ROGER HOWE

ALLEN MOY

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by Roger Howe and Allen Moy

The beauty and simplicity of the representation theory of compact Lie groups has been an inspiration for well over half a century. Indeed, it was Harish-Chandra's desire to carry over the principles and philosophies of representation theory for compact semisimple groups to noncompact semisimple groups which led him to his extraordinary work. Thus, for example, Weyl's beautiful character formula for irreducible representations of compact semisimple groups inspired Harish-Chandra's classification of the discrete series for noncompact semisimple Lie groups [HC].

In another vein, D. Vogan [V] has defined the notion of a minimal or lowest K-type for real reductive groups and shown that minimal K-types can be made to play a role in the representation theory of real reductive groups analogous to the role played by highest weights in Cartan's classification of the finite dimensional irreducible representations of a semisimple Lie algebra.

For reductive p-adic groups, it has been observed (see [Mu]) that in certain cases the restriction of an irreducible admissible representation π of a reductive group G to an open compact subgroup $L \subset G$ contains a representation Ω of L which possesses particularly nice properties. For example, it has been conjectured that every irreducible supercuspidal representation of G is induced from some open compact mod center subgroup. On the other hand, the problem of describing those representations of G which contain a given representation Ω of some compact open group L is very difficult for arbitrary Ω. However, for nice representations Ω, it is possible to give a good accounting of the irreducible admissible representations of G which contain Ω upon restriction to L. In [S,Mc], such a description is given for the trivial representation of a maximal compact group $K \subset G$ by performing an analysis of the structure of the Hecke algebra $\mathcal{H}(G//K)$ of K-spherical functions. More recently, Kazhdan-Lusztig and Ginsburg [KL1,KL2,G] have given a penetrating classification (in the split case) of the representations of G which possess a nonzero Iwahori fixed vector.

In [H], an approach to the classification of representations of $G=GL_n(F)$, F a p-adic field, was proposed based on defining the nice representations in terms of "dual blobs" in $M_n(F)$ satisfying certain geometric conditions. For many of these Ω's it was shown in [HM1] that $\mathcal{H}(G//L,\Omega)$ is in fact isomorphic to a Hecke algebra $\mathcal{H}(G'//L',1)$, where $G'=GL_m(E)$ for some extension E/F with $n=m[E:F]$.

In [My], the dual blobs of [H] were given a more precise formulation. Certain pairs (L,Ω) consisting of an open compact group L and a representation Ω of L were singled out and called minimal K-types. There, it was conjectured that every irreducible admissible representation of G contains a minimal K-type.

Here we prove this conjecture. The proof we present is actually our second proof. Our first proof was combinatorial and somewhat complicated. In searching for a more conceptual argument,the second author conjectured Theorem 2.1 of this paper. The first author later found the proof given in section 2. In the interim, C. Bushnell learned of the conjectured result from the second author, and also was able to prove it. In fact, Bushnell's proof has priority over the one given here. An earlier version of this paper contained both our first and second proofs. However, in the interests of simplicity, the first proof has been deleted. It may be of some interest to experts, and is available from the authors.

A proof of the existence of minimal K-types, based on Theorem 2.1, is given in section 3. Section 4 discusses some complementary results.

In a sequel [HM2], to this paper, we use the existence of minimal K-types to extend considerably the range of the theory of Hecke algebras isomorphisms initiated in [HM1]. This extended theory allows us to give a classification, complementary to that of Bernstein-Zelevinsky, [BZ,Z] of the representations when $n \leq 2p$ (where p is the residual characteristic).

1. Statement of the existence of minimal K-types

In this section we introduce some notation and review the statement of the existence of minimal K-types.

Let R denote the ring of integers in a p-adic field F, $\mathfrak{p}$ the prime ideal of R, and ϖ a prime element in $\mathfrak{p}$. Let q be the order of the residue field $\mathbb{F}_q = R/\mathfrak{p}$. A lattice in $V=F^n$, the space of column vectors, is a free R-submodule L of rank n. A periodic lattice flag $\mathcal{L}$ in V is a sequence of lattices $\{ L_i \mid i \in \mathbb{Z} \}$ such that $L_{i+1} \subsetneq L_i$ and $L_{i+m} = \varpi L_i$ for some fixed positive integer m. The integer m is called the period of the lattice flag $\mathcal{L}$. It is clear that $m \leq n$. For a fixed lattice flag $\mathcal{L}$, let

$$(1.1) \qquad \mathfrak{R} = \{ x \in M_n(F) \mid xL_i \subset L_i \}$$

and

$$(1.2) \qquad \mathfrak{P} = \{ x \in M_n(F) \mid xL_i \subset L_{i+1} \} .$$

The ring $\mathfrak{R}$ is a hereditary order of $M_n(F)$ and $\mathfrak{P}$ is its topological radical.

We refer to [R] for properties of hereditary orders. The group of units J of $\mathfrak{R}$ is a parahoric subgroup of $GL_n(F)$. Define a filtration J_i ($i \geq 0$) of J by

(1.3) $\quad J_0 = J \quad$ and $\quad J_i = \{ 1+x \mid x \in \mathcal{P}^i \}$ for $i>0$.

The J_i's are of course normal in J. They are a special case of similar filtrations defined by Prasad and Raghunathan (see [PR]) for the parahoric subgroups of a reductive group. The minimal K-types are representations of the J_i which are trivial on J_{i+1}. The structure of J_i/J_{i+1} is given as follows:

Case 1: $i=0$. The quotients $V_t = L_t/L_{t+1}$ ($1 \leq t \leq m$) are all the irreducible modules of $\mathfrak{R}$. They can be viewed as $\mathbb{F}_q$-vector spaces and

$$J_0/J_1 = \prod_{1 \leq t \leq m} GL(V_t) .$$

This description of J_0/J_1 as a reductive group over $\mathbb{F}_q$ in particular allows us to speak of cuspidal representations of J_0/J_1.

Case 2: $i \geq 1$. The map $x \longrightarrow 1+x$ from $\mathcal{P}^i$ to J_i gives an isomorphism between $\mathcal{P}^i/\mathcal{P}^{i+1}$ and J_i/J_{i+1}. It is well known that this map in fact allows one to realize the character group of J_i/J_{i+1} as $\mathcal{P}^{-i}/\mathcal{P}^{-i+1}$. One fixes a character χ of F with conductor $\mathfrak{p}$ and identifies a coset $\mathbf{x}=x+\mathcal{P}^{-i+1}$ with the character

(1.4) $\qquad \Omega_{\mathbf{x}}(y) = \chi(\, \mathrm{tr}(x(y-1))\,) \qquad y \in J_i$.

Let $\mathcal{L}$ be a periodic lattice flag. A coset $\mathbf{x} = x+ \mathcal{P}^{-i+1}$ in $\mathcal{P}^{-i}$ is said to be nondegenerate if $\mathbf{x}$ does not contain any nilpotent elements.

We remark that if $\mathbf{x}$ is nondegenerate, then the minimal valuation of the eigenvalues of any representative of $\mathbf{x}$ is $-i/m$ (see the remark after Proposition 2.2).

A minimal K-type is a pair (J_i,Ω), consisting of a parahoric filtration subgroup J_i and an irreducible representation Ω of J_i, trivial on J_{i+1}, and satisfying one of the following criteria:

a) If $i=0$, then Ω is a cuspidal representation

b) If $i>0$, then $\Omega=\Omega_{\mathbf{x}}$ for some nondegenerate coset **x** of $\mathfrak{P}^{-i+1}$ in $\mathfrak{P}^{-i}$.

We now state the main result of the paper.

Theorem 1.1. Given any irreducible admissible representation (π,V) of $G=GL_n(F)$, there is a minimal K-type (J_i,Ω) such that the restriction of π to J_i contains Ω.

We prove Theorem 1.1 in section 3.

2. A theorem on hereditary orders

The main result of this section, Theorem 2.1, was conjectured by the second author. It was first proved by C. Bushnell [B]. The proof presented here was independently but subsequently found by the first author. The fundamental mechanism is the same at that of [B]. However, the authors hope this proof may illuminate certain details of the phenomenon in question. In particular, we point out that the thinning process of Proposition 2.3 and the refinement process of Proposition 2.4 are canonical constructions. Also, Proposition 2.2 shows that the optimal j'/m' does not depend on the fine structure of the lattice flag defining $\mathfrak{R}'$, but only on the eigenvalues of elements of $x+\mathfrak{P}^{j+1}$.

Theorem 2.1 Let $\mathfrak{R}$ be a hereditary order with period m and radical $\mathfrak{P}$. Suppose $\mathbf{x}=x+\mathfrak{P}^{j+1}$ is a coset of $\mathfrak{P}^j$. If $\mathbf{x}$ contains a nilpotent element, then there is a hereditary order $\mathfrak{R}'$ with radical $\mathfrak{P}'$ and period m' such that for some j' we have $x+\mathfrak{P}^{j+1} \subset \mathfrak{P}'^{j'}$ and $j/m < j'/m'$.

In order to prove Theorem 2.1, we begin by establishing some notation and a few preliminary results. Let $\mathcal{L} = \{ L_i \}$ be a periodic lattice flag with period m, and let $\mathfrak{R}$ and $\mathfrak{P}$ be as in section 1. Consider the quotients

$$(2.1) \qquad \bar{L}_i = L_i/L_{i+1} .$$

The L_i are all vector spaces over $\mathbb{F}_q$. Multiplication by ϖ induces isomorphisms

$$(2.2) \qquad \bar{\varpi} : \bar{L}_i \longrightarrow \bar{L}_{i+m} .$$

We use the maps $\bar{\varpi}$ to identify $\bar{L}_i$ and $\bar{L}_{i+m}$. Given $i \in \mathbb{Z}$, let i denote its image in $\mathbb{Z}/m\mathbb{Z}$. Let

$$(2.3) \qquad \bar{\mathcal{L}} = \sum_{\bar{i} \in \mathbb{Z}/m\mathbb{Z}} \bar{L}_{\bar{i}} = \sum_{0 \le i < m} \bar{L}_{\bar{i}} .$$

Of course, $\bar{\mathcal{L}}$ is a vector space over $\mathbb{F}_q$. Consider $\mathrm{End}(\bar{\mathcal{L}})$. We have

$$(2.4) \qquad \mathrm{End}(\bar{\mathcal{L}}) = \sum_{\bar{i},\bar{j}} \mathrm{Hom}(\bar{L}_{\bar{i}},\bar{L}_{\bar{j}}) .$$

Given $\bar{x} \in \mathrm{End}(\bar{\mathcal{L}})$, we say $\bar{x}$ is homogeneous of degree $\bar{k}$ if

$$(2.5) \qquad \bar{x}(\bar{L}_{\bar{i}}) \subset \bar{L}_{i+k}.$$

Denote the space of such $\bar{x}$ by $\mathrm{End}(\bar{\mathcal{L}})_{\bar{k}}$. Then, clearly

$$(2.6) \qquad \mathrm{End}(\bar{\mathcal{L}})_k = \sum_{\bar{i}} \mathrm{Hom}(\bar{L}_{\bar{i}},\bar{L}_{\bar{i}+\bar{k}}) \quad \text{and}$$

$$\mathrm{End}(\bar{\mathcal{L}}) = \sum_{\bar{k}} \mathrm{End}(\bar{\mathcal{L}})_{\bar{k}} .$$

Furthermore, under multiplication in $\mathrm{End}(\bar{\mathcal{L}})$, we have

$$(2.7) \qquad \mathrm{End}(\bar{\mathcal{L}})_{\bar{k}}\, \mathrm{End}(\bar{\mathcal{L}})_{\bar{j}} \subset \mathrm{End}(\bar{\mathcal{L}})_{\bar{k}+\bar{j}} .$$

Given $x \in \mathfrak{P}^j$, it is obvious that x defines a map

(2.8) $\bar{x}_{\bar{i}} : \bar{L}_{\bar{i}} \longrightarrow \bar{L}_{\bar{i}+\bar{j}}$.

It is also clear that

(2.9) $\bar{x}_i \,\bar{\varpi} = \bar{\varpi}\, \bar{x}_{i+m}$;

hence the x_{i+am} ($a \in \mathbb{Z}$) collapse to define a map

(2.10) $\bar{x}_{\bar{i}} : \bar{L}_{\bar{i}} \longrightarrow \bar{L}_{\bar{i}+\bar{j}}$.

Taking the direct sum of the $\bar{x}_{\bar{i}}$, we see that x defines a mapping

(2.11) $\bar{x} = \sum_{\bar{i}} \bar{x}_{\bar{i}} \in \mathrm{End}(\bar{\mathbf{L}})_{\bar{j}} \subset \mathrm{End}(\bar{\mathbf{L}})$.

The following facts are not difficult to verify (see chapter 9 in [R])

(2.12) i) if $v \in L_i - L_{i+1}$, then $\mathfrak{P}^j(v) = L_{i+j}$

ii) the map $x \longrightarrow \bar{x}$ defined by (2.11) is an isomorphism

$$\mathfrak{P}^j / \mathfrak{P}^{j+1} \longrightarrow \mathrm{End}(\bar{\mathbf{L}})_j$$

iii) the diagram

$$\begin{array}{ccc} \mathfrak{P}^j \otimes \mathfrak{P}^k & \longrightarrow & \mathfrak{P}^{j+k} \\ \downarrow & & \downarrow \\ \mathrm{End}(\bar{\mathbf{L}})_{\bar{j}} \otimes \mathrm{End}(\bar{\mathbf{L}})_{\bar{k}} & \longrightarrow & \mathrm{End}(\bar{\mathbf{L}})_{\bar{j}+\bar{k}} \end{array}$$

where the horizontal arrows are given by multiplication and the vertical arrows by the reduction mapping (2.11), commutes.

Statement (2.12ii) says given $\bar{x} \in \mathrm{End}(\bar{\mathbf{L}})_{\bar{j}}$ and $j \in \bar{j}$, there is a unique coset

(2.13) $x + \mathfrak{P}^{j+1}$ in $\mathfrak{P}^j$

such that x reduces to $\bar{x}$ via the mapping (2.11).

Proposition 2.2 Consider $x \in \mathfrak{P}^j$ and the coset $\bar{x} = x + \mathfrak{P}^{j+1}$.

a) If $\bar{x} \in \mathrm{End}(\bar{\mathcal{L}})_{\bar{j}}$ is nilpotent, say $\bar{x}^a=0$, then for any i and $k \geq 0$

(2.14) i) $(x+ \mathcal{P}^{j+1})^{ak} (L_i) \subset L_{i+akj+k}$, in particular

ii) $(x+ \mathcal{P}^{j+1})^{mak} (L_i) \subset \varpi^{(aj+1)k} L_i$.

b) Let L be a lattice in V such that $L_i \supset L \supsetneq L_{i+1}$. Then

(2.15) $$(x+ \mathcal{P}^{j+1}) (L) = x(L) + L_{i+j+1} .$$

In particular, if $L/L_{i+1} \not\subset \ker \bar{x}_i$, then

$$L_{i+j} \supset (x+ \mathcal{P}^{j+1}) (L) \supsetneq L_{i+j+1} .$$

However, if $L/L_{i+1} \subset \ker \bar{x}_i$, then

$$(x+ \mathcal{P}^{j+1}) (L_i) = L_{i+j+1} .$$

Proof. Equation (2.15) is clear from (2.12i), and inclusion (2.14) follows directly from (2.15) and (2.12iii). □

Remark. Suppose $\bar{x}$ is not nilpotent. Then $y=x^m\varpi^{-j} \in \mathcal{R}$, and mod $\mathcal{P}$, y is a nonnilpotent element. The eigenvalues of y are integral over R, and at least one eigenvalue has valuation 0. This means the minimal valuation of an eigenvalue of x is j/m.

Let $\mathcal{L} = \{ L_i \}$ and $\mathcal{L}' = \{ L'_i \}$ be two periodic lattice flags. We say $\mathcal{L}'$ is a <u>refinement</u> of $\mathcal{L}$ if each L_i is an $L'_{i'}$ for suitable i'. Suppose $\mathcal{L}'$ is a refinement of $\mathcal{L}$. We may reparametrize $\mathcal{L}'$ so that $L'_0=L_0$. If m' is the period of $\mathcal{L}'$, then

$$L_m = \varpi L_0 = \varpi L'_0 = L_{m'} .$$

For $0 \leq i \leq m$, there are integers c_i so that $L_i = L'_{c_i}$. Thus, the flag $\{L'_i/\varpi L_0\}$ of subspaces of $L_0/\varpi L_0$ defines a refinement of the flag $\{L_i/\varpi L_0\}$. Conversely, given a refinement of the flag $\{L_i/\varpi L_0\}$, there exists a unique refinement $\mathcal{L}'$ of $\mathcal{L}$ such that

i) $L'_0 = L_0$ and

ii) the flag $\{L'_i/\varpi L_0\}$ is the given refinement of $\{L_i/\varpi L_0\}$.

Again, let $\mathcal{L}'$ be a refinement of $\mathcal{L}$. With c_i as in the previous paragraph, we have

(2.16) a) $\bar{\mathcal{L}} = \sum_{\bar{i} \in \mathbb{Z}/m\mathbb{Z}} \bar{L}_{\bar{i}}$

$\bar{\mathcal{L}}' = \sum_{\bar{i}' \in \mathbb{Z}/m'\mathbb{Z}} \bar{L}'_{\bar{i}'}$

b) $\bar{L}_{\bar{i}} \cong \sum_{c_i \leq i' < c_{i+1}} \bar{L}_{\bar{i}'}$.

If $\mathcal{L}'$ is a refinement of $\mathcal{L}$, we call $\mathcal{L}$ a <u>thinning</u> of $\mathcal{L}'$

Consider $\mathcal{L}$ as above. Let $X \subset M_n(F)$ be a set of operators on V. Assume X is a compact subset of $M_n(F)$. We say X is <u>taut</u> with respect to $\mathcal{L}$ if given any $L_i \in \mathcal{L}$, there is an $L_j \in \mathcal{L}$ so that

(2.17) $X L_i = L_j$.

In other words, if L_j is the smallest element of $\mathcal{L}$ such that $X L_i \subset L_j$, then in fact $X L_i = L_j$. If X is taut with respect to $\mathcal{L}$, then the mapping

(2.18) $\sigma(X) : L_i \to X L_i$

defines a mapping from $\mathcal{L}$ to itself. We denote also by $\sigma(X)$, the map induced by $\sigma(X)$ on the index set $\mathbb{Z}$ of $\mathcal{L}$, i.e. $\sigma(X)L_i = L_{\sigma(X)(i)}$. Since $X L_{i+1} \subset X L_i$ and $X(L_{i+m}) = X(\varpi L_i) = \varpi X(L_i)$ the following two properties of $\sigma(X)$ are clear

(2.19) a) $\sigma(X)$ is order preserving, i.e. $\sigma(X)(i+1) \geq \sigma(X)(i)$

b) $\sigma(X)$ is periodic, i.e. $\sigma(X)(i+m) = \sigma(X)(i) + m$.

Define X to be <u>completely taut</u> with respect to $\mathcal{L}$ if the map $\sigma(X)$ is a bijection.

Proposition 2.3 a) If $X \subset M_n(F)$ is taut with respect to $\mathcal{L}$, there is a thinning $\mathcal{L}'$ of $\mathcal{L}$ such that X is completely taut with respect to $\mathcal{L}'$.

b) If $X \subset M_n(F)$ is completely taut with respect to $\mathcal{L}$, and j is the largest integer such that $X \subset \mathcal{P}^j$, then

$$\sigma(X)(L_i) = XL_i = L_{i+j} \tag{2.20}$$

for all $i \in \mathbb{Z}$.

c) If X is completely taut with respect to $\mathcal{L}$, and j is as in b), then j may be calculated by the formula

$$X^{am}(L_i) = \varpi^{ja} L_i \tag{2.21}$$

for any $i \in \mathbb{Z}$ and any positive integer a. Here, m is the period of $\mathcal{L}$.

Proof. By (2.19b), we see that $\sigma(X)$ factors through the quotient map $\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$, i.e. $\sigma(X)$ is effectively a mapping on the finite set $\mathbb{Z}/m\mathbb{Z}$. A mapping μ of a finite set S to itself is a bijection if and only if it is either a surjection or an injection. Moreover, the restriction of μ to the non-empty μ-invariant set $\bigcap_{a\geq 0} \mu^a(S)$ will always be a bijection. We apply this to $\mathbb{Z}/m\mathbb{Z}$and $\sigma(X)$. Choose in $\mathbb{Z}/m\mathbb{Z}$ any $\sigma(X)$-invariant non-empty set A on which $\sigma(X)$ is bijective. Select from $\mathcal{L}$, the L_i with $\bar{i} \in A$. This gives us a thinning $\mathcal{L}'$ with respect to which X is completely taut. This proves a). If X is completely taut with respect to $\mathcal{L}$, then $\sigma(X)$ defines a bijection of $\mathbb{Z}$ to itself. According to property (2.19a), this bijection is order preserving. Whence, it is the form of a translation by j for some j. This j is just the integer described in b). Finally, (2.21) is an immediate consequence of iterating (2.20) am times. This completes the proof of the proposition. □

Proposition 2.4 Consider $\mathcal{L}$ as in Proposition 2.2. Then, there is a refinement $\mathcal{L}'$ of $\mathcal{L}$ with respect to which $X = x+\mathcal{P}^{j+1}$ is taut.

Proof. Consider the reduced map $\bar{x} \in \mathrm{End}(\bar{\mathcal{L}})_j$. For each summand $\bar{L}_{\bar{i}} \subset \bar{\mathcal{L}}$, consider the flag defined by the subspaces

$$\bar{x}^a(\bar{L}_{\bar{i}-\bar{a}\bar{j}}) \quad a \in \mathbb{Z} \text{ and } a \geq 0 . \tag{2.22}$$

This is a flag because

$$\bar{x}^{a+1}(\bar{L}_{\bar{i}-(\bar{a}+\bar{1})\bar{j}}) = \bar{x}^a(\bar{x}(L_{\bar{i}-(\bar{a}+\bar{1})\bar{j}})) \subset \bar{x}^a(L_{\bar{i}-\bar{a}\bar{j}}) .$$

Moreover, by its definition, we see that x maps the flag in $\bar{L}_{\bar{i}}$ to the analogous flag in $\bar{L}_{\bar{i}+\bar{j}}$, i.e.

$$\bar{x}(\bar{x}^a(\bar{L}_{\bar{i}-\bar{a}\bar{j}})) = \bar{x}^{a+1}(\bar{L}_{\bar{i}+\bar{j}-(\bar{a}+\bar{1})\bar{j}}) . \tag{2.23}$$

According to the discussion of refinements, there is a refinement $\mathcal{L}'$ of $\mathcal{L}$ so that the elements of $\mathcal{L}'$ contained between L_i and L_{i+1} define the flag (2.22) in $\bar{L}_{\bar{i}}$. Equation (2.23) combined with (2.15) show that $x+\mathcal{P}^{j+1}$ is taut with respect to $\mathcal{L}$. ☐

Proof of Theorem 2.1. Indeed, Proposition 2.4 allows us to pass to a refinement $\mathcal{L}''$ of $\mathcal{L}$ with respect to which $x+\mathcal{P}^{j+1}$ is taut. Proposition 2.3 then allows us to pass to a thinning $\mathcal{L}'$ of $\mathcal{L}''$ so that $x+\mathcal{P}^{j+1}$ is completely taut with respect to $\mathcal{L}'$. Choose L_i, L'_h and b so that $L_i \supseteq L'_h \supseteq \varpi^b L_i$. By (2.21) and (2.14ii), for any positive integer k, we have

$$\varpi^{(aj+1)m'k}L_i \supseteq X^{mm'ak}L_i \supseteq X^{mm'ak}L'_h \supseteq \varpi^{j'mak}L'_h \supseteq \varpi^{j'mak+b}L_i .$$

Thus, $j'mak+b \geq (aj+1)m'k$, i.e. $(j'/m') + (b/mm'ak) \geq (j/m) + (1/ma)$. Since k is an arbitrary positive integer, we conclude

$$j'/m' \geq j/m + 1/ma .$$

This completes the proof of Theorem 2.1 ☐

3. Proof of existence of minimal K-types

We refer to section 1 for the context and notation. We begin with

Proposition 3.1 Suppose $V^{J_{i+1}} \neq \{0\}$ and $i \geq 1$, then either

i) there is a nonzero v in $V^{J_{i+1}}$ transforming under J_i/J_{i+1} by a minimal K-type character $\Omega_{\mathbf{x}}$, or

ii) there is a parahoric filtration subgroup $J'_{i'}$ such that $V^{J'_{i'+1}} \neq \{0\}$ and $i'/m' < i/m$, where m (resp. m') are the periods of the lattice flags corresponding to J_i and $J'_{i'}$.

Proof. Pick a nonzero v in $V^{J_{i+1}}$ transforming under J_i/J_{i+1} by a character $\Omega_{\mathbf{x}}$, where $\mathbf{x} = x+\mathfrak{P}^{-i+1}$. If $\mathbf{x}$ does not contain a nilpotent element then $\Omega_{\mathbf{x}}$ is a minimal K-type character. Thus, we can assume $\mathbf{x}$ does contain a nilpotent element. Let m be the period of the lattice flag used to define J_i. By Theorem 2.1, there is a hereditary order $\mathfrak{R}'$ with period m' and an integer -i' such that $\mathbf{x} \subset \mathfrak{P}'^{-i'}$ and $-i/m < -i'/m'$, i.e. $i'/m' < i/m$. Observe that $\mathfrak{P}'^{-i'} \supset x+\mathfrak{P}^{-i+1}$ means $\mathfrak{P}'^{-i'} \supset \mathfrak{P}^{-i+1}$. Then

$$\begin{aligned}\mathfrak{P}'^{i'-(m'-1)} &= \{ y \in M_n(F) \mid tr(y\mathfrak{P}'^{-i'}) \subset R \} \\ &\subset \{ y \in M_n(F) \mid tr(y\mathfrak{P}^{-i+1}) \subset R \} = \mathfrak{P}^{i-1-(m-1)},\end{aligned}$$

and so $\mathfrak{P}'^{i'+1} \subset \mathfrak{P}^{i}$. Thus, $J'_{i'+1} \subset J_i$ and for z in $\mathfrak{P}'^{i'+1}$, we have

$$\pi(1+z)v = \chi(\,tr(xz)\,)\ .$$

Since $\mathbf{x} \subset \mathfrak{P}'^{-i'}$ we have xz lies in $\mathfrak{P}'^{1}$. This means $tr(xz) \in \mathfrak{p}$ and so v is fixed by $J'_{i'+1}$. This proves the proposition. □

We now give a proof of Theorem 1.1.

Proof of Theorem 1.1. The set of periods of hereditary orders in $M_n(F)$

is the set of integers from 1 to n. In particular, the set

$$D = \{\ i/m \mid i \text{ is a positive integer and } m \text{ a period of a hereditary order }\}$$

is a discrete set. Let P_j be a parahoric filtration subgroup for which $V^{P_j} \neq \{0\}$. Combining Proposition 3.1 with the discreteness of D, we see that there is a parahoric subgroup J_i such that either $i \geq 1$ and (π,V) contains a minimal K-type of the form $(J_i,\Omega_{\mathbf{x}})$ or $i=0$ and $V^{J_1} \neq \{0\}$. In the latter case, by philosophy of cusp forms, (π,V) contains a minimal K-type (J_0,Ω). This completes the proof of Theorem 1.1. □

4. Basic properties of minimal K-types

It is natural to ask: when can two minimal K-types (J_i,Ω) and $(J'_{i'},\Omega')$ both occur in an irreducible admissible representation (π,V). The next theorem gives a necessary condition for this to occur.

Theorem 4.1 Suppose (π,V) is an irreducible admissible representation of $G=GL_n(F)$ and the two minimal K-types (J_i,Ω) and $(J'_{i'},\Omega')$ both occur in (π,V). Then either i and i' are both greater than zero, so $\Omega=\Omega_{\mathbf{x}}$, $\Omega'=\Omega_{\mathbf{x}'}$ and some element of **x** is conjugate to some element of **x'**, or both i and i' are zero and $J_0/J_1 \cong J'_0/J'_1$, $\Omega \cong \Omega'$.

Proof. The proof is based on the principle of intertwining [H], which we now recall. Let W_Ω (resp. $W_{\Omega'}$) be irreducible Ω (resp. Ω') subspaces of V. Decompose V as a J'-module, and let $E_{\Omega'}$ be a J'-module projection of V onto $W_{\Omega'}$. Since (π,V) is irreducible, there exists a g in G such that the map

$$I = E_{\Omega'}\, \pi(g) : W_\Omega \longrightarrow W_{\Omega'}$$

is nonzero, and for h in $J_i \cap gJ'_{i'}g^{-1}$ we have $I\Omega(h) = \Omega'(g^{-1}hg)I$. We consider three cases according to whether i and i' are both greater than zero, one is equal to zero, or both are zero.

Case 1: $i, i' \geq 1$. Let $\mathbf{x} = x+\mathcal{P}^{-i+1}$ and $\mathbf{x}' = x'+\mathcal{P}'^{-i'+1}$ be the cosets which give the characters Ω and Ω' respectively. By the intertwining principle just explained, the two characters

$$1+y \longrightarrow \chi(\operatorname{tr}(xy)) \quad \text{and} \quad 1+y \longrightarrow \chi(\operatorname{tr}(x'g^{-1}yg))$$

agree on $J_i \cap gJ'_{i'}g^{-1}$. This means $1 = \chi(\operatorname{tr}((x-gx'g^{-1})y))$ for all y in $\mathcal{P}^i \cap g\mathcal{P}'^{i'}g^{-1}$. It follows that $\operatorname{tr}((x-gx'g^{-1})y)$ must lie in $\mathfrak{p}$ for all y in $\mathcal{P}^i \cap g\mathcal{P}'^{i'}g^{-1}$. Hence, $(x-gx'g^{-1})$ lies in $\mathcal{P}^{-i+1} + g\mathcal{P}'^{-i'+1}g^{-1}$, i.e. $\mathbf{x}$ and $g\mathbf{x}'g^{-1}$ intersect.

Case 2: $i \geq 1$and $i'=0$. Here we need to show that $(J_i, \Omega_{\mathbf{x}})$ and (J'_0,Ω') cannot both occur in (π,V). Observe that if (J'_0,Ω') occurs in (π,V), then the trivial character of J'_1 occurs in π. The trivial character of J'_1 is represented by the coset $\mathcal{P}'^0$ in $\mathcal{P}'^{-1}$. By the same reasoning as in case 1, $\mathcal{P}'^0$ and $g\mathbf{x}'g^{-1}$ must intersect. This is impossible, since the valuations of eigenvalues of elements in $\mathcal{P}'^0$ are greater than or equal 0 and the minimal valuation of the eigenvalues of each element of $\mathbf{x}'$ is $-i/m < 0$.

Case 3: $i, i'=0$. The reasoning is again based on the intertwining principle and indeed it has already been proved by Harish-Chandra using the Bruhat decomposition of G (see [HM1]). □

As an immediate corollary we have

Corollary 4.2 If (J_i,Ω) and $(J'_{i'},\Omega')$ are minimal K-types which both occur in the irreducible admissible representation (π,V) and m,m' are the periods of the lattice flags attached to J_i, $J'_{i'}$ respectively, then

$i/m = i'/m'$.

The argument in the proof Theorem 4.1 is easily adapted to show

Theorem 4.3 Suppose (π,V) is an irreducible admissible representation of $Gl_n(F)$. Let (J_i,Ω) be a minimal K-type which occurs in π. If $J'_{i'+1}$, is another parahoric filtration subgroup such that $V^{J'_{i'+1}} \neq \{0\}$, then $i/m \leq i'/m'$, where m and m' are the periods of the corresponding lattice flags. Moreover, if $i>0$ and equality occurs, then the $J'_{i'}/J'_{i'+1}$-space $V^{J'_{i'+1}}$ is a sum of minimal K-types . If i and i' are zero, there is a parahoric subgroup $J''_0 \subset J_0$ and a cuspidal representation Ω'' of J''_0 such that the J'_0/J'_1 components of $V^{J'_{i'+1}}$ all occur in the induced representation ß of Ω'' from J''_0 to J_0.

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R.HOWE
Yale University
Box 2155
Yale Sta
New Haven , CT 06520
U.S.A.

A.MOY
Rutgers University
New Brunswick, NJ 08904
U.S.A.