1} \) follows from (1.2.2). From (1.2.4) we have \( w(i-1) = \tau_{i-1} + 1. \)

Also \( w(i+1) = r_m \ldots r_{i+1} = r_m \ldots r_{i+1}(i+2) = \ldots = r_m \ldots r_{i+1}(i+2) = \ldots \)

\[ r_{i+1}(i+1), \]

fixes \( i \), \( \tau_i \) \( \tau_{i+1} = \tau_i + 2 \) if \( \tau_{i-1} > \tau_i + 1 \) and \( \tau_{i+2} > \tau_{i+1} = \tau_{i+1} + 2 \) if \( \tau_i = \tau_i + 1 \). The result now follows from the observation that \( r_j \) fixes \( 1, \ldots, \tau_i + 2 \) if \( j > \tau_i + 2. \)

1.3. The following known result (see [6, pg. 156] as reference) is useful.

LEMMA 1.- Let \( w \in S_n \), assume \( 1 \leq p < q \leq n \). We have \( l(ws_{p,q}) < l(w) \) if and only if \( wp > wq \), moreover in this case \( l(w)-l(ws_{p,q}) = 1+2[k \text{ s.t. } p < k < q \text{ and } wp > wk > wq]. \)

As a consequence if \( w > w \) for some fundamental transposition \( s \) and \( l(ws_{p,q}) = l(w) \)

then \( ws(p) > ws(q) \) and there is no \( k \) with \( p < k < q \) and \( ws(p) > ws(q) > ws(k) > ws(q). \)

LEMMA 2.- Let \( w = w_1, w' = w_2 \), be permutations corresponding to tableaux \( T=(a_{p,q}), T'=(a_{p,q}) \) as in the beginning of the section. Suppose there are positive integers \( t, j, k, j < k, \) with \( a_{p,q} = a_{p,q} \) \( a_{p,q} \), \( a_{p,q} = a_{p,q} \), \( a_{p,q} = a_{p,q} \) if \( q \neq t \) or \( p > k, \)

\[ a_{p,q} = a_{p,q} = a_{p,q} = a_{p,q} \text{ if } 1 \leq p < j \text{ and } \]

\[ a_{p,q} = a_{p,q} = a_{p,q} = a_{p,q} \text{ if } j \leq p < k. \]

Then \( w^{-1}w' \) is the cyclic permutation \( (t,b,c), \) where \( b \) is defined by \( wb = j \) and \( c \) by \( w'c = k+1. \)
PROOF.- Write $w = c_1 \ldots c_m$, $w' = c_1' \ldots c_m'$, then $c_p = c_p'$ if $p \neq t$ and $c_t(h) \neq c_t'(h)$ exactly for three values of $h$, namely $h = t, t+j, t+k$. Therefore $w^{-1}w'$ is a cyclic permutation $(a,b,c)$. Moreover $wT = k+1$ and $w'T = c_1 \ldots c_{t-1}(t+j-1) = j$, so we can take $a = t$ and $b, c$ defined by $wb = w'T = j$, $w'c = wT = k+1$.

2. Combinatorial correspondences

2.1. Let $L$ be the set of tableaux $T = (a_{p,q})$ as in section 1, with $w(i+1) = 1$.

Let $M$ be the set of tableaux with $w(i) = 1$ and $a_{1,i} = i+1$.

Let $N$ be the set of tableaux with $w(i-1) = 1$, $a_{1,i-1} = i$ and $a_{1,i} = i+1$.

2.2. Define a map $\psi : L \longrightarrow M$ as follows: $\psi T$ is obtained by replacing the numbers $i, i+1, \ldots, i+\tau_i-1$ in the $i$th column of $T$ by $i-1, i+2, \ldots, i+\tau_i$. Define $\psi : M \longrightarrow N$ similarly (change $i$ for $i-1$). Define $e = e_T$ by $w_T e = \tau_i+1$ ($\tau_i = \tau(T,i)$) if $T \in L$ (similarly if $T \in N$).

If $T \in N$ then $\tau_{i-1} > \tau_i$ by definitions. Define a map $\chi : N \longrightarrow M$ as follows: $\chi T$ is obtained by replacing the numbers $i, i+1, \ldots, i+\tau_i-1$ in the $(i-1)$th column of $T$ by $i-1, i, \ldots, i+\tau_i-1$.

We list a number of results.

2.2.1. We have $\psi \circ \chi$ = identity, in particular $\chi$ is injective.

2.2.2. Suppose $T \in L$, then $e > i+1$, $w_T = \psi_T a_{1,i} s_{i-1,1}$ and $\psi T$ is reduced if $T$ is reduced.

2.2.3. Suppose $T \in N$ is reduced, then $w_T = \chi_T a_{1,i-1,1} s_{i-1,1}$ and $\chi T$ is reduced.

2.2.4. We have that $\psi T = \psi T'$ and $e_T = e_{T'}$, implies $T = T'$.

2.2.5. If $T \in M$ is reduced and $l(ws_{i-1,s_{i+1}}) = l(w)$ for some $t < i$, then $t = i-1$ and $\tau_{i-1} = \tau_{i+1}$, in particular $T = \chi \psi T$, $\psi T \in N$.

2.2.6. Given $T' \in M$ reduced, $e > i+1$, $l(ws_{i-1,s_{i+1}}) = l(w_{T'})$ then there exists a reduced $T \in L$ with $\psi T = T'$ and $e = e_T$, if and only if $w_T e \leq \tau(T', i)+1$.

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2.2.1.* This follows from the constructions of the maps.

2.2.2.* Write $w_T = c_1 \ldots c_m$ as in section 1. We have:

$$e = w_T^{-1}(\tau_i+1) = c_m^{1} \ldots c_1^{-1}(i+1) = c_m^{1} \ldots c_1^{-1}(i+1) = \ldots = c_m^{1} \ldots c_1^{-1}(i+1).$$

Now $c_1+i, \ldots, c_m$ fix $(1, \ldots, i+1)$ therefore $e > i+1$. 260
Applying lemma 2 to \( T \), \( \psi T \) with \( t=i, j=1 \) and \( k=\tau_i \) we obtain \( w_T = w_{\psi T}(i, e, i+1) \) because \( b = i+1 \) and \( c=e \), thus \( w_T = w_{\psi T}^TS_i^S_i,e \).

We shall prove \( w_{\psi T}(i+1) > \tau_i + 1 = w_{\psi T}^T \). Then we shall have \( 1(w_T) < 1(w_{\psi T}^T) = 1(w_{\psi T}^TS_i^S_i,e) \) (c.f. lemma 1), thus \( \psi T \) is reduced if \( T \) is. Write \( w_{\psi T} = r_m \ldots r_i \) then \( w_{\psi T}(i+1) = r_m \ldots r_i(i+1) = r_m \ldots r_{\tau_i+1}(i+\tau_i+1) \), now \( r_m, \ldots, r_{\tau_i+1} \) fix \( 1, \ldots, \tau_i \) and \( w_{\psi T}(i+1) \neq \tau_i+1 \) because \( e > i+1 \).

2.2.3.* We have \( w_T = w_{\psi T}TS_i^S_i-1S_i-1, e = w_{\psi T}TS_i^S_i-1S_i-1, e \), \( e = e_T \) (c.f. 2.2.2.). Now \( w_T = \tau(xT,i-1)+1 \) (definition) and \( \tau(xT,i-1) = \tau(T,i)-1 \) by construction. We shall prove \( w_T(i+1) = \tau(T,i)+2 \) then \( e = i+1 \) and \( w_T = w_{\psi T}TS_i^S_i-1S_i-1, e \). We have:

\[
 w_T(i+1) = c_1 \ldots c_m(i+1) = c_1 \ldots c_i(i+1) = c_1 \ldots c_i-1(i+\tau_i+1) = c_1 \ldots c_i-2(i+\tau_i),
\]

\( T \) is reduced and \( \tau_{i-2} > \tau_{i-1} \) (c.f. 1.2.5.), hence

\[
 c_1 \ldots c_i-2(i+\tau_i) = c_1 \ldots c_i-3(i+\tau_i-1) = \ldots = c_1(\tau_i+3) = \tau_i+2
\]

Also \( 1(w_T) < 1(w_{\psi T}TS_i^S_i-1S_i+1, e) = 1(w_{\psi T}TS_i^S_i) \) (use that \( w_T(i-1) = 1 \) c.f. lemma 1), thus \( \chi T \) is reduced.

2.2.4.* If \( T, T' \in \mathcal{L} \) and \( \psi T = \psi T' \), \( e_T = e_{T'} \), implies \( \tau(T,i) = \tau(T',i) \) (definition of \( e \)). Then the construction of \( \psi T = \psi T' \) shows \( T = T' \).

2.2.5.* We have \( w_{S_1}(i+1) = w_i = 1 \). We deduce \( t = i-1 \) (c.f. 1.2.4., and lemma 1). We must have also (c.f. lemma 1) \( w_{S_1}(i) = w(i+1) > w_{S_1}(i-1) = w(i-1) \), then (1.2.5) implies \( \tau_{i-1} = \tau_{i+1} \).

2.2.6.* If \( T = \psi T \), \( e = e_T \) then by definition \( w_{\psi T} = e_T \), \( e = e_{\psi T} \ tau(i+1)+1 \) (construction). Conversely if \( w_{\psi T} = \tau(i+1)+1 \) put \( k+1 = w_{\psi T} \), and obtain \( T \) from \( T' \) by replacing the numbers \( i+1, \ldots, i+k \) in the \( i \)th column of \( T' \) by the numbers \( i, \ldots, i+k-1 \) (\( T \) is a strict tableau because \( \tau(T',i-1) > \tau(T',i) \) (c.f. 1.2.5.).) Then \( \psi T = T' \), \( \psi T = T' \), \( w_{\psi T} = w_T, e = k+1 = \tau(T,i)+1, i.e. e = e_T, and \( T \) is reduced because \( w_T = w_{\psi T}TS_i^S_i, e \) (c.f. 2.2.2.).

2.3. Let \( T \in \mathcal{L} \) be reduced and let \( t<i \) be such that \( 1(w_{S_1}TS_t,i-1) = 1(w) \), then \( t \) is uniquely determined by \( T \) (c.f. 1.2.4. and lemma 1). We have also (lemma 1 and 1.2.4) \( w(i+1) < wT = \tau_{t+1} \).

In this situation construct \( \lambda T \in \mathcal{L} \) in the following way:

Replace the numbers \( t+w(i+1)-1, \ldots, t+\tau_{t-1} \) in the \( t \)th column of \( T \) by the numbers \( t+w(i+1), \ldots, t+\tau_{t-1} \). (We obtain a strict tableau because \( w(t+1) < w(i+1) \) (c.f. lemma 1).)
Define $k = k_T$ by $w_{\lambda_T}^k = \tau(T,t)+1$.

We list a number of results.

2.3.1. We have $k > i+1$ and $w_{1s}s_{t,i}^k = w_{1s}s_{1i,k}$. 

2.3.2. $\lambda_T$ is reduced and there is no reduced $T' \in L$ with $\psi_T' = \psi_T$ and $k = e_{T'}$. 

2.3.3. Suppose we have $\lambda_T = \lambda_T'$, $k_T = k_{T'}$, for two reduced $T,T' \in M$ with $l(w_{1s}s_{1i,k}) = l(w_{1s}s_{1i,k}) = l(w_{1s}s_{1i,k}) = l(w_{1s}s_{1i,k})$. Then $t = t'$ and $T = T'$. 

2.3.4. Let $T \in M$ be reduced, let $e > i+1$ be such that $l(w_{1s}s_{1i,e}) = l(w_{1s}s_{1i,e})$, and suppose that there is no reduced $T' \in L$ with $w_{1s}s_{1i,t'} = w_{1s}s_{1i,t'}$. Then there is a reduced $T' \in M$ and $t' < i$ with $l(w_{1s}s_{1i,t'}^k) = l(w_{1s}s_{1i,t'}^k)$, $T = T'$ and $e = e_{T'}$. 

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2.3.1.* Write $w_{\lambda_T} = c_1 \ldots c_m$ then

$$k = w_{\lambda_T}^{-1}(\tau+1) = c_1 \ldots c_{m-1}(\tau+1) = c_1 \ldots c_t(\tau+1) = c_1 \ldots c_t+(\tau+1).$$

Observe that $c_j^{-1}(s) > j+1$ if $j < i$ and $s > j+1$. Also $c_i \ldots c_i$ fix $(1, \ldots, i)$, we conclude $k > i$.

Applying lemma 2 to $T, \lambda_T, t, w(i+1)$, $\tau_i$ we obtain $w_T = w_{\lambda_T}(t,c,b)$ where $b = i+1$ and $w_{\lambda_T} = c_i \ldots c_m$. Therefore $k \neq i+1$ and $w_{\lambda_T} = c_i \ldots c_m$. 

2.3.2.* We have $l(w_{\lambda_T}^{-1}_T) = l(w_{\lambda_T}^{-1}_T) = l(w_{\lambda_T})$ (cf. 2.3.1). We shall prove $w_{\lambda_T}^{-1}(i+1) > \tau_t + 1 = w_{\lambda_T}$. Then $l(w_{\lambda_T}^{-1}s_i,i_k) = l(w_{\lambda_T})$ (lemma 1) and $\lambda_T$ is reduced.

Write $w_T = c_i \ldots c_m$. Also $w(i+1) < \tau_t+1$ by definition of $t$. Then $w_T = c_i \ldots c_t(t+w(i+1))$ and $w_{\lambda_T}(i+1) = c_i \ldots c_t(c_t(w(i+1))) = c_i \ldots c_t(u)$ where $u > t+\tau_t+1$, hence $w_{\lambda_T}(i+1) > \tau_t + 2$.

We have $w_{\lambda_T}k = \tau_t + 1 > \tau_t + 1$ (cf. 1.2.4, 1.2.5). Hence the second part of the statement follows from (2.2.6).

2.3.3.* In this situation $w_{\lambda_T}^{-1}_T = w_{\lambda_T}^{-1}_T$ (cf. 2.3.1). If $t \neq t'$ suppose for instance $t' < t$, then $w_{\lambda_T}t' > w_{\lambda_T}$ (cf. 1.2.4) and $w_{\lambda_T}t' = w_{\lambda_T}(i+1) > w_T$, $t' = w_T$ against the definition of $t'$.

Therefore $t = t'$ and $w_T = w_T$. We conclude $T = T'$ because they can only differ in one column.
2.3.4.* In this situation we > T^+1 (c.f. 2.2.6). We deduce \( \tau_{i-1} = \tau_i + 1 \) because if \( \tau_{i-1} > \tau_i + 1 \) then \( w(i+1) = \tau_i + 2 \) (c.f. 1.2.5) against we < w(i+1) (c.f. lemma 1). We have in fact we > \( \tau_{i-1} \) because \( e \neq i-1 \).

Define t to be the smallest number with we > \( \tau_t + 1 \).

We claim: the \( t \) column of T is of the form, \( t, t+1, \ldots \), \( t+x^2 \), \( t+x^3, \ldots \). If \( w = c_1 \ldots c_m \) this is equivalent to \( c_t (t+\tau_t+1) > t+we \).

Now we show \( c_{j+1} \ldots c_m (i+1) = c_i (i+1) = i+\tau_i + 1 = i+\tau_{i-1} = j+\tau_j + 1 \).

Next we show \( c_j (j+\tau_j + 1) = j+1 \) if \( t < j < i \) by decreasing induction.

If the \( j \) column of T has the form \( j, j+1, \ldots, j+\tau_j, j+\tau_j + 1, j+\tau_j + 2, \ldots \) then \( \tau_{j-1} > \tau_j + s + 1 \) (c.f. 1.2.2). If \( \tau_{j-1} > \tau_j + s + 1 \) then \( w(i+1) < w(j-l) = \tau_{j-1} + 1 \), this is against \( w(i+1) > \tau_t + 1 \) (c.f. 1.2.4). We have \( \tau_{j-1} = \tau_j + s + 1 \) and the result follows.

Now construct \( T' \) by replacing the numbers \( t+\tau_t + 1, \ldots, t+we-1 \) in the \( t \) column of T by the numbers \( t+\tau_t, \ldots, t+e-2 \).

We have \( wt = \tau_{t+1} \) (c.f. 1.2.4), \( w_T t = \tau_1 \) (by construction of \( T' \)) and \( w_T (i+1) = \tau_{t+1} \) (we had \( c_{t+1} \ldots c_m (i+1) = t+\tau_t + 1 \)). Then (by lemma 2) \( w_T = w(i+1, t, e) \) and \( s_{i-1} e = w_T s_{i-1} t, i' \).

We have \( l(w_T, s_{i-1} t, i') = l(w, s_{i-1} e) = l(w) \) and \( we = w_T, t > w_T, (i+1) = \tau_{t+1} \), then \( l(w_T, s_{i-1}) > l(w) \) (c.f. lemma 1) and \( T' \) is reduced.

Also \( l(w_T, s_{i-1} t, i') = l(w_T) \). One see easily \( \lambda T' = T \) (we have \( w_T, (i+1) = \tau_{t+1} \)) and \( e = k_T \).

3. The main result

3.1. We start with a unipotent u in the general linear group \( Gl(n, \mathbb{C}) \), which in Jordan normal form has a block of size \( n-m \) and \( m \) blocks of size one.

Let us recall that the variety \( \beta_u \) can be identified with the variety of flags fixed by u.

It is known [11], that the standard tableaux of shape \((n-m, 1, \ldots, 1)\) parametrize the components of \( \beta_u \). Here we shall follow the convention that a standard tableau has strictly decreasing rows and columns.
3.1.1. Theorem.- The expression of the homology class of a component $C$ of $\beta$ corresponding to the tableau $T_C$ in terms of Schubert classes is $[C] = \sum_{T \in \mathcal{T}_{\beta}} [X^T]$, where $T$ runs over the set of reduced tableaux $T = (a_{p,q})$ with $a_1 \leq q < m$.

3.1.2 Example.- The component $C$ corresponding to the tableau $T_b$ is non-singular and is not a Schubert cycle. There is only one reduced tableau corresponding to the component, namely:

\[
\begin{pmatrix}
6 & 4 & 2 \\
5 & 1 \\
3 & 2 & 4 \\
1 & 3 & 5 \\
3 & 3 & 3
\end{pmatrix}
\]

Thus $[C] = [X_w]$ with $w = s_3 s_2 s_1 s_4 s_3 s_5$ ($w = 415263$) and $[C]$ is a single Schubert class. The corresponding Schubert cycle is singular and has different Poincaré polynomial and intersection homology Poincaré polynomial $(P_C = (q^2+q+1)(q+1)^4, P_w = (q^3+3q^2+2q+1)(q+1)^3, \text{IHP}_w = (q+1)^6)$.

3.2. Let us recall some results on the components and on the action of the Weyl group.

Let $\Delta$ be the root system of a reductive group; $\Pi$ is a system of simple roots; $\Delta_+$ is the set of positive roots; $<,>$ is the duality pairing between roots and coroots; $\delta'$ is the coroot associated to the root $\beta$; $s_\beta \in W$ is the reflection defined by $\beta$ [3].

3.2.1. According to Bernstein-Gelfand-Gelfand [1 th. 3.12] and Demazure [4 pg. 80] the action of a simple reflection $s = s_\alpha, \alpha \in \Pi$, on Schubert classes is given by:

\[
s[X_w] = \begin{cases}
-X_w & \text{if } ws < w \\
\sum_{\beta \in \Delta_+ - \{\alpha\}, \beta < \delta'} [X_{ws_\beta}] & \text{if } ws > w \\
1(\text{ws}_\beta) = 1(w) & \text{if } ws = w
\end{cases}
\]

Here $[X_w]$ is the Schubert class corresponding to $w \in W$.

3.2.2 Let $P_\beta$ be the variety of parabolic lines of type $s$ [15], and $\Pi: \beta \mapsto P_\beta$ the natural projection. Following Hotta [7] we say that a pair of components $(C,C')$ form an $s$-pair if $\Pi(C') \subset \Pi(C)$ but $\Pi(C') \neq \Pi(C)$; in particular $C$ and $C'$ intersect in codimension one.

The action of $s$ on the homology classes of the components is given by:

\[
s[C] = \begin{cases}
-C & \text{if } \dim \Pi(C) < \dim C \\
C + \sum n_{C,C'} [C'] & \text{if } \dim \Pi(C) = \dim C
\end{cases}
\]

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Here the summation is over all the components $C'$ with $(C,C')$ and s-pair and the numbers $n_{C,C'}$ are strictly positive integers [8].

REMARK.- In the formulas of Demazure and Hotta we observe

i) $w_s < w$ if and only if the Schubert cycle corresponding to $w$ contains lines of type $s$.

ii) $\dim H(C) < \dim C$ if and only if the component $C$ contains lines of type $s$.

3.3. In our case the Weyl group $W$ is $S_n$ and the set of positive roots $\Lambda_+$ is in one-to-one correspondence with set of transpositions in $S_n$.

Denote by $\beta_{i,j}$ the positive root corresponding to the transposition $s_{i,j} = (i,j)$. The roots $e_i = \beta_{i,i+1}$ $1 \leq i \leq n-1$ corresponding to the fundamental transpositions $s_i = (i,i+1)$ form a system of simple roots. Say that a component contains lines of type $i$ if it contains lines of type $s_i$. Say that two components form an i-pair if they form an $s_i$-pair.

3.3.1. PROPOSITION [12 pg. 87].- The component corresponding to the tableau $T_C$ contains exactly lines of type $\{a_1, a_2, \ldots, a_m\}$.

The intersection pattern of these components is known [16], in particular we have:

3.3.2. PROPOSITION.- Two components intersect in codimension one if and only if the corresponding tableaux differ by a transposition of consecutive integers not lying in the same row or column.

As a consequence there are for given $i$ and $C$ at most two components $C'$ with $(C,C')$ an $i$-pair.

We see that the homology classes of the components are a very special basis for the action of the Weyl group (Springer representations), i.e. the matrices of the fundamental reflections have 1's and -1's in the diagonal and 0's or positive integers outside. For "hook" components we have:

3.3.3. PROPOSITION.- All the integers $n_{C,C'}$ in Hotta's formula (2) are 1.

PROOF.- Take a component $A$ whose tableau has $i$ in the first column and $i+1 \leq n-1$ in the first row. Let $B$ be the component obtained by interchanging $i$ and $i+1$. Assume:

a) $i \geq 2$ and $i-1$ is in the first column. Let $C$ be the component whose tableau is obtained by interchanging $i$ and $i-1$ in $B$. 

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b) \(i+1 < n-1\) and \(i+2\) is in the first column. Let \(D\) be the component whose tableau is obtained by interchanging \(i+1\) and \(i+2\) in \(A\).

Then

\[
s_1[B] = [B] + n_{BA}^i[A] + n_{BC}^i[C],
\]
\[
s_{i+1} s_1[B] = -[B] + n_{BA}^i([A] + n_{AB}^i[B] + n_{AD}^i[D]) - n_{BC}^i[C],
\]
\[
s_1 s_{i+1} s_1[B] = -[B] - n_{BA}^i[A] - n_{BC}^i[C] - n_{BA}^i[A] + n_{BA}^i n_{AB}^i[B] + n_{BA}^i [A] + n_{BC}^i[C] + n_{BC}^i[C] - n_{BA}^i n_{AD}^i[D],
\]
\[
s_{i+1}[B] = -[B],
\]
\[
s_{i+1} s_1[B] = -[B] - n_{BA}^i[A] - n_{BC}^i[C],
\]
\[
s_{i+1} s_{i+1} s_1[B] = [B] - n_{BA}^i([A] + n_{AB}^i[B] + n_{AD}^i[D]) + n_{BC}^i[C]
\]

But \(s_{i+1} s_1 s_{i+1} [B] = s_{i+1} s_1 s_{i+1} [B]\) and the homology classes of the components form a basis, so comparing coefficients:

\(-1 + n_{BA}^i n_{AB}^i + 1 = -n_{BA}^i n_{AB}^i + 1\) and \(n_{BA}^i n_{AB}^i + 1\) therefore \(n_{BA}^i = 1\)

If the assumptions a) and b) are not satisfied the components \(C\) or \(D\) do not appear but the proof is the same.

3.4. The proof of theorem 3.1.1. is by double induction on the length of the first row and on the integer in the upper right-hand corner of the tableau.

Read the tableau \(\begin{array}{cccc}
n \ldots b_1 \\
a_m \\
\vdots \\
a_1 \\
\end{array}\)

as the "word" \(a_1 \ldots n \ldots b_1\).

i) case \(1 \ldots n\). If the length of the first row of the tableau is one, the unipotent is the indentity and \(\beta_u = \beta\). On the other hand \(s_{n-1} \ldots s_1 s_{n-1} \ldots s_2 \ldots s_1\) is \(w_0\) the longest element in \(S_n\) which corresponds to \([\beta]\).

ii) case \(a_1 \ldots a_m \ldots b_3 \ldots b_1\). If the number in the upper right-hand corner of the tableau is 1, all the flags in the component have for one-dimensional subspace a fixed line [12].

The component is isomorphic to the component given by the tableau \(a_{1-1}, a_2-1, \ldots, n-1, \ldots, b_3-1, b_2-1\); the isomorphism is given by the natural isomorphism between Flags \((n-1)\) and the Schubert variety in Flags \((n)\) of flags
that contain the fixed line, and through this isomorphism the Schubert cycle corresponding to the permutation \( w' \in S_n \) goes to the one corresponding to \( w \in S_n \), \( w(i) = w'(i-1) + 1 \) if \( i > 1 \). (See for instance [6, Chapter III §4]).

If one writes \( w' = s_{a_1} \cdots s_{a_k} \) as a product of fundamental transpositions then

\( w = s_{a_1+1} \cdots s_{a_k+1} \). The result now follows immediately.

iii) Case 12...i i+1...n. By induction we have now a fixed component

A corresponding to a tableau of this form (if the tableau is 12...m n-1... m+1

the proof is the same as in the first step of the induction).

Let B the component corresponding to the tableau which we obtain by

interchanging i and i+1 in the tableau of A.

We will assume \( i > 2 \) and we shall not treat the case \( i = 1 \) separately,

(because the proof in that situation is a particular case of the proof for \( i > 2 \)).

Let C be the component corresponding to the tableau which we obtain by

interchanging i-1 and i in the tableau of B.

Put \( s = s_i \), and let \( e = e_i \) be the corresponding fundamental root (i is

fixed).

Define \( \Gamma_A \) as the set of reduced tableaux \( T = (a_{p,q}) \) with \( a_{1, q} = a_{1, q - 1} \), \( 1 \leq q \leq m \).

Define \( \Gamma_B, \Gamma_C \) similarly.

We have \( \Gamma_A \subseteq L, \Gamma_B \subseteq M, \Gamma_C \subseteq N \).

Hotta formula gives \( \mathfrak{s}[B] = [B] + [A] + [C] \).

By the induction hypothesis the main result is true for B and C, so we have:

\[ [B] = \sum_{T \in \Gamma_B} [X_T^{w_B}] \]
\[ [C] = \sum_{T \in \Gamma_C} [X_T^{w_C}] \]

Now \( \Gamma_B \subseteq M \) then \( w_B^T = 1 \) if \( T \in \Gamma_B \) (c.f. 1.2.3) and \( w_T \cdot s > w_T \). Thus by Demazure formula

\[ [A] = \sum_{T \in \Gamma_A} [X_T^{w_T}] \]

is equivalent to:

\[ (*) \sum_{T \in \Gamma_A} [X_T^{w_T}] + \sum_{T \in \Gamma_B} [X_T^{w_T}] = \sum_{(T, \beta) \in I} \langle e, \delta \rangle [X_T^{w_T}] \]

where I is the set of pairs \( (T, \beta) \in \Gamma_B, \beta \in \Delta^+ - \{e\} \) and \( 1(w_T) \cdot s \in (w_T - \beta) \).

We prove (*) by "counting" terms in both sides of the equality. Put

\( I = I_1 \cup I_- \) where \( (T, \beta) \in I_1 \) if \( \langle e, \delta \rangle = 1 \) and \( (T, \beta) \in I_- \) if \( \langle e, \delta \rangle = -1 \).
We shall construct two maps:

\[ \phi: \Gamma_A \cup \Gamma_C \rightarrow I_+ \) injective and satisfying \( w_{\Gamma_A} = w_{\Gamma_C} \) if \( (T,v) = \phi(T') \)

\[ \psi: I_+ \rightarrow I_+ \text{ im} \phi \) bijective and satisfying \( w_{T,v} = w_{T',v'} \) if \( (T,\delta) = \psi(T',\delta') \)

Define \( \phi \) by:

\[ \phi(T) = \begin{cases} \psi(T,\delta_{i+1}) & \text{if } T \in \Gamma_A \\ (\lambda T, \delta_{i+1}) & \text{if } T \in \Gamma_C \end{cases} \]

\( \phi \) is well defined and satisfies the previous requirement (c.f. 2.2.2, 2.2.3).

Suppose \( (T,v) \) is in \( I_+ \). Then \( \delta \) has the form \( \delta = \delta_{t,i} \) with \( t < i \) (if \( \delta \) has the form \( \delta = \delta_{i+1, e} \) with \( e \geq i+1 \), then because \( w_{T,i+1,e} < 1 \), \( w_{T,s} < 1 \), \( w_{T,i+1} < 1 \) (c.f. lemma 1)).

Define \( \psi \) by:

\[ \psi(T,\delta) = (\lambda T, \delta_i) \]

\( \psi \) is well defined and satisfies the requirement (c.f. 2.3.2, 2.3.1). \( \psi \) is injective (c.f. 2.3.3).

If \( (t,\delta) \in I_+ \) and \( \delta \) has the form \( \delta = \delta_{t,i+1} \) then \( (T,\delta) \) is in \( \text{im} \phi \) (c.f. 2.2.5), it follows from (2.3.4) that \( \psi \) is surjective.

Example.- We have a non trivial example for the tableau

\[
\begin{array}{cccc}
9 & 7 & 6 & 4 \\
8 & & & \\
5 & & & \\
2 & & & \\
1 & & & \\
\end{array}
\]

Here the cardinalities of the sets \( \Gamma_A, \Gamma_B, \Gamma_C \) are 10, 4, 3 respectively. Moreover \( I_- \) is a non empty set.
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