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SHALIKA GERMS ON $GSp(4)$

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This paper calculates all of the Shalika germs of the group $GSp(4)$ and its inner forms over a local p-adic field of characteristic zero. As a consequence we conclude that the conjectures of Langlands and Shelstad [LS2], relating linear combinations of germs on a reductive group G to germs on the endoscopic groups of G, are valid for $G = GSp(4)$ or one of its inner forms. More generally, when $G = Sp(4)$ or one of its inner forms we show that the germs associated to the regular and subregular unipotent classes satisfy the conjectures of Langlands and Shelstad.

§1. A REVIEW OF IGUSA THEORY

The approach to calculating Shalika germs used here was introduced by R.P. Langlands in [L]. This paper fixes the group $GSp(4)$ and studies all of its germs. This should be contrasted with [H] which studies a particular germ, that associated to the subregular unipotent classes, for all reductive groups. The essential ingredient is a theorem of Igusa on asymptotic expansions. For the sake of completeness we restate the theorem.

1. Let X be a smooth variety over a curve Γ ($X \xrightarrow{\varphi} \Gamma$) with X, Γ and φ defined over a local p-adic field F of characteristic zero.

2. Suppose that there is a point x_0 on $\Gamma(F)$ such that X is smooth outside $\varphi^{-1}(x_0)$ and $\varphi^{-1}(x_0)$ is a divisor with normal crossings with the property that any irreducible component of $\varphi^{-1}(x_0)$ with an F-rational point is defined over F. Let $\mathcal{E}$ be the set of irreducible components of $\varphi^{-1}(x_0)$ defined over F. Let $a(E)$ for $E \in \mathcal{E}$ be the multiplicity of E in $\varphi^{-1}(x_0)$.

3. Let ω be a form of maximal degree on X which is defined over the algebraic closure $\overline{F}$ of F which is nonvanishing and regular outside $\varphi^{-1}(x_0)$. Write the divisor of ω as $D + \sum_{E \in \mathcal{E}} (b(E) - 1)E$ with $b(E) \in \mathbf{Z}$. We may ignore the term D having no F-rational points.

4. Suppose that there is a torus T over F which is split by a Galois extension E/F and rational functions $t_\sigma \in T(K_E)$ for $\sigma \in Gal(E/F)$ and K_E the field of rational functions on $X \times_{Spec(F)} Spec(E)$. Suppose that t_σ defines a cohomology class $[t_\sigma]_p$ of $H^1(Gal(E/F), T(E))$ for F-rational points p in a Zariski open set of X and that $p \mapsto [t_\sigma]_p$ extends to a locally constant function on the F-rational points of $X \setminus \varphi^{-1}(x_0)$. For any character κ of $H^1(Gal(E/F), T(E))$ and divisor $D \in \mathcal{E}$ we define a character κ_D of $F^\times$ ($\kappa_D : F^\times \to \mathbf{C}^\times$) as follows. Pick local p-adic coordinates $\mu_1, \ldots, \mu_n$ over F at $p_0 \in D(F)$ such that $\mu_1 = 0$ defines D locally. Choose, if possible, κ_D so that $\kappa(t_\sigma)/\kappa_D(\mu_1)$ extends to a function constant in a neighborhood of p_0. If p_0 lies in no other divisor of $\mathcal{E}$ then such a character exists and is independent of $p_0 \in D(F)$ and the choice of local coordinates.

5. If θ is a character of finite order of $F^\times$ and $\beta \in \mathbf{Q}$ let $\mathcal{E}(\theta, \beta)$ be the set of divisors in $\mathcal{E}$ such that $\theta^{a(E)} = \kappa_E$ and $b(E)/a(E) = \beta$. Let $e(\theta, \beta)$ be the maximum number of divisors of $\mathcal{E}(\theta, \beta)$ with non-empty intersection.

6. Let f be a locally constant function on X whose support is proper over Γ.

7. Normalize the valuation on F by the additive Haar measure dx so that $d(ax) = |a|dx$ and set $m(\lambda) = -log_q|\lambda|$ where q is the order of the residue field of F. Extend, whenever necessary, $|\cdot|$ to extensions E of F. Normalize the Haar measure dx so that $\int_{|x| \leq 1} dx = 1$. Finally let λ be a local F-parameter on Γ such that $\lambda = 0$ defines the point x_0.

PROPOSITION 1.1. *In the above situation, for $|\lambda|$ sufficiently small there is an expansion*

$$F(\lambda) \stackrel{def}{=} \int_{\varphi^{-1}(\lambda)} \kappa(t_\sigma) f \frac{|\omega|}{|\varphi^*(d\lambda)|} = \sum \theta(\lambda)|\lambda|^{\beta-1} \sum_{r=1}^{e(\theta,\beta)} m(\lambda)^{r-1} F_r(\theta,\beta).$$

The first sum runs over (θ,β) with $\mathcal{E}(\theta,\beta)$ nonempty. $F_r(\theta,\beta)$ is independent of λ but depends on r,θ,β,f,κ.

PROOF: See [L] for a proof and details. The only difference in our presentation is that [L] incorporates $\kappa(t_\sigma)$ directly into the definition of f which is not assumed to be locally constant.

We also need the explicit formula for $F_r(\theta,\beta)$ when $e(\theta,\beta) = 1,2$. Begin with $e(\theta,\beta) = 1$. By construction $\kappa(t_\sigma)/\theta(\lambda)$ extends generically to $E \in \mathcal{E}$. Let $m_{\theta,E}$ be its restriction to E. By construction $\omega/(\lambda^{\beta-1}d\lambda)$ extends generically to E. Let ω_E be its restriction. Since β is rational ω_E is defined up to a root of unity. Then we have

$$F_1(\theta,\beta) = \sum_{E\in\mathcal{E}(\theta,\beta)} PV \int_E m_{\theta,E} f|\omega_E|. \tag{1.2}$$

A principal value integral is required since ω_E may have poles and $m_{\theta,E}$ might not be locally constant. Now let $e(\theta,\beta) = 2$. We proceed as before but along the intersection of two divisors E, E' in $\mathcal{E}(\theta,\beta)$ the forms ω_E and $\omega_{E'}$ have simple poles, and the principal value integrals $PV \int_E m_{\theta,E} f|\omega_E|$ diverge. The difficulty stems from the fact that the form dx/x is scale invariant. The problem is overcome by truncating the integral on E and E' near the pole and adding a principal value integral on $E \cap E'$. We continue to write $F_1(\theta,\beta)$ as a sum of integrals over divisors but when $e(\theta,\beta) = 2$ the integrals must be regularized in this manner.

LEMMA 1.3. *Suppose that $\lambda = \alpha\mu_1^{a_1}\ldots\mu_n^{a_n}$ in local p-adic F-coordinates on a patch U containing $p \in D(F), D = E_1 \cap E_2$, that $\mu_i = 0$ defines E_i, $i = 1,2$, $a_1 = a_2 = 1$, $b_1 = b_2$, and $E_i \in \mathcal{E}(\theta,\beta)$ $i = 1,2$, for some (θ,β). Set*

$$\omega_D = Residue_D \omega_{E_1} = \left.\frac{\omega_{E_1}}{\frac{d\mu_2}{\mu_2}}\right|_D = Residue_D \omega_{E_2}, \text{ and } m_{\theta,D} = m_{\theta,E_1}|_D = m_{\theta,E_2}|_D.$$

Suppose that U is chosen small enough that $|\alpha|$ is constant on U. Suppose that the integrals on E_i are truncated by $|\mu_i| \geq q^{-m_i}$, $i = 1,2$, then the contribution to the term $F_1(\theta,\beta)$ from the pole D is given locally by

$$(1-\frac{1}{q}) \int_{U\cap D} m_{\theta,D}(1-M) f|\omega_D|.$$

where $M = m(\alpha) + a_1 m_1 + a_2 m_2 + \sum_{i>2} a_i m(\mu_i)$.

PROOF: This is a special case of the general formula found in [L].

§2. Background on Shalika Germs

Let G be a reductive group over a p-adic local field F of characteristic zero, and let T be a Cartan subgroup over F. For every unipotent orbit O in $G(F)$ let μ_O denote an invariant measure on O. Let dg be an invariant measure on $T(F) \setminus G(F)$. Shalika [Sh] has shown that there exist functions $\Gamma_O(\gamma)$ called germs defined on the regular elements of $T(F)$ for all unipotent classes O in $G(F)$ such that for every $f \in C_c^\infty(G)$, the space of locally constant functions of compact support on $G(F)$, there is a neighborhood V_f of the identity element in $T(F)$ in which the expansion

$$\int_{T(F)\setminus G(F)} f(x^{-1}\gamma x)dg = \sum_O \mu_O(f)\Gamma_O(x) \text{ holds for } \gamma \text{ regular in } V_f.$$

If $g \in (T \setminus G)(F)$, then $\sigma(g)g^{-1}$ for $\sigma \in Gal(\overline{F}/F)$ defines a cocycle of $H^1(Gal(\overline{F}/F), T)$. Now let dg denote an invariant measure associated to an invariant form on $T \setminus G$. Similarly normalize measures μ_O on unipotent classes belonging to the same stable conjugacy class by fixing an invariant form on the stable conjugacy class. For $h \in G(\overline{F})$ such that $\sigma(h)h^{-1}$ gives a cocycle of $Gal(\overline{F}/F)$ in Z the center of G, define f_h by $f_h(x) = f(h^{-1}xh)$. Let κ be a character on $H^1(Gal(\overline{F}/F), T)$.

We form a κ-orbital integral and take its germ expansion:

$$\Phi^{T,\kappa}(\gamma, f) \stackrel{def}{=} \int_{(T\setminus G)(F)} \kappa(\sigma(g)g^{-1}) f(g^{-1}\gamma g)dg = \sum_O \mu_O(f)\Gamma_O^{T,\kappa}(\gamma) \qquad \gamma \text{ in } V_f.$$

Comparing the κ-orbital integrals of f and f_h it follows easily that

$$\Gamma_{O^h}^{T,\kappa} = \Gamma_O^{T,\kappa}\kappa(\sigma(h)h^{-1}). \tag{2.1}$$

The character κ restricted to $H^1(Gal(\overline{F}/F), Z)$ depends only on the endoscopic group H defined by (T, κ) and not directly on T (see for instance [H]). For background on endoscopic groups see [L2]. We may therefore write for an *adjoint* conjugacy class O (that is, the inverse image of an F-orbit in the adjoint group)

$$\mu_O^H \stackrel{def}{=} \sum \mu_{O'} \kappa(\sigma(h)h^{-1}) \tag{2.2}$$

The sum runs over all F-classes O' in the adjoint class, and h is determined by $O' = O_1^h$ for a fixed F-class O_1 in O. Using this fixed choice $O_1 \subseteq O$ write $\Gamma_O^{T,\kappa}$ for $\Gamma_{O_1}^{T,\kappa}$. Then the germ expansion becomes

$$\Phi^{T,\kappa}(\gamma, f) = \sum_{O \text{ adjoint}} \mu_O^H(f)\Gamma_O^{T,\kappa}(\gamma). \tag{2.3}$$

All ordinary orbital integrals may be recovered as linear combinations of κ-orbital integrals. One advantage of considering κ-orbital interals instead of ordinary orbital integrals is that one is able to group F-classes belonging to the same adjoint class together in this way. In all that follows a germ is associated to an *adjoint* unipotent conjugacy class using the measures μ_0^H.

We will make use of the following results from Harish-Chandra and Rogawski. Say that an orbit O is r-regular if $r = (\dim\ C_G(u) - rank\ G)/2$ for $u \in O$. If $r = 0, 1$ the classes are also called regular and subregular respectively. Let Z_G^0 be the connected center of G. For details on normalizations of measures and proofs we refer to [H-Ch] and [R].

PROPOSITION 2.4. *(1) If $z \in Z_G^0(F)$ and $\gamma \in T(F)$ lie sufficiently close to the identity then $\Gamma_O^{T,\kappa}(z\gamma) = \Gamma_O^{T,\kappa}(\gamma)$.*

(2) If $X \in Lie\ G(F)$ is regular and $exp(X)$ is sufficiently close to the identity then

$$\Gamma_O^{T,\kappa}(exp(t^2X)) = |t|^{2(r-r_0)}\Gamma_O^{T,\kappa}(exp(X))$$

for $t \in F^\times$ sufficiently small for every r-regular class O, $r_0 = (\dim\ G - rank\ G)/2$.

(3) Let M be the connected centralizer of a semisimple element γ_0 in T. Then for every $f \in C_c^\infty(G)$ there exists $f^M \in C_c^\infty(M)$ such that

$$\Phi_G^T(\gamma, f) = \Phi_M^T(\gamma, f^M)$$

for regular elements γ in a sufficiently small neighborhood V_f of γ_0 in $T(F)$.

(2.5) Statement (2.4.1) tells us that the centers are mostly irrelevant to the study of germs. By passing to the derived group and then to the simply connected cover we may assume that G

is simply connected or semi-simple. Notice that the function $\kappa(\sigma(g)g^{-1})$ on $(T \setminus G)(F)$ always pulls back to the simply connected cover G_{sc} of the derived group and that $T_{sc} \setminus G_{sc} \to T \setminus G$ is an isomorphism over F. Also (T_{sc}, κ_{sc}) defines an endoscopic group H' of G_{sc} which differs from the endoscopic group H of G defined by (T, κ) only by a central factor. The simply connected semi-simple group is more difficult to deal with than say the adjoint group because there are more endoscopic groups associated to the simply connected groups.

(2.6) By combining (2.4.3) with (2.4.1) writing $T = T_1 Z_M$ where $T_1 \subseteq M_{der}$ for sufficiently small $\gamma_1 \in T_1(F)$, and sufficiently small $z_1, z_0 \in Z_M^0(F)$ we have

$$\Phi_G^T(\gamma_1 z_1 z_0, f) = \Phi_G^T(\gamma_1 z_0, f)$$

provided $C_G(z_0)^0 = M$. Thus we may consider the germs of the expansion of $\Phi_G^T(\gamma, f)$ near z_0 as functions on T_1 alone instead of T.

(2.7) Langlands and Shelstad [LS2] have defined transfer factors $\Delta_G^{T,\kappa}(\gamma)$ and have conjectured that for every $f \in C_c^\infty(G)$ there exists a function $f^H \in C_c^\infty(H)$ on the endoscopic group H associated to (T, κ) such that identifying Cartan subgroups in H and G we have

$$\Delta_G^{T,\kappa} \Phi^{T,\kappa}(\gamma, f) \stackrel{?}{=} \Delta_H^{T,st} \Phi^{T,st}(\gamma, f^H) \tag{2.8}$$

for all (T, κ) associated to H. The integral on the right is a *stable* orbital integral on H (that is, the character κ is trivial). We define $\Delta_G^{T,\kappa}$ for γ regular and sufficiently close to 1 for G quasi-split by the condition

$$\Delta_G^{T,\kappa} \Phi_G^{T,\kappa}(\gamma, f) = \mu_0^H(f) \tag{2.9}$$

for 0 the regular adjoint unipotent class and f supported on regular elements of G. The factor $\Delta_H^{T,st}$ is defined similarly. Implicit in our definition of transfer factors is a choice of measure μ_0^H. In [LS2] the factors $\Delta_G^{T,\kappa}$ and $\Delta_H^{T,st}$ are combined into a canonically defined single factor but they show the definition by (2.9) is equivalent to theirs (up to a scalar). For the reduction of this problem (2.8) to the problem of matching germ expansions of $\Delta^{T,\kappa}\Phi^{T,\kappa}$ and $\Delta_H^{T,st}\Phi^{T,st}$ near the identity element see [LS3].

Proposition (2.4) remains true with minor modifications when the transfer factor is included. From this point on we shift notation to let $\Gamma_0^{T,\kappa}$ be the germ of $\Delta^{T,\kappa}\Phi^{T,\kappa}$. By (2.9) we conclude that (2.4.1) holds for G quasi-split. More generally, [LS2] shows that there is a character θ of Z_G^0

such that $\Delta(z\gamma) = \theta(z)\Delta(\gamma)$. By the explicit description of the transfer factor in [LS2] (2.4.2) holds with $r_0 = 0$. A version of (2.4.3) with transfer factors is proved in (2.11).

Statement (2.4.1) gives a necessary condition for the transfer of germs to an endoscopic group. Identifying the connected center Z_H^0 of H with a subgroup $T \subseteq G$ it says that the germs of κ-orbital integrals on G associated to H should be invariant by Z_H^0.

The rest of this section shows, roughly speaking, that to prove (2.6) for a fixed (T, κ) when H is a product $H = H_1 \times H_2$ it is often enough to prove that the germs of $\Delta^{T,\kappa}\Phi^{T,\kappa}(\gamma, f)$ have a decomposition of the form $\Gamma_O^{T,\kappa} = \sum a_i b_i$ where a_i are functions on $T \cap (H_1 \times \{1\})$ and b_i are functions on $T \cap (\{1\} \times H_2)$. Since the germs on a product of groups equal the products of germs on the individual groups it is clear that such a decomposition is a necessary condition for (2.8) to hold. This section does *not* show that the choice of f^H can be made independently of (T, κ). This will be shown later in the special case $G = GSp(4)$.

For any reductive group G let G_{qs} denote a quasi-split inner form of G. It is unique up to an isomorphism over F. There is an injection of stable conjugacy classes of Cartan subgroups in G to stable conjugacy classes of Cartan subgroups in G_{qs}. If T is a Cartan subgroup over F in G, write T_{qs} for an image in G_{qs} and for $\gamma \in T$ write γ_{qs} for the corresponding element in G_{qs}. T and T_{qs} are then isomorphic over F and this isomorphism may be used to identify characters on $H^1(Gal(\overline{F}/F), T)$ and $H^1(Gal(\overline{F}/F), T_{qs})$.

If S is a torus over F in G let $C(S)$ denote its centralizer in G.

DEFINITION 2.10. *The κ-orbital integrals on G are said to have quasi-split reduction (QSR) if for every triple (S, T, κ), $1 \neq S \subseteq T$, S torus over F, T Cartan subgroup over F, κ character on $H^1(Gal(\overline{F}/F), T)$ and for every function $f \in C_c^\infty(C(S))$ there exists a function $f_{qs} \in C_c^\infty(C(S)_{qs})$ such that*

$$\Phi_{C(S)}^{(T,\kappa)}(\gamma, f) = \Phi_{C(S)_{qs}}^{(T_{qs},\kappa)}(\gamma_{qs}, f_{qs})$$

for every $\gamma \in T'(F)$ (the regular elements of T). We note that this condition is independent of the choice of T_{qs}.

PROPOSITION 2.11. *Suppose that G is quasi-split and the κ-orbital integrals of G have quasi-split reduction. Let (S, T, κ) be a triple as in the definition of QSR. Let $f \in C_c^\infty(G)$ and let $s \in S(F)$ be an element sufficiently close to the identity such that $C_G(s) = C(S)$. Then there exists a neighborhood V (depending on f) of s in $T(F)$ and a function f_{qs} in $C_c^\infty(C(S)_{qs})$ (depending on s) such that*

$$\Delta_G^{T,\kappa}\Phi_G^{(T,\kappa)}(\gamma,f)=\Delta_{C(s)_{qs}}^{T_{qs},\kappa}\Phi_{C(S)_{qs}}^{(T_{qs},\kappa)}(\gamma_{qs},f_{qs})\qquad \textit{for}\quad \gamma\in V\cap T'(F).$$

PROOF: This is no more than (2.4.3) combined with the definition of QSR. A few details will make this clear.

Set $M=C(S)$. The map $i:T\hookrightarrow M$ gives $i_*:H^1(Gal(\overline{F}/F),T)\to H^1(Gal(\overline{F}/F),M)$. If $Tg\in(T\setminus G)(F)$ then the cocycle $i_*(\sigma(g)g^{-1}),\sigma\in Gal(\overline{F}/F)$ belonqs to $H^1(Gal(\overline{F}/F),M)$. Let $g_1,\dots,g_r\in G(\overline{F}),Tg_1,\dots,Tg_r\in(T\setminus G)(F)$ be such that $i_*(\sigma(g_i)g_i^{-1}),i=1,\dots,r$ are representatives of the classes in $H^1(Gal(\overline{F}/F),M)$ so obtained. Then M^{g_i} is an inner form of M. Set $T_i=T^{g_i},M_i=M^{g_i},\gamma_i=\gamma^{g_i}$.

It is easy to check that $(T\setminus G)(F)$ is a disjoint union of $X_i=g_i((T_i\setminus M_i)(F))G(F)$, $i=1,\dots,r$. The κ-orbital integral on G is an integral over $(T\setminus G)(F)$ which breaks up as a sum of integrals over the X_i. First we prove a version of the proposition for each X_i, i.e., that there exists $(f_i)_{qs}\in C_c^\infty(M_{qs})$ such that

$$\int_{(g_i^{-1}X_i)}\kappa_i(\sigma gg^{-1})f(g^{-1}\gamma_i g)dg=\Phi_{M_{qs}}^{T_{qs},\kappa}(\gamma_i,(f_i)_{qs})$$

where κ_i is the character on $H^1(Gal(\overline{F}/F),T_i)$ obtained by identifying T and T_i by the isomorphism $\gamma\mapsto\gamma_i$.

Let $\epsilon_1^i,\dots,\epsilon_t^i$ be the elements of the image of $(T_i\setminus M_i)(F)$ in $H^1(Gal(\overline{F}/F),T_i)$ and let $m_1^i,\dots,m_t^i$ be representatives in$(T_i\setminus M_i)(F)$. Then dropping super and subscript i's on $m_j^i\ \epsilon_j^i,T_i$ we have, using the definition of X_i

$$\begin{aligned}\int_{g_i^{-1}X_i}&=\sum_j\kappa_i(\epsilon_j)\int_{T^{m_j}(F)\backslash G(F)}f(g^{-1}\gamma_i^{m_j}g)dg\\&=\sum_j\kappa_i(\epsilon_j)\int_{T^{m_j}(F)\backslash M_i(F)}f_i(m^{-1}\gamma_i^{m_j}m)dm\qquad\text{(by 2.4.3)}\\&=\Phi_{M_i}^{T,\kappa}(\gamma_i,f_i)\\&=\Phi_{M_{qs}}^{T_{qs},\kappa}(\gamma_{qs},(f_i)_{qs})\qquad\text{by the definition of } QSR\ .\end{aligned}$$

Thus combining the X_i and $(f_i)_{qs}$

$$\Delta_G^{T,\kappa}\Phi_G^{T,\kappa}(\gamma,f)=\delta\ \Delta_{M_{qs}}^{T_{qs},\kappa}\Phi_{M_{qs}}^{T_{qs},\kappa}(\gamma_{qs},f'_{qs})=\Delta_{M_{qs}}^{T_{qs},\kappa}\Phi_{M_{qs}}^{T_{qs},\kappa}(\gamma_{qs},\delta f'_{qs}),\delta=\Delta_G^{T,\kappa}/\Delta_{M_{qs}}^{T_{qs},\kappa}.$$

Fix s small enough so that (2.9) holds, then this equation becomes $\mu_0^H(f)=\delta\mu_{O'}^{H'}(f'_{qs})$ where O' is the regular unipotent class of M_{qs} and (T_{qs},κ) defines the endoscopic group H'. Thus δ and hence $f_{qs} = \delta f'_{qs}$ depends on $\gamma \in T(F)$, $\gamma_{qs} \in T_{qs}(F)$ only through $s \in S(F)$ in a sufficiently small neighborhood of the identity and sufficiently near s. The proposition follows.

As above let G be a reductive group over a p-adic field F of characteristic zero, and let H be an endoscopic group of G. Suppose that up to a finite center H is a product $H_1 \times H_2$ over F. Then we associate to G and H an intermediate group M as follows.

Select a Cartan subgroup T over F in H. Corresponding to $H_1 \times H_2$ we write $T = T_1T_2$ with $T_1 \cap T_2$ finite. Identify $T = T_1T_2$ with a Cartan subgroup in G. It is determined up to stable conjugacy. Let $M = (C_G(T_1) \times C_G(T_2))/\{(x, x^{-1}) : x \in T\}$. For example, if $G = Sp(2n)$, $H = Sp(2i) \times SO(2(n-i))$, then $M = Sp(2i) \times Sp(2(n-i))$.

LEMMA 2.12. *M_{qs} is independent of the choice of $T \subseteq H$.*

PROOF: It is enough to check that T and $T' = T_1T'_2$ give the same M_{qs}. Select $g \in Q(\overline{F})$ such that $T_2^g = T'_2$ where $Q = C_G(T_1)$. $C_{C_G(T_2)}(T_1T_2) = C_G(T_1T_2) = T_1T_2$ so T_1T_2 is a Cartan subgroup of $C_G(T_2)$. It follows that there is a Cartan subgroup T_{der} of $C_G(T_2)_{der}$ contained in T_1.

If $m \in C_G(T_2)_{der}$ $\sigma \in Gal(\overline{F}/F)$ then $\sigma(m)^{\sigma(g)g^{-1}g} = \sigma(m^g)$, $\sigma(g)g^{-1} = w_\sigma \in N_Q(T_2)$. By the definition of Q, $t^{w_\sigma} = t$ for $t \in T_{der}(\overline{F})$ $(\subseteq T_1(\overline{F}))$. Since T_{der} is a Cartan subgroup, $w_\sigma \in T_{der}(\overline{F})$ so that $C_G(T_2)_{der}$ and $C_G(T'_2)_{der}$ are inner forms. Using $TC_G(T_2)_{der} = C_G(T_2)$, $TC_G(T'_2)_{der} = C_G(T'_2)$ the result follows easily.

The following simple lemma is the key to what follows.

LEMMA 2.13. *Let k be any field. Let X, Y be sets endowed with decreasing filtrations $X = X_0 \supseteq X_1 \supseteq \ldots, Y = Y_0 \supseteq Y_1 \supseteq \ldots$. Let $\mathcal{F}_X, \mathcal{F}_Y$ be k-vector spaces of k-valued functions on X and Y respectively. Suppose a function φ on $X \times Y$ has the form $\varphi = \sum_{i=1}^p a_ib_i, a_i \in \mathcal{F}_X, b_i \in \mathcal{F}_Y$, $i = 1, \ldots, p$. Suppose further that there exist functions $a'_i \in \mathcal{F}_X, b'_i \in \mathcal{F}_Y$, $i = 1, \ldots, q$ and an integer $n \geq 0$ such that for every $x \in X_n$ there is a j_x such that*

$$\varphi(x, \cdot) = \sum_{i=1}^{q} a'_i(x)b'_i \quad \text{as functions on } Y_{j_x}.$$

Then there exists a positive integer N such that $\varphi = \sum_{i=1}^q a'_ib'_i$ on $X_N \times Y_N$. Moreover, if the $a'_1, \ldots, a'_p$ are linearly independent over k on X, and if $b_1, \ldots, b_q$ are linearly independent over k on Y, then there exists a matrix of constants $e_{ij} \in k$, $i = 1, \ldots, p$ $j = 1, \ldots, q$ and a positive integer N' such that

$$\varphi = \sum_{i,j}^{p,q} a'_i e_{ij} b_j \text{ on } X_{N'} \times Y_{N'}.$$

PROOF: Choose $n_1 \geq n$ so that the dimension of the span of the functions $\{a_i\}_{i=1}^p$ or $\{a'_i\}_{i=1}^q$ (resp. $\{b_i\}$ or $\{b'_i\}$) is constant on X_i (resp. Y_i) for $i \geq n_1$. By combining terms in the sum we may assume that the $\{b_i\}$ (resp. the $\{a'_i\}$) are k-linearly independent for all Y_i, $i \geq n_1$ (resp. X_i, $i \geq n_1$). Assuming now that $a'_1, \ldots, a'_q$ are linearly independent on X_i, $i \geq n_1$, we may select $x_1, \ldots, x_q \in X_{n_1}$ such that the matrix $A' = (a'_j(x_i))$ is invertible. Then if $A = (a_j(x_i)), b = (b_1, \ldots, b_p)^t$, $b' = (b'_1, \ldots, b'_q)^t$ it is clear that $Ab = A'b'$ on Y_j for $j \geq n_2 \overset{def}{=} \max_{x \in S}(j_x)$, $S = \{x_1 \ldots x_q\}$ or that $b' = A'^{-1}Ab$. Set $a = (a_1, \ldots, a_p)$, $\quad a' = (a'_1, \ldots, a'_q)$. Suppose there exists $x \in X_{n_1}$ such that $a(x) - a'(x)A'^{-1}A \neq 0$. Then by the assumed linear independence of the b's we have on Y_j, $j \geq max(j_x, n_2)$ $0 \neq (a(x) - a'(x)A'^{-1}A)b = a(x)b - a'(x)b' = 0$. This contradiction shows that $a = a'A'^{-1}A$ on X_{n_1}. Consequently for $N = max(n_1, n_2)$ and on $X_N \times X_N$, $\varphi = \sum a_i b_i = ab = a'A'^{-1}Ab = a'b' = \sum a'_i b'_i$. For the second statement of the lemma we take $e_{ij} = (A'^{-1}A)_{ij}$.

Suppose that up to finite center H is equal to $H_1 \times H_2$. Let (T, κ) be a pair associated to H and let $f \in C_c^\infty(G)$. Write $T = T_1 T_2$, $T_1 \cap T_2$ finite as above. Let $A_1, \ldots, A_j$ be the germs of (T_{qs}, κ) on $C(T_2)_{qs}$ normalized with transfer factors as in (2.9). These may be considered functions on T_1. Let $B_1, \ldots, B_k$ be the normalized germs of (T_{qs}, κ) on $C(T_1)_{qs}$, again considered as functions on T_2.

PROPOSITION 2.14. *Suppose G is quasi-split and that the κ-orbital integrals on G have quasi-split reduction. Suppose further that κ-orbital integrals have the form*

$$\varphi \overset{def}{=} \Delta_G^{T,\kappa} \Phi_G^{(T,\kappa)}(\gamma, f) \;=\; \sum a_i(\gamma_1) b_i(\gamma_2) \qquad \gamma = \gamma_1 \gamma_2 \qquad \gamma_i \in T_i$$

in a small neighborhood of the identity $1 \in T(F)$.

(1) There exist constants e_{ij} such that

$$\varphi = \sum_{i,j} A_i(\gamma_1) e_{ij} B_j(\gamma_2)$$

in some possibly smaller neighborhood of the identity.

(2) $\varphi = \Delta_{M_{qs}}^{T,\kappa} \Phi_{M_{qs}}^{T,\kappa}(f^M)$ for some function $f^M \in C_c^\infty(M_{qs})$ on a sufficiently small neighborhood of the identity.

PROOF: (1) Proposition 2.11 shows that for every $\gamma_1 \in T_1(F)$ such that $C_G(\gamma_1) = C(T_1)$ there exists a function $F = F_{\gamma_1}$ such that

$$\varphi = \Delta^{T_{qs},\kappa}_{C(T_1)_{qs}} \Phi^{(T_{qs},\kappa)}_{C(T_1)_{qs}}(\gamma_{qs}, F_{\gamma_1})$$

expanding the right side in a germ expansion

$$\varphi = \sum_i B_i \mu_i(F_{\gamma_1})$$

where μ_i are measures on unipotent orbits of $C(T_1)_{qs}$. These expansions hold in some neighborhood $V \subseteq T_2(F)$ depending on γ_1. Thus Lemma 2.13 holds with $a'_i(\gamma_1) = \mu_i(F_{\gamma_1})$ $\quad b'_i = B_i$, so that

$$\varphi = \sum A'_i B_i \qquad \text{for some } A'_i$$

on some neighborhood of the identity. Interchanging the roles of T_1, T_2 we apply the lemma again to conclude

$$\varphi = \sum A_i B'_i \qquad \text{for some } B'_i$$

on some neighborhood of the identity. By combining terms we may assume that the A_i are linearly independent on every sufficiently small neighborhood of the identity. Likewise for B_i. We apply the last part of the lemma this time with

$$a'_i = A_i, \quad b'_i = B'_i, \quad b_i = B_i, \quad a_i = A'_i$$

to obtain $\varphi = \sum A_i e_{ij} B_j$ in some neighborhood of the identity.

(2) The germs on M_{qs} are $A_i B_j$. The result is immediate.

§3. A VARIETY WHICH COMPUTES SHALIKA GERMS

This section reviews a geometric approach to Shalika germs. If X is a variety over F we often write $x \in X$ instead of $x \in X(\overline{F})$.

Fix a curve Γ in T whose tangent direction at the identity does not lie in a singular hyperplane. Let λ be a local parameter on Γ such that $\lambda = 0$ gives the identity element of T. Suppose that $\Gamma(\lambda)$ is regular for $\lambda \neq 0$, and let $\Gamma^0 = \Gamma \setminus \{0\}$. There is a variety Y_Γ over Γ which fits into the diagram

$$\begin{array}{ccccc} \Gamma^0 \times T \setminus G & \xrightarrow{i} & Y_\Gamma & \xrightarrow{\pi} & G \\ & & \downarrow \varphi & & \\ & & \Gamma. & & \end{array}$$

The following properties of Y_Γ are known [H],[L].

(3.1) If T and Γ are over F then $Y_\Gamma, \varphi, \pi, i$ are also defined over F.

(3.2) $\pi : Y_\Gamma \to G$ is proper.

(3.3) i embeds $\Gamma^0 \times T \setminus G$ as an open subvariety of Y_Γ.

(3.4) If G is made to act on $\Gamma^0 \times T \setminus G$ by translations on the second factor, on G by inner automorphisms and trivially on Γ then there is a G action on Y_Γ which makes i, φ, π into G-equivariant maps.

(3.5) $\pi \circ i(\gamma, g) = \gamma^g; \qquad \varphi \circ i(\gamma, g) = \gamma.$

(3.6) Let E be an irreducible component of $\varphi^{-1}(0)$. Then $\pi(E)$ is the closure of a stable unipotent class O in G. Call E an O-divisor. If O is regular (resp. subregular) we also call E a regular (resp. subregular) divisor. There is exactly one irreducible component E_0 such that $\pi(E)$ coincides with the unipotent variety in G. E_0 is isomorphic to the Springer resolution $\{(u, B) | u \in B,\, u \text{ unipotent in } G,\, B \text{ a Borel subgroup } \}$ of the unipotent variety. Identifying E_0 and the Springer resolution, π becomes $\pi(u, B) = u$.

(3.7) A Zariski open patch of Y_Γ may be described in local coordinates as follows. It depends on a choice of opposite Borel subgroups (B_∞, B_0). Let Φ be the root system of G with positive roots Φ^+, let $p = |\Phi^+|$ be the number of positive roots and let Δ be the set of positive simple roots. Consider the affine space $\mathbf{A}^{2p+1}$ the coordinates being labelled

$$\begin{array}{ll} w(\gamma) & \gamma \in \Phi^+ \setminus \Delta \\ z(\alpha) & \alpha \in \Delta \\ x(\gamma) & \gamma \in \Phi^+ \\ \lambda & \text{(identifying } \lambda \text{ with its pull-back to } Y_\Gamma). \end{array}$$

There is an open set $Y_U = Y_U(B_\infty, B_0)$ of Y_Γ which is isomorphic to an open set of the product $\mathbf{A}^p \times Z$ where $Z = Z_\Phi$ is the Zariski closure in $\mathbf{A}^{2p+1}$ of the variety defined for $\lambda \neq 0$ by the equations

$$\lambda w(\gamma) = x(\gamma) \Pi z(\alpha)^{m(\alpha)} \quad \gamma = \sum m(\alpha)\alpha \quad \alpha \in \Delta,$$

where $w(\alpha) = 1$ for $\alpha \in \Delta$.

We write $v(\gamma) \quad \gamma \in \Phi^+$ for coordinates on the factor $\mathbf{A}^p$ of $\mathbf{A}^p \times Z$.

(3.8) There is a G-invariant differential form of maximal degree ω_Y which is non-vanishing for $\lambda \neq 0$ which on Y_U equals

$$\omega_Y = d\lambda \bigwedge_{\gamma \in \Phi^+} dx(\gamma) \bigwedge_{\gamma \in \Phi^+} dv(\gamma).$$

(3.9) Let E be a splitting field of T and suppose that T is defined over F. Then the function $\sigma(g)g^{-1}$ for $\sigma \in Gal(E/F)$, $g \in (T \setminus G)(F)$ pushes forward to a function $t'_\sigma \in T(K_E)$, K_E the rational functions on $Y_\Gamma \times_{Spec(F)} Spec(E)$. There is a cocycle $\delta^T_{\Gamma,\lambda}$ in $T(E(\lambda)) \subseteq T(K_E)$ such that $t_\sigma =^{def} \delta^T_{\Gamma,\lambda} t'_\sigma$ satisfies condition (4) preceding Proposition 1.1. The factor $\delta^T_{\Gamma,\lambda}$ is determined for quasi-split groups by the condition that $\kappa(t_\sigma)$ extends generically to the regular divisor E_0 and

$$\int_{E_0} \pi^*(f)\kappa(t_\sigma)|\omega_{E_0}| = \mu^H_O(f) \quad \forall f \in C^\infty_c(G)$$

with O the unique adjoint regular unipotent class and μ^H_O determined as in (2.2).

$$F(\lambda) = \int_{\varphi^{-1}(\lambda)} \pi^*(f)\kappa(t_\sigma)\frac{|\omega_Y|}{|\varphi^*(d\lambda)|} \;=\; \Delta^{T,\kappa}\Phi^{T,\kappa}(\Gamma(\lambda), f) \tag{3.10}$$

for λ sufficiently small and non-zero.

(3.11) The variety Y_Γ, form ω_Y, functions t_σ, $\pi^*(f)$,etc., satisfy all of the conditions of Proposition 1.1 *except* that Y_Γ is not in general smooth, and the irreducible components of $\varphi^{-1}(0)$ are not in general divisors with normal crossings. By blowing up Y_Γ one obtains a variety $\widetilde{Y}_\Gamma$ proper over Y_Γ for which all of the conditions of Proposition 1.1 are met.

It is often convenient to consider all possible tangent directions at the identity in T simultaneously rather than fixing one direction. We introduce parameters $T_1, \ldots, T_\ell$ $\ell = \dim T$, and let the vector $(T_1, \ldots, T_\ell)$ denote the tangent direction (in Lie T). Let R be the field of rational functions in $T_1, \ldots, T_\ell$ over E. There is then a natural action of the Weyl group on R fixing F. We often consider the varieties Y_Γ, G, Γ, over R instead of F. Set $K_R = K_E \otimes_E R$.

(3.12) Another system of coordinate patches are $Y_1(B_\infty, B_0, \Sigma)$ parametrized by pairs (B_∞, B_0) of opposite Borel subgroups together with a map $\Sigma : \Delta \to \mathcal{W}$ where $\mathcal{W}$ is the set of Weyl chambers. As B_∞, B_0 and Σ vary, the patches $Y_1(B_\infty, B_0, \Sigma)$ cover Y_Γ. We also let $Y_1(B_\infty)$ denote the union of these patches for a fixed B_∞. The coordinates on $Y_1(B_\infty, B_0, \Sigma)$ are

$$\begin{array}{lll} z_1(W,\alpha) & w \in \mathcal{W} & \alpha \in \Delta \quad \textit{where } z_1(\Sigma(\alpha),\alpha) = \frac{\alpha(\gamma)-1}{\lambda}, \quad \gamma = \Gamma(\lambda). \\ z(\alpha) & \alpha \in \Delta & \\ x(\gamma) & \gamma \in \Phi^+ & \\ v(\gamma) & \gamma \in \Phi^+ & \\ \lambda & & \end{array}$$

The relation between $\Gamma^0 \times T\backslash G$, Y_U and $Y_1(B_\infty, B_0, \Sigma)$ is the following. Let T_0 be the intersection of B_0 and B_∞. Let N_0 and N_∞ be the unipotent radicals of B_0 and B_∞. Fix a Borel subgroup

B containing T. The other Borel subgroups containing T are then $B(W) \stackrel{def}{=} B^{\omega}$, where $W = \omega^{-1}W_+ \in \mathcal{W}$ and W_+ is the positive Weyl chamber with respect to B. Then for $g \in T\backslash G(\overline{F})$ such that $B(W)^g$ is opposite B_∞ for all W we may write for $t = \Gamma(\lambda)$ regular

$$(t^g, B(W)^g) = (t_0 \cdot n, B_0^{n_W})^v, \quad \text{with } n_W, v \in N_\infty, \quad t_0 \in T_0, \quad n \in N_0.$$

Fixing an order on root vectors X_γ we may write

$$n = \Pi exp(x(\gamma)X_\gamma), v = \Pi exp(v(\gamma)X_{-\gamma})$$

$$n_W = exp(z(\alpha_k)z_1(W_k,\alpha_k)X_{-\alpha_k}) \cdots exp(z(\alpha_1)z_1(W_1,\alpha_1)X_{-\alpha_1})$$

where $W = \omega^{-1}W_+$ with $\omega = \sigma_{\alpha_k} \cdots \sigma_{\alpha_1}$, a wall of type α_j separates the chambers W_{i+1} and W_i and $W_1 = W_+$. This gives the relation to $\Gamma^0 \times T\backslash G$.

To relate this to the patch Y_U, pick the function $\Sigma : \Delta \to \mathcal{W} \quad \Sigma(\alpha) = W_+ \quad \forall \alpha \in \Delta$. Select g_0 so that $T^{g_0} = T_0$, $B(W_+)^{g_0} = B_0$, $B(W)^{g_0} = B_0(W)$, and set $t^{g_0} = t_0$. The set $g_0T_0N_0N_\infty$ is open in G. If g lies in this open set we have the decomposition $g = g_0t_0'n'v$ which implies that $t^g = t_0^{n'v}$. We obtain

$$(t_0 \cdot n, B_0^{n_W})^v \;=\; (t_0^{n'}, B_0(W)^{n'})^v.$$

Define $x(\gamma)$, $z(\alpha)$ and $v(\gamma)$ as above and set $t_0^{-1}t_0^{n'} = \Pi\ exp((w(\gamma)\lambda)/(\Pi z(\alpha)^{m(\alpha)})X_\gamma)$ where $\gamma = \Sigma m(\alpha)\alpha$. By the relation $B_0^{n_W} = B_0(W)^{n'}$ we may consider the $w(\gamma)$ as rational functions of the variables $z_1(W,\alpha)$. The variables $w(\gamma)$ turn out to be independent of $z(\alpha)$. We identify the variables $x(\gamma)$, $v(\gamma)$, $z(\alpha)$ on $Y_U = Y_U(B_\infty, B_0)$ and $Y_1(B_\infty, B_0, \Sigma)$ and relate $w(\gamma)$ on Y_U to $z_1(W,\alpha)$ on $Y_1(B_\infty, B_0, \Sigma)$ through these relations.

(3.13) When G is split on an inner form of a split group, the action of $Gal(\overline{F}/F)$ on the variety Y_Γ is given as follows. Fix a choice of (B_∞, B_0); we assume that $T_0 = B_0 \cap B_\infty$ is a Cartan subgroup defined over F. Let G_* be a split inner form of G. Let E be a Galois extension of F which splits T_0 and T. Let $\varphi : G \to G_*$ be an isomorphism defined over E carrying T_0 to T_*. We may assume that T_* is split and that $T^{g_0} = T_0$ for some $g_0 \in G(E)$. We identify Weyl groups and Weyl chambers of $T_0, T_1\ T_*$ by g_0 and φ.

Since G_* is an inner form of G we may assume $\varphi^{-1} \circ \sigma^{-1} \circ \varphi \circ \sigma$ is an automorphism of G of the form ad A_σ^{-1}, $A_\sigma \in N_{G_{\mathrm{adj}}}(T_0)$ and that $A_\sigma = 1$ if $\sigma \in Gal(\overline{F}/E)$. In other words, we may take $A_{\sigma^{-1}}$ to be a cocycle of $Gal(E/F)$.

Let B be a Borel subgroup containing T, for $\sigma \in Gal(E/F)$ define σ_T in the Weyl group W of T by $\sigma(B^w) = B^{w\sigma_T^{-1}}$ for all $w \in W$. Notice that σ_T exists because G is an inner form of a split group. Also $\sigma \mapsto \sigma_T$ is a homomorphism. Let W_T be the image of $Gal(E/F)$ in W. It is often convenient to identify W_T (using g_0) with a subgroup of the Weyl group of T_0. A different choice of g_0 will lead to a conjugate subgroup of the Weyl group of T_0.

Define an action of $\sigma \in Gal(\overline{F}/F)$ on Y_U by

$$\sigma\left[(g,(B_0^{nw}))^v\right] \stackrel{def}{=} \left(\sigma(g),(\sigma(B_0)^{\sigma(n_{\sigma_T^{-1}w})})\right)^{\sigma(v)}.$$

This extends to an action on Y_Γ. Notice that if $g^v = t^{g'}$, and $B_0^{nw\,v} = B^{wg'}$ with $t \in \Gamma^0(F)$ $g' \in (T\setminus G)(F)$ then $\sigma(t^{g'},(B^{wg'})) = (t^{g'},(\sigma(B)^{w\sigma_T\sigma(g')})) = (t^{g'},(B^{wg'}))$ is an F-rational point. This action may be more conveniently described by twisting an action which is independent of T. To this end, define an action σ_* of $\sigma \in Gal(\overline{F}/F)$ on Y_U by

$$\sigma_*\left[(g,B_0^{nw})^v\right] \stackrel{def}{=} \left(\varphi^{-1}\circ\sigma\circ\varphi(g),\ B_0^{\varphi^{-1}\circ\sigma\circ\varphi(nw)}\right)^{\varphi^{-1}\circ\sigma\circ\varphi(v)}.$$

T_* is split and $\varphi^{-1}\circ\sigma\circ\varphi(B_0) = B_0$. Then by the definition of A_σ

$$\sigma_*^{-1}\sigma\left[(g,\ B_0^{nw})^v\right] = \left(g,\ (B_0^{n_{\sigma_T^{-1}}w})\right)^{vA_\sigma}.$$

The action of A_σ by conjugation commutes with the action of σ_T by permutation of Borel subgroups.

The action of σ through σ_T may be expressed as a W_T action on K_R. If we wish to make the dependence of the coordinates $x(\gamma)$ on the elements $b \in B_0$ explicit we write $x_\gamma(b)$ for $x(\gamma)$. When A_σ is trivial and σ_T is a reflection $\sigma_{\alpha'}$ through a simple root α' then $\sigma_{\alpha'}$ acts on the coefficients $x_{\gamma'}(b)$ of b by

$$\sigma_{\alpha'} : x_{\gamma'}(b) \mapsto x_{\gamma'}(b') \quad b' = b^{n_{W(\sigma_{\alpha'})}^{-1}} \quad \gamma' \text{ a positive root.}$$

$$n_{W(\sigma_{\alpha'})} = exp(z(\alpha')z_1(W_+,\alpha')X_{-\alpha'}).$$

This together with $\sigma(\lambda) = \lambda$ and (3.7) determines the action of $Gal(\overline{F}/F)$ on Y_U through σ_T.

To calculate the effect of A_σ when $\sigma_T = 1$ and A_σ maps to a simple reflection, write $vA_\sigma = b_0v'$ with $b_0 \in B_0$, $v' \in N_\infty$ by using the matrix identity

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 0 & -\varsigma a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1/x & -\varsigma \\ 0 & -\varsigma x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/\varsigma x & 1 \end{pmatrix}.$$

(3.14) We consider cocycles on $W \times A$ instead of $Gal(\overline{F}/F)$ whenever convenient (meaning almost always). (We should warn the reader that sometimes we treat $W \times A$ as Galois groups and even apply results of class field theory to $W \times A$ with the understanding that the map from $Gal(\overline{F}/F)$ to $W \times A$ may be used to give precise meaning to our statements.) We have a function t''_σ given on generators by

$$\sigma_{\alpha'} \times A_{\alpha'} \mapsto \left(\frac{\xi_{\alpha'} + \alpha'(T)z(\alpha')}{z(\alpha')}\right)^{-\alpha'^v} \qquad \sigma_{\alpha'} \mapsto (z(\alpha'))^{\alpha'^v} \qquad \alpha' \text{ simple}$$

where $A_{\alpha'}$ is an element of A in the normalizer of T_0 which maps to the simple reflection $\sigma_{\alpha'}$ in the Weyl group. Here $\xi_{\alpha'}$ is the element such that $v = exp(\xi_{\alpha'} X_{-\alpha'})v^{\alpha'}$, $v^{\alpha'} \in N_{-\alpha'}$ the unipotent radical of the parabolic subgroup containing B_∞ associated to the root $-\alpha'$. Also $\alpha'(T)$ is the root evaluated on the tangent direction. Using the cocycle relation $b_{\sigma\tau} = \sigma(b_\tau)b_\sigma$ we have

$$A_{\alpha'} \mapsto (\xi_{\alpha'})^{\alpha'^v}, \quad \sigma_{\alpha'} \mapsto (z(\alpha'))^{\alpha'^v}.$$

We have $t_\sigma = t''_\sigma b_\sigma$ where t_σ is the cocycle of (3.9) and b_σ depends on points of Y_Γ only through λ. To determine b_σ for quasi-split groups we use the fact that the cycycle t_σ restricted to the divisor E_0 becomes constant (3.9) on each regular unipotent class.

LEMMA 3.15. *Define b'_σ by $\sigma_{\alpha'} \longmapsto (1/\lambda)^{\alpha'^v}$, $A_{\alpha'} \longmapsto 1$. Then there is a factor b''_σ which is independent of λ with values in $T(E)$ such that $b'_\sigma b''_\sigma t''_\sigma = t_\sigma$.*

PROOF: $t''_\sigma b'_\sigma$ is given by

$$\sigma_{\alpha'} \mapsto (1/x(\alpha'))^{\alpha'^v}, \; A_{\alpha'} \mapsto (\xi_{\alpha'})^{\alpha'^v} \qquad \alpha' \quad \text{simple}.$$

So it is enough to verify that the action of $W \times A$ on $x(\alpha')$, $\xi_{\alpha'}$ is independent of the tangent direction. For $\sigma \in W$ the action is given by (3.13). But $n_{W(\sigma_\alpha)} = 1$ on E_0 so that by 3.13 $\sigma_\alpha(x(\alpha')) = x(\alpha')$. By (3.13) the action of $\sigma \in W$ on v is given by $\sigma : v \mapsto n_{\sigma_T^{-1} W_+} v = v$ (since $n_{\sigma_T^{-1} W_+} = 1$ on E_0). The action of A is given by (3.13) as $(b, B_0)^v \mapsto (b, B_0)^{vA_\sigma}$ where A_σ is an element of the normalizer of T_0 independent of the tangent direction. So on E_0 we have $(u, B_0)^v \mapsto (u, B_0)^{vA_\sigma}$ which is clearly independent of the tangent direction.

A factor $b''_\sigma \in T(E)$ may be needed to make t_σ agree with the cocycles of the transfer factor [LS2] for various reasons. The boundary of $t''_\sigma \in T(K_R)$ lies in $T(E)$ but is not necessarily zero. Thus b''_σ is needed to make $b'_\sigma b''_\sigma t''_\sigma$ a cocycle. Also the choice of measure μ^H_O is made by selecting an F-class O_1 in O the regular class unipotent orbit for split groups. A factor b''_σ will be needed to insure that $\kappa(t''_\sigma b'_\sigma b''_\sigma) = 1$ on $O_1 \subseteq O \cap E_0$. Finally when G is not quasi-split, our analysis has not been complete: the factor b''_σ will be chosen by the normalization of measures μ_O^H for an orbit O which contains an F-rational element. At any rate, to prove the transfer of orbital integrals to an endoscopic group all that is needed is that b''_σ is constant which has been shown.

§4. Symplectic Groups

The argument in the remainder of the paper is divided into the following steps.

(1) List the endoscopic groups H of G and the pairs (T, κ) associated to each H. Determine the stable and adjoint unipotent orbits of H and G.

(2) Show that the patch Y_U is regular in codimension 1 and determine its divisors.

(3) Obtain an explicit resolution of singularities of Y_U.

(4) Fix a stable unipotent class O and look at all divisors E meeting Y_U which are O-divisors. By looking at the data defining the principal value integral

$$PV \int_E f\, m_{\theta,E} |\omega_E|$$

on E either show that the principal value integral is zero or show that there is a decomposition on $Y_U \cap E$ of the type described in the hypotheses of (2.14). The following simple implication of (1.1) and (2.4.2) is often used. If E is an O-divisor, O is r-regular, and if $b(E)/a(E) - 1 \neq r$ then E makes no contribution to the germ of O. Also by (2.4.2) no logarithmic terms appear in the expansion so that $F_r(\theta, \beta) = 0$ for $r > 1$.

(5) Show that the decomposition on Y_U of step (4) extends to all points of the divisor E.

(6) Resolve the singularities on the rest of Y_Γ and show that the principal value integral $\int_E f\, m_{\theta,E}|\omega_E|$ is zero for any divisor not meeting Y_U.

The remainder of this section carries out steps (1), (2), (3) (step (2) for an arbitrary symplectic group). Sections 5 and 6 carry out step 4 for the subregular and 2-regular unipotent classes. Section 7 contains the arguments needed to extend the decomposition obtained in §5, §6 to points outside of the patch Y_U. The fact that the principal value integrals considered in this paper are not birational invariants means that it is not enough to restrict ourselves to the patch Y_U. However, the arguments become much more technical outside of Y_U and these details, including steps 5 and 6 are relegated to the final section.

We ignore the regular unipotent class in all that follows. By (2.9) we see that the transfer factors are chosen so that the matching of regular germs on G and H is a triviality. We also ignore the identity element. By [R2] the germ associated to the identity element is known explicitly. In particular, it is zero for a κ-orbital integral if κ is nontrivial.

We begin with the simply connected semi-simple symplectic group $Sp(2n)$ of arbitrary rank (or its inner form) and specialize to the rank 2 case toward the end of the section. If all of the Cartan subgroups of H may be identified with Cartan subgroups in a Levi factor M of G then (2.4.1, 2.4.3) shows that the κ-orbital integrals on G may be realized inside M (since M_{der} is then also simply connected). We may then inductively assume that H is *cuspidal*, that is, the L-group of H is not contained in a parabolic subgroup of ${}^LG$. The possibilities for H have been computed [H]. They are:

$$ {}^F H_i = Sp(2i) \times SO(2(n-i)) \tag{4.1}$$

$$ {}^E H_i = Sp(2i) \times {}^E SO(2(n-i)). $$

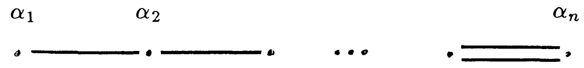
${}^F H_n$ is the quasi-split inner form of G, ${}^F H_{n-1}$ degenerates to a Levi factor and may be excluded. ${}^E H_n$ is also degenerate and may be excluded. Here E is the quadratic field extension which splits the endoscopic group. Fixing i, the Cartan subgroups T in G_{qs} associated to ${}^F H_i$ are all conjugate to a Cartan subgroup of

$$ M_i = Sp(2i) \times Sp(2(n-i)) \subseteq Sp(2n) $$

and every Cartan subgroup T in $M_i \subseteq Sp(2n)$ is associated to an endoscopic group ${}^E H_i$ for an appropriate choice of quadratic field extension E. Given T, E is determined as follows.

Identify W_{H_i} with the product of $W_{Sp(2i)}$ and the subgroup of $W_{Sp(2(n-i))}$ generated by *short* reflections. Then if $T \subseteq M_i$ and $\mathbf{T}$ is a split group in M_i, pick $T^m = \mathbf{T}$ and let E be the quadratic extension associated to the homomorphism $Gal(\overline{F}/F) \to \sigma(m)m^{-1} \in W_{M_i}/W_{H_i} \xrightarrow{\sim} \{\pm 1\}$. Then T may be identified with a stable conjugacy class of Cartan subgroups in ${}^E H_i$. $H^1(Gal(\overline{F}/F), {}^E H_i)$ has two elements. Fix the non-trivial character κ on $H^1(Gal(\overline{F}/F), {}^E H_i)$. Then pulling κ back to T, (T, κ) is associated to the endoscopic group ${}^E H_i$. (This is the only possible choice of κ because for anisotropic T one is guaranteed a non-trivial character on $H^1(Gal, T)$ which must be trivial in $H^1(Gal(\overline{F}/F), {}^E H_i)$. When $n = 2$ we have ${}^E H_0 = {}^E SO(4)$, and ${}^E H_1 = SL(2) \times U_E(1)$. where the letter E indicates the quadratic extension splitting the group and $U_E(1)$ is a one dimensional torus.

Next we determine some prime divisors of $\varphi^{-1}(\lambda)$ in $Z = Z_\Phi$ for $\Phi = C_n$. See (3.7). Label the simple roots Δ of Φ by



Recall that the positive roots have the form $\gamma(k,\ell) \overset{def}{=} \alpha_k + \alpha_{k+1} + \cdots + \alpha_\ell \quad \ell > k$ or $\gamma^+(k,\ell) \overset{def}{=} \alpha_k + \alpha_{k+1} + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_\ell \quad \ell \geq k$. Let $\mathcal{S}$ be the set of possibly empty pairs $(S_1, S_2) \subseteq \Delta \times \Delta$, $S_1 = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ $(i_1 < i_2 < \cdots < i_r)$, $S_2 = \{\alpha_{j_1}, \ldots, \alpha_{j_q}\}$ $(j_1 < \cdots < j_q)$ satisfying $i_r < j_1$ and $j_q \neq n$. For each $(S_1, S_2) \in \mathcal{S}$ we define an open patch $Z_{(S_1,S_2)}$ of Z by

(i) $\quad z(\alpha) \neq 0 \quad$ if $\quad \alpha \notin S_1 \cup S_2 \quad$ (write $z(\alpha_{i_k}) = z'_k \qquad z(\alpha_{j_k}) = z_k$).

(ii) $x(\alpha) \neq 0$ if $\alpha \in S_1$

(iii) $x_k \overset{def}{=} x(\gamma(j_k, j_{k+1})) \neq 0 \qquad k \neq q$

$$x_q \overset{def}{=} x(\gamma^+(j_q, j_q)) \neq 0.$$

(iv) $w_k \overset{def}{=} w(\gamma(j_k, j_{k+1})) \neq 0 \qquad k \neq q$

$$w_q \overset{def}{=} w(\gamma^+(j_q, j_q)) \neq 0.$$

Then on $Z_{(S_1,S_2)}$ if $S_2 \neq \emptyset$ we have by (3.7)

(i) $$w(\delta) = \frac{x(\delta) w(\gamma) \Pi z(\alpha)^{m(\alpha)}}{x(\gamma)}$$

if $x(\gamma) \neq 0$ and $\delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0$

(ii) $x(\gamma) = \dfrac{\lambda w(\gamma)}{\Pi z(\alpha)^{m(\alpha)}} \qquad$ if $\gamma = \Sigma m(\alpha)\alpha \qquad \alpha \notin S_1 \cup S_2$

(iii) $\lambda = \dfrac{x_q}{w_q} z_q^2 (\Pi z(\alpha)^{m(\alpha)}) \qquad 2\alpha_{j_q} + \Sigma m(\alpha)\alpha = \gamma^+(j_q, j_q)$

(notice that $x_q \neq 0, \Pi z(\alpha)^{m(\alpha)} \neq 0$, so $\lambda = 0$ if and only if $z_q = 0$).

(iv) $z_k = z_{k+2}\left(\dfrac{x_{k+1}}{w_{k+1}}\dfrac{w_k}{x_k}\dfrac{\Pi z(\alpha)^{m(\alpha)}}{\Pi z(\alpha)^{m'(\alpha)}}\right)$
where $\Sigma m(\alpha)\alpha + \alpha_{j_{k+2}} = \gamma(j_{k+1}, j_{k+2})$ and $\alpha_{j_k} + \Sigma m'(\alpha)\alpha = \gamma(j_k, j_{k+1})$

$$z_{q-1} = z_q\left(\frac{w_{q-1}x_q\Pi z(\alpha)^{m(\alpha)}}{x_{q-1}w_q\Pi z(\alpha)^{m'(\alpha)}}\right)$$
where $2\alpha_{j_q} + \Sigma m(\alpha)\alpha = \gamma^+(j_q, j_q)$ and $\alpha_{j_{q-1}} + \alpha_{j_q} + \Sigma' m(\alpha)\alpha = \gamma(j_{q-1}, j_q)$.

(v) $z'_\ell = \dfrac{\lambda}{x(\alpha_\ell)} = \dfrac{x_q z_q^2(\Pi z(\alpha)^{m(\alpha)})}{w_q x(\alpha_\ell)}$ where $2\alpha_{j_q} + \Sigma m(\alpha)\alpha = \gamma^+(j_q, j_q)$, $\alpha_\ell \in S_1$.

Notice that (iv), (v) give $z(\alpha) = f_\alpha z_q^\epsilon$ for some f_α regular and non-vanishing and $\epsilon \in \{0, 1, 2\}$.

(vi) Suppose $\gamma = \Sigma m(\alpha)\alpha$ with $\Pi z(\alpha)^{m(\alpha)} = f_\gamma z_q$, f_γ regular and non-vanishing then

$$x(\gamma) = \frac{x_q z_q(\Pi z(\alpha)^{m'(\alpha)})w(\gamma)}{f_\gamma w_q}, \qquad 2\alpha_{j_q} + \Sigma m'(\alpha)\alpha = \gamma^+(j_q, j_q).$$

Z is defined by p equations and is consequently $p+1$-dimensional. A count of the equations in $(i)\ldots(vi)$ reveals p independent equations. In fact for $\gamma = \Sigma m(\alpha)\alpha \in \Phi^+$ let $e(\gamma) \in \mathbf{Z}$ be the exponent which makes $\dfrac{\Pi z(\alpha)^{m(\alpha)}}{z_q^{e(\gamma)}}$ regular and invertible on $Z_{(S_1,S_2)}$ (by (iv) such an exponent exists). Let $q' = \#\{\gamma \in \Phi^+ \mid e(\gamma) = 0 \text{ or } 1\}$. Then the above equations show that $z(\alpha)$, $\alpha \in \Delta \setminus (S_1 \cup S_2)$; z_q; $w(\gamma)$, $(e(\gamma) = 0, 1)$; $x(\gamma)$ $(e(\gamma) \geq 2)$ are $p + 1 = (n - r - q) + (1) + (q' - n + q + r) + (p - q')$ variables which generate the ring of $Z_{(S_1,S_2)}$. Hence $Z_{(S_1,S_2)}$ is regular. The divisor $\lambda = 0$ on $Z_{(S_1,S_2)}$ has the form $(\lambda) = a(E_{(S_1,S_2)})E_{(S_1,S_2)}$ with $a(E_{(S_1,S_2)}) = 2$ where $E_{(S_1,S_2)}$ is the prime divisor given by $z_q = 0$.

If $S_2 = \emptyset$ then on $Z_{(S_1,\emptyset)}$ we have by (3.7)

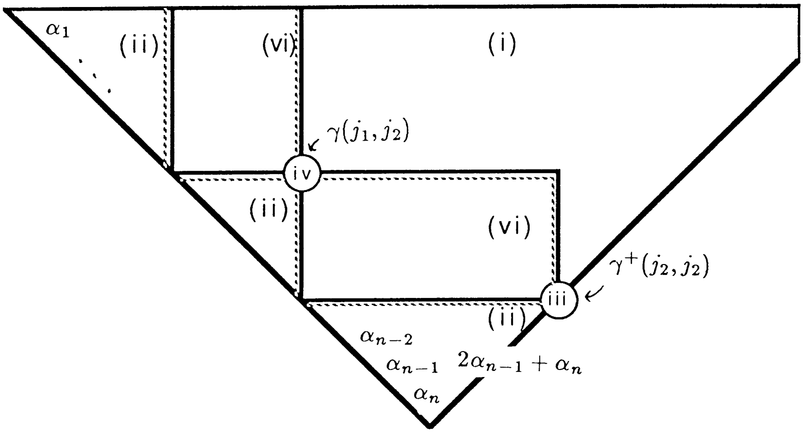
(i) $w(\delta) = \dfrac{x(\delta)w(\gamma)\Pi z(\alpha)^{m(\alpha)}}{x(\gamma)}$ if $x(\gamma) \neq 0$ and $\delta = \gamma + \Sigma m(\alpha)\alpha \quad \alpha \geq 0$.

(ii) $x(\gamma) = \dfrac{\lambda w(\gamma)}{\Pi z(\alpha)^{m(\alpha)}} \qquad \gamma = \Sigma m(\alpha)\alpha \quad \alpha \notin S_1$.

(iii) $\lambda = z'_r x(\alpha_{i_r})$.

(iv) $z'_\ell = z'_r\left(\dfrac{x(\alpha_{i_r})}{x(\alpha_{i_\ell})}\right) \qquad \ell \neq r \quad \alpha_{i_\ell} \in S_1$.

For $\gamma = \Sigma m(\alpha)\alpha \in \Phi^+$ define $e'(\gamma) \in \mathbf{Z}$ to be the exponent which makes $\dfrac{\Pi z(\alpha)^{m(\alpha)}}{z_r'^{e'(\gamma)}}$ regular and



This diagram illustrates the p equations for $S_1 = \emptyset$ and $|S_2| = 2$

invertible on $Z_{(S_1,\phi)}$. Let $q' = \#\{\gamma \in \Phi^+ \mid e'(\gamma) = 0\}$. Then (i)-(iv) show that the coordinate ring of $Z_{(S_1,\phi)}$ is generated by $z(\alpha)$, $\alpha \in \Delta \setminus S_1$; z'_r; $w(\gamma)$, $(e'(\gamma) = 0)$; $x(\gamma)$ $(e'(\gamma) \geq 1)$ and hence by $p + 1 = (n - r) + (1) + (q' + r - n) + (p - q')$ variables. Again by a dimension count we see that $Z_{(S_1,\emptyset)}$ is regular and that the divisor $\lambda = 0$ on $Z_{(S_1,\emptyset)}$ has the form $(\lambda) = a(E_{(S_1,\emptyset)})E_{(S_1,\emptyset)}$ with $a(E_{(S_1,\emptyset)}) = 1$ and $E_{(S_1,\emptyset)}$ prime.

We extend the prime divisors $E_{(S_1,S_2)}$ to all of Z by taking their closures. Given one of the divisors E above, S_1 is determined as the set of simple roots for which $x(\alpha) \neq 0$ generically on E, and S_2 is determined as the set of simple roots not in S_1 for which $z(\alpha) \equiv 0$ on E. Hence the map from pairs (S_1, S_2) to divisors is injective.

LEMMA 4.2. *(1) The codimension of the complement of* $\bigcup_{(S_1,S_2)\in S} Z_{(S_1,S_2)}$ *in* Z *is at least two.*

(2) Z *is regular in codimension one.*

(3) The divisor (λ) *on* Z *is given by*

$$(\lambda) = \sum_{(S_1,S_2)} a(E_{(S_1,S_2)})E_{(S_1,S_2)}$$

$$\textit{with} \quad a(E_{(S_1,S_2)}) = \begin{cases} 1 & S_2 = \emptyset \\ 2 & \textit{otherwise.} \end{cases}$$

PROOF: By the preceding remarks (2) and (3) follow immediately from (1). Z is clearly regular for $\lambda \neq 0$ so we study λ near points of $\lambda = 0$.

Consider a point p in Z where $\lambda = 0$. Let S_1 be the set of simple roots such that $x(\alpha) \neq 0$. Let $S_{\min}$ be the set of positive roots such that $x(\gamma) \neq 0$ and such that whenever $\gamma = \beta + \Sigma m(\alpha)\alpha$, $m(\alpha) \geq 0$ then $x(\beta) = 0$. Clearly $S_1 \subseteq S_{\min}$. We may write $\Phi^+ = S_{\min} \cup S^+ \cup S^-$, where $S^+ = \{\delta : \delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0 \text{ for some } \gamma \in S_{min},\ \delta \notin S_{min}\}$. $S^- = \{\beta : x(\beta) = 0\}$.

We have equations

(i) $\quad x(\beta) = 0 \qquad \beta \in S^-$

(ii) $\quad w(\delta) = \dfrac{w(\gamma)x(\delta)\Pi z(\alpha)^{m(\alpha)}}{x(\gamma)} \qquad \delta \in S^+,\ \gamma \in S_{\min},\ \delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0.$

(iii) if $\gamma = \gamma(k,\ell)$ or $\gamma^+(k,\ell) \in S_{\min}\backslash S_1$.

$$w(\gamma - \alpha_k)z(\alpha_k) = \frac{w(\gamma)x(\gamma - \alpha_k)}{x(\gamma)} = 0 \qquad \text{by the definition of } S_{\min}$$

(iv) $\quad z(\alpha) = \dfrac{\lambda}{x(\alpha)} = 0$ if $\alpha \in S_1$

(v) $\quad \lambda = 0.$

Counting the number of equations we have at least $p + 1$ independent equations. For the proof of (1) we may exclude sets of codimension 2. Thus we may assume that (i)-(v) give exactly $p + 1$ independent solutions. In particular, we may assume that S^-, S^+ and $S_{\min}$ are disjoint and that in (iii) if $w(\gamma - \alpha_k) = 0$ then $z(\alpha_k) \neq 0$.

Suppose that $\delta = \gamma(k,\ell)$ or $\gamma^+(k,\ell) \in S_{\min} \setminus S_1$, $\ell \geq k$. Then as in (iii) we have $w(\delta - \alpha_\ell)z(\alpha_\ell) = 0$. Suppose that $z(\alpha_\ell) \neq 0$. Then $w(\delta - \alpha_\ell) = 0$. This gives $p + 2$ equations unless

$\delta' = \gamma(k-1, \ell-1)$ or $\gamma^+(k-1, \ell-1) \in S_{\min} \setminus S_1$ and the equation (iii) associated to δ' is $w(\delta' - \alpha_{k-1}) = w(\delta - \alpha_\ell) = 0$ (and so $z(\alpha_{k-1}) \neq 0$). But then

$$w(\delta) = \frac{x(\delta) z(\alpha_\ell) w(\delta')}{x(\delta') z(\alpha_{k-1})}$$

gives $p+2$ equations. This contradiction shows that $z(\alpha_\ell) = 0$.

Now if $\gamma^+(r, \ell) \in S_{\min} \setminus S_1$ then $z(\alpha_\ell) = 0$. This must be a consequence of (iii) so that $\gamma(\ell, m)$ or $\gamma^+(\ell, m) \in S_{\min}$. But this contradicts the definition of $S_{\min}$ unless $r = \ell = m$. Also by the definition of minimality there is at most one ℓ such that $\gamma^+(\ell, \ell) \in S_{\min} \setminus S_1$. Let $\gamma(j_1, j_2) \in S_{\min} \setminus S_1$, then $z(\alpha_{j_2}) = 0$ so there exists a $\gamma(j_2, j_3)$ or $\gamma^+(j_2, j_2) \in S_{\min} \setminus S_1$. Continuing this way we obtain a chain $\gamma(j_1, j_2), \gamma(j_2, j_3) \ldots \gamma^+(j_q, j_q) \in S_{\min} \setminus S_1$, which we may assume to be maximal in length. If also $\gamma(k_1, k_2) \in S_{\min} \setminus S_1$ we see that a new chain could be formed which must also end in $\gamma^+(k_{q'}, k_{q'})$ $k_{q'} = j_q$. But then $\gamma(k_{q'-1}, k_{q'}), \gamma(j_{q-1}, j_q) \in S_{\min} \setminus S_1$ and by the definition of S_{min} $\gamma(k_{q'-1}, k_{q'}) = \gamma(j_{q-1}, j_q)$. Continuing in this manner one finds that $\gamma(k_1, k_2) = \gamma(j_{q-q'+1}, j_{q-q'+2})$. Thus $\{\gamma(j_1, j_2), \cdots \gamma^+(j_q, j_q)\} = S_{\min} \setminus S_1 \overset{def}{=} S_2$. Since $\gamma^+(j_q, j_q) \notin S_1$, $j_q \neq n$. Since (i)-(v) give all equations which hold identically at p (excluding sets of codimension 2) the inequalities defining $Z_{(S_1, S_2)}$ hold at the generic point of the variety defined by (i)-(v) together with $x(\gamma) \neq 0$, $\gamma \in S_{min}$. This completes the proof.

Turn to the case $G = C_2$. We introduce new notation for thse divisors. Set $\alpha = \alpha_1$, $\beta = \alpha_2$.

(4.3)

S_1	S_2	$E_{(S_1,S_2)}$	$\pi(E_{(S_1,S_2)})$
$\{\alpha, \beta\}$	ϕ	E_0	regular
$\{\alpha\}$	ϕ	E_β	subregular Richardson class of P_β
$\{\beta\}$	ϕ	E_α	subregular Richardson class of P_α
ϕ	$\{\alpha\}$	E_2	2-regular
ϕ	ϕ	E_{id}	identity.

The projection of each divisor to G is the closure of a stable unipotent conjugacy class. The final column lists that conjugacy class. Write $\gamma \overset{def}{=} \alpha + \beta$, $\delta = 2\alpha + \beta$.

There are 4 stable unipotent classes in Sp(4) [Sp]: the regular, the subregular, the 2-regular and the 4-regular (the identity element). For a complete discussion see [Sp]. There is only one adjoint regular class, one adjoint 2-regular class, and one adjoint 4-regular class. The adjoint subregular classes are in bijection with quadratic extensions (or elements of $F^\times/F^{\times 2}$). To determine the quadratic extension associated to a subregular unipotent element $u \in G(F)$, conjugate u by an element of $G(F)$ so that it has the form

$$\begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X \in M_2(F),$$

where $\mathrm{Sp}(4)$ is defined using the skew form $J_0 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det X \in F^\times/F^{\times 2}$ depends only on the adjoint conjugacy class of u. Equivalently, $\mathcal{B}_u$, the variety of Borel subgroups containing u, is a union of three projective lines. The lines of type β are defined over a quadratic extension of F depending only on the adjoint conjugacy class of u.

When G is the inner form of $\mathrm{Sp}(4)$ the regular, 2-regular, and 1-regular adjoint class associated to the trivial quadratic extension $F^{\times 2} \subseteq F^\times/F^{\times 2}$) do not exist. If $u \in G(F)$ is unipotent regular then there is a unique Borel subgroup over F containing u. If $u \in G(F)$ is 2-regular unipotent, then the line of type β in $\mathcal{B}_u$ is defined over F giving a parabolic of type β over F; but the minimal parabolic of G is of type α. If $u \in G(F)$ is 1-regular corresponding to the trivial extension then the Borel subgroup corresponding to the intersection of a line of type α and a line of type β in $\mathcal{B}_u$ is defined over F. Thus such unipotent classes do not exist for the inner form G.

LEMMA 4.4. *(1) The variety Z_Φ $\Phi = C_2$ is regular except possibly at points p such that $\pi(p)$ is the identity element of G or at points p where*

$$\lambda = x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0.$$

(2) The singularity at $\lambda = x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0$ may be resolved by blowing up once along the subvariety $x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0$. Let E_B be the divisor introduced by blowing up.

(3) The divisors on the desingularized variety and their Igusa constants are given as follows (assume $\pi(p) \neq$ identity).

E	$a(E)$	$b(E)$	$\pi(E)$	$\beta(E)-1$
E_0	1	1	0-reg	0
E_α	1	2	1-reg	1
E_β	1	2	1-reg	1
E_2	2	6	2-reg	2
E_B	2	5	2-reg	3/2

PROOF: The divisors $E_0, E_\alpha, E_\beta, E_2, E_{\mathrm{id}}$ are defined by

$E_0 : z(\alpha) = z(\beta) = w(\gamma) = w(\delta) = 0$

$E_\alpha : x(\alpha) = z(\beta) = 0, z(\alpha)x(\gamma) = w(\gamma)x(\beta), w(\delta)x(\gamma) = w(\gamma)x(\delta)z(\alpha)$

$E_\beta : x(\beta) = z(\alpha) = 0, z(\beta)x(\gamma) = w(\gamma)x(\alpha), w(\delta) = 0$

$E_2 : z(\alpha) = x(\alpha) = x(\beta) = x(\gamma) = 0$

$E_{\mathrm{id}} : x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0.$

We consider several patches, showing that the coordinate ring on each is regular, and at the same time we identify the divisors on each patch. The defining relations (3.7) of Z are used repeatedly. The expression $\lambda = z(\beta)x(\beta) \quad (E_0E_\beta)$ indicates that $z(\beta) = 0$ defines the divisor E_0, that $x(\beta) = 0$ defines the divisor E_β, and so forth.

(If $x(\alpha) \neq 0$) $\quad \lambda = z(\beta)x(\beta)\ (E_0E_\beta), \quad z(\alpha) = z(\beta)x(\beta)/x(\alpha),$
$w(\gamma) = x(\gamma)z(\beta)/x(\alpha), w(\delta) = x(\delta)z(\alpha)z(\beta)/x(\alpha)$
generators of local ring: $z(\beta), x(\beta), x(\gamma), x(\delta), x(\alpha)$.

(If $x(\beta) \neq 0$) $\quad \lambda = z(\alpha)x(\alpha) \quad (E_0E_\alpha), \quad z(\beta) = z(\alpha)x(\alpha)/x(\beta),$
$w(\gamma) = x(\gamma)z(\alpha)/x(\beta), w(\delta) = x(\delta)z(\alpha)^2/x(\beta)$
generators of local ring: $z(\alpha), x(\alpha), x(\beta), x(\gamma), x(\delta)$.

(If $x(\gamma) \neq 0$) $\lambda = x(\alpha)x(\beta)w(\gamma)/x(\gamma)\ (E_\alpha E_0 E_\beta)$, $z(\alpha) = w(\gamma), x(\beta)/x(\gamma)$,
$z(\beta) = w(\gamma)x(\alpha)/x(\gamma)$, $w(\delta) = w(\gamma)x(\delta)z(\alpha)/x(\gamma)$
generators: $x(\beta), x(\alpha), w(\gamma), x(\gamma), x(\delta)$.

(If $z(\alpha) \neq 0$) $\lambda = z(\beta)x(\beta)\ (E_\alpha E_{\mathrm{id}})$, $x(\alpha) = z(\beta)x(\beta)/z(\alpha)$,
$x(\gamma) = w(\gamma)x(\beta)/z(\alpha), x(\delta) = x(\beta)w(\delta)/z(\alpha)^2$,
generators: $x(\beta), z(\beta), w(\gamma), z(\alpha), w(\delta)$

(If $w(\delta) \neq 0$) $\lambda = z(\alpha)^2 z(\beta)x(\delta)/w(\delta)\ (E_2^2 E_\alpha E_{\mathrm{id}})$
$x(\alpha) = z(\alpha)z(\beta)x(\delta)/w(\delta)$, $x(\beta) = z(\alpha)^2 x(\delta)/w(\delta)$, $x(\gamma) = z(\alpha)w(\gamma)x(\delta)/w(\delta)$
generators: $z(\alpha), z(\beta), x(\delta)w(\gamma), w(\delta)$

Since we are assuming $\pi(p) \neq 1$ and since the variety has now been shown to be nonsingular for $x(\alpha), x(\beta)$ or $x(\gamma) \neq 0$, we may assume in the remaining cases that $x(\delta) \neq 0$.

(If $w(\gamma) \neq 0$) $\lambda = x(\gamma)^2 w(\delta)z(\beta)/(w(\gamma)^2 x(\delta))\ (E_2^2 E_\beta E_\alpha)$
$z(\alpha) = w(\delta)x(\gamma)/(w(\gamma)x(\delta))$, $x(\alpha) = z(\beta)x(\gamma)/w(\gamma)$, $x(\beta) = x(\gamma)^2 w(\delta)/(w(\gamma)^2 x(\delta))$
generators: $x(\gamma), x(\delta), z(\beta), w(\gamma), w(\delta)$.

(If $z(\beta) \neq 0$) $\lambda = w(\delta)x(\alpha)^2/(z(\beta)x(\delta))$ $(E_\beta E_2^2)$ $x(\beta) = w(\delta)x(\alpha)^2/(z(\beta)^2 x(\delta))$,
$x(\gamma) = w(\gamma)x(\alpha)/z(\beta)$, $z(\alpha) = w(\delta)x(\alpha)/(x(\delta)z(\beta))$
generators: $x(\alpha), x(\delta), z(\beta), w(\gamma), w(\delta)$.

This completes the proof of (1). The validity of (2) is easily checked using the same seven patches. Details are omitted.

The only points that are not clear in (3) are the value of the constants $b(E)$ and the value of $a(E_B)$. $b(E) - 1$ is given as the order to which the form (3.8)

$$\omega_Y = d\lambda dx(\alpha)dx(\beta)dx(\gamma)dx(\delta)dv$$

$$dv = dv(\alpha)dv(\beta)dv(\gamma)dv(\delta)$$

vanishes along E. For example, the form when expressed in terms of the coordinates on the patch $(w(\gamma) \neq 0)$ becomes

$$\omega_Y = x(\gamma)^5 w(\delta)z(\beta)/(x(\delta)^2 w(\gamma)^4)dz(\beta)d(1/w(\gamma))dw(\delta)dx(\gamma)dx(\delta)dv \quad (E_2^{6-1}E_\beta^{2-1}E_\alpha^{2-1}).$$

Thus $b(E_2) = 6, b(E_\beta) = 2, b(E_\alpha) = 2$. The Igusa constants $b(E)$ for $E = E_B, E_0$ are similarly calculated using a blown up region. Introduce projective coordinates $X_\alpha, X_\beta, X_\gamma, Z_\alpha, W_\delta, Z_\beta, W_\gamma$. On $X_\alpha = 1$ we have $X_\beta x(\alpha) = x(\beta)$, $X_\gamma x(\alpha) = x(\gamma)$. The patch $(x(\alpha) \neq 0)$ becomes (dropping the assumption that $x(\alpha) \neq 0$).

$$\lambda = x(\alpha)^2 Z_\beta X_\beta \ (E_B^2 E_0 E_\beta), \ Z_\alpha = Z_\beta X_\beta, \ W_\gamma = X_\gamma Z_\beta,$$

$$W_\delta = x(\delta)Z_\alpha Z_\beta \text{ so that } a(E_B) = 2.$$

$$\omega_Y = x(\alpha)^{5-1} Z_\beta^{1-1} X_\beta^{2-1} dZ_\beta dx(\alpha) dX_\beta dX_\gamma dx(\delta)dv$$

$$\text{so that } b(E_B) = 5, \ b(E_0) = 1, \ b(E_\beta) = 2.$$

This completes the proof.

§5. The subregular germs on $G =$ Sp(4)

We begin with a summary of the results of this section. Let G be Sp(4) or its inner form. By (2.5) we will have determined all of the germs of $GSp(4)$ once we have calculated those of $Sp(4)$. Details on normalizations of measures are found below. Let α' be a root of Sp(4), let E be a quadratic extension (possibly trivial of F), and let H be an endoscopic group. Let $T(\alpha')$ be a

Cartan subgroup corresponding to the homomorphism $\mathrm{Gal}(\overline{F}/F) \to W$ given by $\sigma \in \mathrm{Gal}(E/F)$ $(\sigma \neq 1)$ $\sigma \mapsto \sigma_{\alpha'} \in W$ (or let $T(\alpha')$ be split if $E = F$). Let $U_E(1)$ be a one dimensional unitary group split by E.

Let $O_{E'}$ be the subregular adjoint unipotent class associated to the quadratic extension E' of F (see discussion following 4.3). Let $\Gamma(E', E, \alpha')$ be the germ of $O_{E'}$ for the pair $(T(\alpha'), st)$ where st is the trivial character on $H^1(Gal(\overline{F}/F), T(\alpha'))$. Set $\Gamma(E', E, \alpha') = 0$ if $T(\alpha')$ does not exist. The function $\Gamma_1(E', E'', \beta)$ is defined in (5.18). Set $\delta_G = 1$ if G is quasi-split, 0 otherwise; set $\epsilon_G = 2\delta_G - 1$; and set $\delta(E, E') = 1$ if $E = E'$, 0 otherwise. Then we have for various (T, κ) in G associated to the nondegenerate cuspidal endoscopic groups H (4.1)

(5.1)

$\underline{H}$	$\underline{T}$	Germ of $O_{E'}$ $E' \neq F$	Germ of O_F
${}^{E}H_1$ $E \neq F$	$U_{E''}(1) \times U_E(1)$	$\delta(E, E')\delta_G \Gamma_1(E', E'', \beta)$	0
${}^{E}H_0$ $E \neq F$	*arbitrary*	0	0
${}^{F}H_2$	$T/(\pm 1) =$ $U_{E''}(1) \times U_E(1)$	$\Gamma(E', E, \alpha)$ $+\epsilon_G \Gamma(E', E'', \gamma)$	$\Gamma(F, E, \alpha)$ $+\epsilon_G \Gamma(F, E'', \gamma)$
${}^{F}H_0$	*arbitrary*	$\epsilon_G \Gamma^{T_{qs}, st}_{O_{E'}}$	$\delta_G \Gamma^{T_{qs}, st}_{O_F}$.

Quasi-split reduction will be useful in identifying the germs of the subregular unipotent classes. For a group of semi-simple rank 2, quasi-split reduction is a statement about the matching of inner forms of rank 1 groups. This situation is well understood. See for example [LS]. In particular it follows immediately from the well-known matching of orbital integrals on a rank 1 group with orbital integrals on its quasi-split inner form that G has quasi-split reduction.

The variety of Borel subgroups containing a given subregular unipotent element in a group of type C_2 is a union of three projective lines and is called the Dynkin curve. There is one line corresponding to the short root and two lines corresponding to the long root. If u is contained in $G(F)$ then $\mathrm{Gal}(\overline{F}/F)$ permutes the lines ℓ_β, ℓ'_β corresponding to the long root. We say that a subregular class is distinguished if the quadratic extension associated to it is the trivial extension, that is, all three lines are defined over F. Most of the analysis of this section will be devoted to the classes which are not distinguished.

LEMMA 5.2. *Suppose that H is an endoscopic group of G which is split by a nontrivial quadratic extension E of F. Let (T, κ) be a pair associated to H. Then the subregular germ of the unipotent class $O_{E'}$ associated to the quadratic extension E' for the pair (T, κ) is zero unless $E' = E$.*

PROOF: Let Z be the center of G. Let $u \in O(F)$ and let $C_G(u)_{red}$ be the reductive centralizer of u. If κ is nontrivial on $\ker(H^1(Gal(\overline{F}/F), Z) \to H^1(Gal(\overline{F}/F), C_G(u)_{red}))$ pick h so that $O^h = O$, $\kappa(\sigma(h)h^{-1}) \neq 1$ so that by (2.1) $\Gamma_O^{T,\kappa} = 0$. It was mentioned in (2.1) that κ restricted to $H^1(Gal(\overline{F}/F, Z)$ depends only on the endoscopic group associated to to (T, κ). We see this directly as follows in the case that G is the inner form of a split group and $Z = \{\pm 1\}$ and H split by a quadratic extension E of F. Directly from the definition of endoscopic groups we see easily that $E = F$ if and only if H (up to a central factor) is an endoscopic group of G which in turn is true if and only if κ restricted to $H^1(Gal(\overline{F}/F), Z)$ is trivial. If E is a nontrivial extension then κ is nontrivial but trivial over the quadratic extension E of F. In other words, κ is trivial on the image of the corestriction map from $H^1(Gal(\overline{F}/E), Z)$ to $H^1(Gal(\overline{F}/F, Z)$. Identifying $H^1(Gal(\overline{F}/F), Z)$ with $F^\times/F^{\times 2}$ it follows that κ may be identified with the nontrivial character on $F^\times/NE$ where NE is by definition the group of norms of nonzero elements of E. To complete the proof it is sufficient to show that $\ker(H^1(Gal(\overline{F}/F), Z) \to H^1(Gal(\overline{F}/F),\ C_G(u)_{red}))$ is $NE'/F^{\times 2} \subseteq F^\times/F^{\times 2}$. Note that Z fixes the lines ℓ_β, ℓ'_β so that the image of $H^1(Gal(\overline{F}/F), Z)$ lies in $H^1(Gal(\overline{F}/F), C_G(u)'_{\mathrm{red}})$ where $C_G(u)'_{\mathrm{red}}$ is the subgroup of $C_G(u)_{\mathrm{red}}$ which fixes the lines ℓ_β, ℓ'_β. In the situation at hand it is known [Sp] that $C_G(u)'_{\mathrm{red}} = C_G(u)^0_{\mathrm{red}}$ the identity component of $C_G(u)_{\mathrm{red}}$, so that it is sufficient to work with $C_G(u)^0_{\mathrm{red}}$.

The connected reductive centralizer is computed when G is split as follows. Let B be a Borel subgroup over F in the line of type α in the Dynkin curve of u. Conjugating B to the upper triangular matrices we see that

$$u = \begin{pmatrix} I_2 & XJ \\ 0 & I_2 \end{pmatrix}$$

where $X = {}^tX \in \mathrm{GL}_2(F)$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows immediately that

$$C_G(u) = \begin{pmatrix} {}^tA & B \\ 0 & JA^{-1}J \end{pmatrix} \quad \text{and} \quad C_G(u)_{\mathrm{red}} = \begin{pmatrix} {}^tA & 0 \\ 0 & JA^{-1}J \end{pmatrix}$$

where ${}^tAXA = X$. Thus $C_G(u)^0_{\mathrm{red}} \xrightarrow{\sim} U_{E'}(1)$ a one dimensional torus split by E'. $ker(H^1(Z) \to H^1(C_G(u)^0_{\mathrm{red}}))$ is then identified with $NE'/F^{\times 2} \subseteq F^\times/F^{\times 2}$ and the lemma follows for G split.

When G is not split we make the following modifications. G contains a parabolic subgroup P_α over F which is identified modulo a Borel subgroup B with the line of type α in the Dynkin curve of u. Thus we may assume that u has the same form. Over the quadratic extension E' of F G splits and u becomes distinguished so that over E' it follows that $C_G(u)^0_{\mathrm{red}} \xrightarrow{\sim} \mathbf{G}_m$. But if $C_G(u)^0_{\mathrm{red}}$ were isomorphic to $\mathbf{G}_m$ over F we would have a split torus in $\begin{pmatrix} {}^tM & 0 \\ 0 & JM^{-1}J \end{pmatrix}$, where

$det\ M = 1$, which would contradict the hypothesis that G is not split. Thus $C_G(u)^0_{\text{red}} \xrightarrow{\sim} U_{E'}(1)$ as before, and $E' \neq F$. This completes the proof of (5.2).

We have seen there are two divisors which contribute to the subregular germ. On the patch Y_U given by (3.7)

$$\begin{aligned} \lambda &= z(\alpha)x(\alpha) \\ \lambda &= z(\beta)x(\beta) \\ \lambda w(\gamma) &= z(\alpha)z(\beta)x(\gamma) \\ \lambda w(\delta) &= z(\alpha)^2 z(\beta)x(\delta). \end{aligned}$$

E_α is described by the equations $x(\alpha) = z(\beta) = x(\beta)w(\gamma) - z(\alpha)x(\gamma) = x(\beta)w(\delta) - z(\alpha)^2 x(\delta) = 0$ and E_β is described by the equations $x(\beta) = z(\alpha) = w(\delta) = x(\alpha)w(\gamma) - z(\beta)x(\gamma) = 0$.

LEMMA 5.3. *If u_0 is not distinguished then E_β contains no F-rational points over u_0 and consequently makes no contribution to the germ associated to u_0.*

PROOF: The lemma is an immediate consequence of two facts. First, if $(u_0, (B(W))) \in E_\beta$ then every Borel subgroup $B(W)$ lies in the same line ℓ_β of type β in the Dynkin curve. For select local coordinates as in (3.12) such that B_0 lies in ℓ_β and ℓ_β contains $B(W_+)$. Then $B(W) = B_0^{n_W v}$ where n_W has the form

$$n_W = exp(z(\alpha)z_k X_{-\alpha})exp(z(\beta)z_{k-1}X_{-\beta}) \cdots exp(z(\alpha)z_1 X_{-\alpha}) \tag{5.4}$$

for appropriate values of $z_1, \ldots, z_k$. So $z(\alpha) = 0$ implies that the product (5.4) collapses to $n_W = exp(z(\beta)z' X_{-\beta})$ for some z'. Also $B_0^v = B(W_+) \in \ell_\beta$ and B_0, $B(W_+) \subseteq P_\beta$ where $B\backslash P_\beta = \ell_\beta$ so $P_\beta^v = P_\beta$ which implies that $v \in N_\infty \cap P_\beta$ or $v = exp(\xi\ X_{-\beta})$ for some ξ, where N_∞ is the unipotent radical of B_∞. Thus $n_W v \in P_\beta$ and $B(w) = B_0^{n_W v} \in P_\beta$. Thus if ℓ_β and ℓ'_β are the lines of the Dynkin curve of u_0 we may separate points $(u_0, (B(W)))$ of E_β into ℓ_β-points and ℓ'_β-points according to where the Borel subgroups lie. Second, the action of $\text{Gal}(\overline{F}/F)$ on the points of E_β interchanges ℓ_β-points with ℓ'_β-points. For the action (3.13) is given by $(u_0, (B(W))) \mapsto (u_0, (\sigma(B(\sigma_T^{-1}W))))$ for $\sigma \in Gal(\overline{F}/F)$, $u_0 \in G(F)$. If $B(W)$ lies in ℓ_β then $B(\sigma_T^{-1}W)$ lies in ℓ_β and $\sigma(B(\sigma_T^{-1}W))$ lies in $\sigma(\ell_\beta) = \ell'_\beta$ provided σ has nontrivial image in $\text{Gal}(E/F)$ where E is the quadratic extension trivializing the action on the lines ℓ_β, ℓ'_β.

We now devote our attention to the divisor E_α. We fix u_0 a subregular element which is not distinguished and let B_0 be a Borel subgroup at the intersection of the line of type α and a line of type β in the Dynkin curve of u_0. As in the discussion of the divisor E_β we find that if $(u, B_0^{nw})^v$ lies in E_α over the subregular element u_0 then $u^v = u_0$ and $v \in P_\alpha \cap N_\infty$ so that $v = exp(\xi X_{-\alpha})$ for some ξ.

Select a Cartan subgroup $T_0 \subseteq B_0$ over F such that if σ_α is the simple reflection associated to the simple root α, $B_0^{\sigma_\alpha}$ is equal to the Borel subgroup at the intersection of ℓ_α and ℓ'_β (the second line of type β in the Dynkin curve). Such a Cartan subgroup may be found by selecting any Cartan over F in B_0 and in a Levi factor over F of P_α where $B \setminus P_\alpha = \ell_\alpha$. Let B_∞ be the Borel subgroup opposite B_0 through T_0. We consider the patch $Y_U = Y_U(B_\infty, B_0)$ for this choice of (B_∞, B_0).

The choice of B_0 forces u_0 to have the form $u_0 = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix}$, $X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. The choice of B_∞ forces $\sigma_\alpha u_0 \sigma_\alpha^{-1}$ to have the same form which implies that $y = 0$ so that

$$u_0 = \begin{pmatrix} I_2 & xI_2 \\ 0 & I_2 \end{pmatrix} \tag{5.5}$$

We choose a Haar measure on O_{u_0} as follows. We select coordinates $v' = \begin{pmatrix} I_2 & 0 \\ N & I_2 \end{pmatrix}$, $y = \begin{pmatrix} I_2 & Y \\ 0 & I_2 \end{pmatrix}$, $N = \begin{pmatrix} n_\gamma & n_\beta \\ n_\delta & n_\gamma \end{pmatrix}$, $Y = \begin{pmatrix} y_\gamma & y_\delta \\ y_\beta & y_\gamma \end{pmatrix}$ so that $y^{v'}$ describes points on an open set of O_{u_0}. Then

$$\frac{1}{2}\, dy\ dv' \overset{def}{=} \frac{1}{2}\, dy_\beta dy_\gamma dy_\delta dn_\beta dn_\gamma dn_\delta \tag{5.6}$$

is an invariant measure on O_{u_0}.

We consider the patch $x(\gamma) \neq 0$ on Y_U so that the equations (3.7) take the form

$$\lambda = \frac{x(\alpha)x(\beta)w(\gamma)}{x(\gamma)}, \quad z(\alpha) = \frac{x(\beta)w(\gamma)}{x(\gamma)}, \quad z(\beta) = \frac{x(\alpha)w(\gamma)}{x(\gamma)}, \quad w(\delta) = \frac{x(\beta)w(\gamma)^2 x(\delta)}{x(\gamma)^2} \tag{5.7}$$

By (1.1), (3.8), and a change of variables

$$\omega_{E_\alpha} = \frac{x(\gamma)}{x(\beta)}\,\frac{dw(\gamma)}{w(\gamma)^2}\,dx(\beta)dx(\gamma)dx(\delta)dv = \frac{\omega_Y}{\lambda^2 \frac{dx(\alpha)}{x(\alpha)}}.$$

We divide by the Haar measure $\omega_{O_{u_0}} = (1/2)dx\ dv' = (1/2)dy\ dv'$ $(dx = dx(\beta)\ dx(\gamma)\ dx(\delta))$ to obtain a differential form $\omega_{E_\alpha(u_0)}$ on the fibre $E_\alpha(u_0)$ over u_0. Making use of the fact that $v = exp(\xi X_{-\alpha})$ on $E_\alpha(u_0)$ this gives

$$\omega_{E_\alpha(u_0)} = \frac{2x(\gamma)}{x(\beta)}\,\frac{dw}{w^2}d\xi \qquad w = w(\gamma). \tag{5.8}$$

Next we note that if $(u, B(W))^v$ is a point of $E_\alpha(u_0)$ with coefficients $x_\beta(u) = x(\beta)$ $\quad x_\gamma(u) = x(\gamma)$, $x_\delta(u) = x(\delta)$ we must have $u^v = u_0$ with $v = exp(\xi X_{-\alpha})$. By (5.5) this implies that $x(\delta) = 0$, and $x(\gamma)/x(\beta) = 1/2\xi$ so that (5.8) becomes

$$\omega_{E_\alpha(u_0)} = \frac{dw}{w^2}\,\frac{d\xi}{\xi} \tag{5.9}$$

Note that it is always possible to recover the action of Gal$(\overline{F}/F)$ from the action of $W \times A$ and the action σ_* for a split Cartan subgroup and B_0 over F. The equation $\sigma_{\alpha'}(x_\gamma(b)) = x_\gamma(b')$ of (3.13) gives for an *arbitrary* divisor.

(5.10)

$$\begin{aligned}
\sigma_\alpha(x(\alpha))/x(\alpha) &= 1\\
\sigma_\alpha(x(\beta))/x(\beta) &= 1 + 2(T_1 - T_2)w(\gamma) + (T_1 - T_2)^2 w(\delta)\\
\sigma_\alpha(x(\gamma))/x(\gamma) &= 1 + (T_1 - T_2)z(\alpha)x(\delta)/x(\gamma) \;=\; 1 + (T_1 - T_2)w(\delta)/w(\gamma)\\
\sigma_\alpha(x(\delta))/x(\delta) &= 1
\end{aligned}$$

$$\begin{aligned}
\sigma_\beta(x(\alpha))/x(\alpha) &= 1 - 2T_2 w(\gamma)\\
\sigma_\beta(x(\beta))/x(\beta) &= 1\\
\sigma_\beta(x(\gamma))/x(\gamma) &= 1\\
\sigma_\beta(x(\delta))/x(\delta) &= 1.
\end{aligned}$$

Here T_1, T_2 represents the tangent direction of the curve Γ in T at the identity and σ_α and σ_β are simple reflections associated to the simple roots α and β. From the choice of B_0 made following lemma 5.3 the image of the group A in the Weyl group is the subgroup of order two generated by the reflection σ_α. Let σ_0 be the representative of σ_α in A. Then (3.13) and (5.10) specializing to the divisor E_α give

$$\begin{aligned}
\sigma_\beta(\xi) &= \xi \\
\sigma_\alpha(\xi) &= w_A \xi \\
\sigma_0(\xi) &= 1/\varsigma\xi \\
\sigma_\beta(w) &= w/w_B \\
\sigma_\alpha(w) &= w/w_A \\
\sigma_0(w) &= -w/w_A
\end{aligned} \tag{5.11}$$

where we introduce abbreviations

$$\begin{aligned}
w_A &= 2(T_1 - T_2)w + 1 \\
w_B &= -2T_2 w + 1 \\
w_D &= 2T_1 w + 1.
\end{aligned} \tag{5.12}$$

and ς is an element of $F^\times$. Since σ_0, σ_α and σ_β generate $W \times A$ we easily deduce the following charts:

(5.13)

$\underline{\sigma}$	$\underline{\sigma(w)}$	$\underline{\sigma(w_A)}$	$\underline{\sigma(w_B)}$	$\underline{\sigma(w_D)}$	$\underline{\sigma(\xi)}$	$\underline{t_\sigma}$
1	w	w_A	w_B	w_D	ξ	$(1,1)$
σ_α	$\frac{w}{w_A}$	$\frac{1}{w_A}$	$\frac{w_B}{w_A}$	$\frac{w_D}{w_A}$	$w_A\xi$	$(\xi w,\ \frac{1}{\xi w})$
$\sigma_\beta\sigma_\alpha$	$\frac{w}{w_D}$	$\frac{w_B}{w_D}$	$\frac{1}{w_D}$	$\frac{w_A}{w_D}$	$\frac{w_D\xi}{w_B}$	$(\frac{\xi w}{w_B},\ \frac{-w}{w_B x(\gamma)})$
$\sigma_\alpha\sigma_\beta\sigma_\alpha$	$\frac{w}{w_D}$	$\frac{w_B}{w_D}$	$\frac{w_A}{w_D}$	$\frac{1}{w_D}$	$\frac{w_A w_D\xi}{w_B}$	$(\frac{\xi w^2}{w_B x(\gamma)},\ \frac{w_A}{w_B})$
$\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$	$\frac{w}{w_A}$	$\frac{1}{w_A}$	$\frac{w_D}{w_A}$	$\frac{w_B}{w_A}$	$\frac{w_A w_D\xi}{w_B}$	$(\frac{-w^2\xi}{w_B x(\gamma)},\ \frac{-1}{x(\gamma)w_D\xi})$
$\sigma_\beta\sigma_\alpha\sigma_\beta$	w	w_A	w_D	w_B	$\frac{w_D\xi}{w_B}$	$(\frac{-w}{w_D x(\gamma)},\ \frac{-w}{w_B x(\gamma)})$
$\sigma_\alpha\sigma_\beta$	$\frac{w}{w_B}$	$\frac{w_D}{w_B}$	$\frac{w_A}{w_B}$	$\frac{1}{w_B}$	$w_A\xi$	$(\frac{w}{w_A x(\gamma)},\ \frac{1}{\xi w})$
σ_β	$\frac{w}{w_B}$	$\frac{w_D}{w_B}$	$\frac{1}{w_B}$	$\frac{w_A}{w_B}$	ξ	$(1,\ \frac{1}{\xi x(\gamma)})$

$\underline{\sigma}$	$\underline{\sigma(w)}$	$\underline{\sigma(w_A)}$	$\underline{\sigma(w_B)}$	$\underline{\sigma(w_D)}$	$\underline{\sigma(\xi)}$	$\underline{t_\sigma}$
σ_0	$\frac{-w}{w_A}$	$\frac{1}{w_A}$	$\frac{w_D}{w_A}$	$\frac{w_B}{w_A}$	$\frac{1}{\varsigma\xi}$	(ξ, ξ^{-1})
$\sigma_\alpha\sigma_0$	$-w$	w_A	w_D	w_B	$\frac{1}{\varsigma w_A\xi}$	$(\frac{w}{w_A}, \frac{w_A}{w})$
$\sigma_\beta\sigma_\alpha\sigma_0$	$\frac{-w}{w_B}$	$\frac{w_D}{w_B}$	$\frac{w_A}{w_B}$	$\frac{1}{w_B}$	$\frac{w_B}{(\varsigma w_D\xi)}$	$(\frac{w}{w_D},\ \frac{w}{w_D}\xi)$
$\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_0$	$\frac{-w}{w_B}$	$\frac{w_D}{w_B}$	$\frac{1}{w_B}$	$\frac{w_A}{w_B}$	$\frac{w_B}{(\varsigma w_A w_D\xi)}$	$(\frac{w^2}{w_A w_D},\ \frac{-1}{w_D\xi})$
$\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$	$-w$	w_A	w_B	w_D	$\frac{w_B}{(\varsigma w_A w_D\xi)}$	$(\frac{w^2}{w_A w_D},\ \frac{w_A}{x(\gamma)w_B})$
$\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$	$\frac{-w}{w_A}$	$\frac{1}{w_A}$	$\frac{w_B}{w_A}$	$\frac{w_D}{w_A}$	$\frac{w_B}{(\varsigma w_D\xi)}$	$(\frac{w\xi}{x(\gamma)w_B},\ \frac{w}{w_D x(\gamma)\xi})$
$\sigma_\alpha\sigma_\beta\sigma_0$	$\frac{-w}{w_D}$	$\frac{w_B}{w_D}$	$\frac{1}{w_D}$	$\frac{w_A}{w_D}$	$\frac{1}{(\varsigma w_A\xi)}$	$(\frac{w\xi}{x(\gamma)},\ \frac{w_A}{w})$
$\sigma_\beta\sigma_0$	$\frac{-w}{w_D}$	$\frac{w_B}{w_D}$	$\frac{w_A}{w_D}$	$\frac{1}{w_D}$	$\frac{1}{\varsigma\xi}$	$(\xi, \frac{1}{x(\gamma)})$

ς is a constant in $F^\times$. It is a norm of an element in the quadratic extension E (associated to the unitpotent element u_0) if and only if G is split. To see this we argue as follows. $w = 0$ defines the intersection of E_α with E_0. By (3.6) the fibre $E_\alpha(u_0)) \cap E_0$ may be identified with a subvariety of the Springer variety. This fibre is 1-dimensional, closed, and consists of $(u_0, (B(W))) \in E_\alpha(u_0) \cap E_0$ with $B(W) = B(W_+) \in \ell_\alpha$ for all W, and hence it is isomorphic to ℓ_α with coordinate ξ. Pull the action of $W \times A$ back to $Gal(\overline{F}/F)$ (3.13) and note that since when $w = 0$, then $w_A = w_B = w_D = 1$ it follows that ξ is a coordinate over the quadratic extension E of F whose absolute Galois group maps to the identity of A. The condition $\sigma^2 = 1$, for $1 \neq \sigma \in Gal(E/F)$ together with $\sigma_0(\xi) = \frac{1}{\varsigma \xi}$ forces $\varsigma \in F^\times$ and the elements of $E_\alpha(u_0) \cap E_0 \xrightarrow{\sim} \ell_\alpha$ are in bijection with elements of E with norm ς. Also ℓ_α has F-rational points if and only if $B \setminus P_\alpha$ contains a Borel subgroup over F which is true if and ony if G is quasi-split. Thus ς is a norm if and only if G is quasi-split. This fact will account for the signs ϵ_G (5.1) that enter into the formulas for the germs.

It is shown in [H] that κ is trivial on the cocycle

$$\sigma_\alpha \mapsto \lambda^{\alpha^\vee} \quad \sigma_\beta \mapsto 1 \quad \sigma_0 \mapsto 1. \tag{5.14}$$

and κ is trivial on the cocycle $\sigma_\alpha \mapsto 1$ $\sigma_\beta \mapsto \lambda^{\beta^\vee}$ $\sigma_0 \mapsto 1$ if and only if H is split.

Let E be the quadratic extension associated to the fixed unipotent element u_0. We define a cocycle taking values in the 1-dimensional torus $U_E(1)$ split by E as follows. For a fixed $w = w_0$, the fibre over w, which is isomorphic over $\overline{F}$ to $\mathbf{P}^1$, does not necessarily have any F-rational points. It does if and only if the cocycle depending on w

$$a_\sigma : \sigma \mapsto \sigma[\xi][\xi]^{-1} \in U_E(1) \tag{5.15}$$

is a coboundary when evaluated at $w = w_0$. The germ, being a principal value integral over $E_\alpha(u_0)$ may be expressed as a double integral over ξ and then w if we integrate only over those points for which (5.15) is a coboundary. Let η_E be the non-trivial character on $H^1(Gal(\overline{F}/F), U_E(1))$. Then the subregular germ is equal to

$$|\lambda| \int_{E_\alpha(u_0)} \left|\frac{dw}{w^2}\,\frac{d\xi}{\xi}\right| \kappa(t_\sigma) = |\lambda| \int_{Nm1} \frac{d\overline{\xi}}{|\overline{\xi}|} \int_{\mathbf{P}^1(F)} \left(\frac{1+\eta_E(a_\sigma)}{2}\right) \frac{dw}{|w|^2}\,\kappa(t_\sigma) \tag{5.16}$$

$$= \frac{|\lambda|}{2} \int_{Nm1} \frac{d\overline{\xi}}{|\overline{\xi}|} \int_{\mathbf{P}^1(F)} \kappa(t_\sigma) \frac{dw}{|w|^2} + \frac{|\lambda|}{2} \int_{Nm1} \frac{d\overline{\xi}}{|\overline{\xi}|} \int_{\mathbf{P}^1(F)} \kappa(t_\sigma)\eta_E(a_\sigma) \frac{dw}{|w|^2}.$$

Here $\overline{\xi}$ ranges over the set of norm one elements in E: $Nm1 =^{def} \{\xi \in E \mid \sigma(\xi)\xi = 1$ for $\sigma \in Gal(E/F)\ \sigma \neq 1\}$. (In particular $\int_{Nm1}$ is independent of the tangent direction.) Notice that in (5.16) we integrate over $\mathbf{P}^1 \times \mathbf{P}^1$. To justify integration over $\mathbf{P}^1 \times \mathbf{P}^1$ rather than over the rational surface $E_\alpha(u_0)$ we must consider coordinate patches other than Y_U and check that the principal value integral is unaffected. This is done in [H]. It is also a simple consequence of calculations done in section 7 on other coordinate patches.

Identify roots in ${}^E SO(2n)$ and Sp$(2n)$ as in the beginning of §4.

PROPOSITION 5.17. *Let (T,κ) be a pair associated to the non-split group ${}^E SO(4)$. Then the subregular germs on G for the pair (T,κ) are zero.*

REMARK: The group ${}^E SO(4)$ has no F-rational subregular elements. Proposition 5.17 gives the transfer of the subregular germ in this case.

PROOF: By (5.2) we may assume that ${}^E SO(4)$ is split by the quadratic extension E' associated to the unipotent class $O_{E'}$ (that is E'=E). Thus the group $W_T \times A$ is equal to a subgroup of

$$W_T \times A \subseteq \left\{ \begin{matrix} 1,\ \sigma_\alpha,\ \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta,\ \sigma_\beta\sigma_\alpha\sigma_\beta \\ \sigma_\beta\sigma_0,\ \sigma_\beta\sigma_\alpha\sigma_0,\ \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_0,\ \sigma_\alpha\sigma_\beta\sigma_0 \end{matrix} \right\} \stackrel{def}{=} \Omega_1$$

where $W_T = \{\sigma_T \mid \sigma \in Gal(\overline{F}/F)\}$.

We introduce the new variable $\xi' = w_D\xi$ and remark that $\sigma(\xi') = \xi'$ or $1/\varsigma\xi'$ if $\sigma \in \Omega_1$ (by the chart (5.13)) so that the cocycle a_σ is trivial if G is split and non-trivial if G is not split. The factor $\dfrac{1+\eta_E(a_\sigma)}{2}$ of (5.16) is then identically 1 if G is split, 0 otherwise. This proves the proposition when G is not split.

Let $t = (\dfrac{w}{w_D}, 1)$, $b_\sigma = \sigma(t)t^{-1}$. It is given on generators of Ω_1 by $\sigma_\alpha \mapsto (\frac{w_D}{w}, \frac{w}{w_D})$ $\sigma_\beta\sigma_0 \mapsto (-w_D, 1)$. t_σ and $b_\sigma t_\sigma$ have the same class and $b_\sigma t_\sigma$ is given by

$$\sigma_\alpha \mapsto (\xi', \xi'^{-1}) \quad \sigma_\beta\sigma_0 \mapsto (-\xi', 1/x(\gamma)).$$

It follows that $b_\sigma t_\sigma$ for $\sigma \in \Omega_1$ is independent of w. We may integrate in (5.16) first over w. But $\int_{\mathbf{P}^1(F)} \dfrac{dw}{|w|^2} = 0$ [LS1] so that the germ is equal to zero. This completes the proof. We remark that the proof does not use the fact that κ is a nontrivial character.

Next we consider the endoscopic group $H = SL(2) \times U_E(1)$. By (5.2) the subregular germ for pairs (T,κ) associated to H are zero except when H is split by the quadratic extension associated to the unipotent class. We assume that E splits H. We identify the nontrivial element of the Weyl group of H with the simple reflection σ_β and the automorphism which twists the torus $U_E(1)$ with the element $\sigma_0\sigma_\alpha\sigma_\beta\sigma_\alpha$. The factor σ_0 reflects the fact that by (5.2) we may assume $E' = E$. Then we have $W_T \times A \subseteq \{1, \sigma_\beta, \sigma_0\sigma_\alpha\sigma_\beta\sigma_\alpha, \sigma_0\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\} \stackrel{def}{=} \Omega_1$.

PROPOSITION 5.18. *(1) Let (T,κ) be a pair associated to the non-split endoscopic group $H = SL(2) \times U_E(1)$. Then the nonzero subregular germ on G for the pair (T,κ) considered as a function of the tangent direction (T_1, T_2) of the curve Γ at the identity element depends only on T_2 not T_1.*

(2) The subregular germ of the unipotent class O_E on G for pairs (T,κ) associated to $SL(2) \times U_E(1)$ are given by the first row of (5.1) where

$$\Gamma_1(E, E'', \beta) = |\lambda| \int_{Nm1} \frac{d\overline{\xi}}{|\xi|} \int_{\mathbf{P}^1} \eta_E(w^2/w_B) \frac{dw}{|w|^2}$$

Here the action of $Gal(\overline{F}/F)$ on w is determined by the rule $\sigma_\beta(w) = w/w_B$, $\sigma_0\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(w) = -w$. $Nm1$ denotes the group of norm 1 elements of E.

(3) The transfer (2.8) of subregular germs near the identity is compatible for various choices of (T,κ) associated to $SL(2) \times U_E(1)$. (In other words the function f^H may be chosen independent of (T,κ).)

PROOF: The character κ of $H^1(Gal(\overline{F}/F), T)$ may be identified with the nontrivial character on $H^1(Gal(\overline{F}/F), U_E(1))$ under the projection $T \subseteq H = SL(2) \times U_E(1) \to U_E(1)$. We then see using (5.13) that $\kappa(t_\sigma)$ equals η_E applied to $w^2/w_A w_D \in F^\times$. (We use η_E indiscriminately for the nontrivial character on $U_E(1)$ and the nontrivial character on $F^\times$ trivial on the norms of E.) Again by (5.13) $\eta_E(a_\sigma)$ of (5.15) is equal to η_E applied to $\dfrac{w_A w_D}{w_B}\varsigma \in F^\times$. By (5.16) the germ is equal to

$$\frac{|\lambda|}{2} \int \frac{d\overline{\xi}}{|\xi|} \int \eta_E\left(\frac{w^2}{w_A w_D}\right) \frac{dw}{|w|^2} + \frac{|\lambda|}{2} \int \frac{d\overline{\xi}}{|\xi|} \int \eta_E\left(\frac{w^2\varsigma}{w_B}\right) \frac{dw}{|w|^2}. \tag{5.19}$$

$w_B = -2T_2 w + 1$ depends only on T_2 not T_1 and by (5.13) $\sigma(w), \sigma(w_B)$ for $\sigma \in \Omega_1$ depend only on T_2 not T_1. Thus the second term of (5.19) depends only on T_2.

To show that the first term of (5.19) is independent of T_1 we introduce the variable $w' = w/2T_1 w + 1 = w/w_D$. Simple calculations give

$$\frac{dw}{w^2} = \frac{dw'}{w'^2}, \quad \frac{w^2}{w_A w_D} = \frac{w'^2}{(-2T_2 w' + 1)}, \tag{5.20}$$

and writing $w'_B = -2T_2 w' + 1$,

(5.21)

	$\sigma(w)$	$\sigma(w')$	$\sigma(w_B)$	$\sigma(w'_B)$
σ_β	w/w_B	w'/w'_B	$1/w_B$	$1/w'_B$
$\sigma_0\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$	$-w$	$-w'$	w_B	w'_B

We see that the first term is independent of T_1 (because w' $w'_B, \sigma(w')$ and $\sigma(w'_B)$ are). In fact, the second term of (5.19) is equal to $\eta_E(\varsigma)$ times the first. In particular, if G is not quasi-split the germ is zero; if G is quasi-split then the germ is

$$|\lambda| \int \frac{d\bar{\xi}}{|\bar{\xi}|} \int \eta_E(\frac{w^2}{w_B}) \frac{dw}{|w|^2}.$$

(3) To prove compatibility it would be easy enough at this point to identify the principal value integrals with those of [LS]. But instead, we use the fact that the subregular germ is invariant in the direction of T_1. Thus a function supported on regular and subregular elements satisfies $\Delta_G^{T,\kappa}\Phi_G^{T,\kappa}(\gamma, f) = \Delta_G^{T,\kappa}\Phi_G^{T,\kappa}(z_0\gamma, f)$ (assuming that $z_0 = exp(zT_1)$ and γ are sufficiently small), so that the germ expansion at the identity coincides with the germ expansion at z_0. By (2.11) $\Delta_G^{T,\kappa}\Phi^{T,\kappa}(z_0\gamma, f) = \Delta_{M_{qs}}^{T_{qs},\kappa}\Phi_{M_{qs}}^{T_{qs},\kappa}(\gamma, f)$. κ is trivial on $(T\setminus M)(F)$ and $M_{der} \xrightarrow{\sim} H_{der}$ so the result follows.

Next we consider the case that $H = Sp(4)$, G is not quasi-split and the integrals are all stable.

PROPOSITION 5.22. *If $(T,1)$ is a Cartan subgroup of G and G is not quasi-split, $\Gamma_{qs}^{(T,1)}$ is the germ on the quasi-split inner form and $\Gamma^{(T,1)}$ is the germ on G. Then $\Gamma^{(T,1)} = -\Gamma_{qs}^{(T,1)}$.*

PROOF: The factor $\kappa(t_\sigma)$ of (5.16) is constant so that by the result $\int_{\mathbf{P}^1(F)} \frac{dw}{|w|^2} = 0$ it follows that the first term of (5.16) is zero. The cocycle a_σ for G and a_σ^{qs} for the quasi-split inner form differ only by the factor ς. Thus we have

$$\Gamma^{(T,1)} = \frac{|\lambda|}{2} \int \frac{d\bar{\xi}}{|\bar{\xi}|} \int_{\mathbf{P}^1(F)} \eta_E(a_\sigma) \frac{dw}{|w|^2} = \frac{|\lambda|}{2} \eta_E(\varsigma) \int \frac{d\bar{\xi}}{|\bar{\xi}|} \int \eta_E(a_\sigma^{qs}) \frac{dw}{|w|^2}$$
$$= \eta_E(\varsigma)\Gamma_{qs}^{(T,1)} = -\Gamma_{qs}^{(T,1)}$$

since $\eta_E(\varsigma) = -1$ because ς is not a norm if G is not quasi-split.

Finally we consider the endoscopic group $H = SO(4)$. Since H is split we must consider all subregular unipotent classes including the distinguished one. The Weyl group of H may be identified with $\{1, \sigma_\alpha, \sigma_\beta\sigma_\alpha\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\}$. The group $W_T \times A$ is a subgroup of

$$\Omega_1 \stackrel{def}{=} \left\{ \begin{matrix} 1, & \sigma_\alpha, & \sigma_\beta\sigma_\alpha\sigma_\beta, & \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \\ \sigma_0, & \sigma_\alpha\sigma_0, & \sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0, & \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0 \end{matrix} \right\}.$$

PROPOSITION 5.23. *(1) Let (T,κ) be a pair associated to the endoscopic group $H = SO(4)$. Then the germ on G for the pair (T,κ) and a subregular unipotent class associated to a nontrivial*

quadratic extension E of F breaks into a sum of two terms. The first term depends on $T_1 + T_2$ but not $T_1 - T_2$. The second term depends on $T_1 - T_2$ but not $T_1 + T_2$.
(2) These germs are given as in (5.1) and the transfer (2.8) to the endoscopic group $SO(4)$ is valid for those germs.

PROOF: As in the discussion of the endoscopic group $H = {}^E SO(4)$ we use coordinates $\xi' = w_D \xi$ and w instead of $\xi$and w. We have by (5.13) $\sigma_\alpha(\xi') = \xi'$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(\xi') = \xi'$, $\sigma_0(\xi') = (\frac{w_B w_D}{w_A \varsigma})(\frac{1}{\xi'})$. Thus η_E evaluated on the cocycle a_σ of (5.15) equals $\eta_E(\frac{w_B w_D}{w_A \varsigma})$, $\frac{w_B w_D}{w_A \varsigma} \in F^\times$.

$H = SO(4)$ is an endoscopic group (up to a central factor) even if G is of adjoint type, so we may calculate the cocycle in the adjoint group. We form a cocycle in $U_{E'}(1)$ (where E' is the invariant field of σ_α and σ_0) by replacing the cocycle $t_\sigma : \sigma \mapsto (t_{1\sigma}, t_{2\sigma})$ by $e_\sigma : \sigma \mapsto t_{1\sigma} t_{2\sigma} \in U_{E'}(1)$. The fact that $W_T \times A$ is a subgroup of Ω_1 insures that this is well-defined. By (5.13) we find that e_σ is given by

$$\sigma_\alpha \mapsto 1, \quad \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \mapsto w^2/x(\gamma)^2 w_B w_D \in F^\times, \quad \sigma_0 \mapsto 1. \tag{5.24}$$

Thus $\kappa(t_\sigma) = \eta_{E'}(e_\sigma)$. Since $x(\gamma) \in F^\times$, $x(\gamma)^2 \in F^{\times 2}$ and it may be removed without affecting $\eta_{E'}(e_\sigma)$. Write e'_σ for the cocycle so obtained.

We make the change of coordinates $w = w'/(2T_2 w' + 1)$. Then e'_σ becomes $\sigma_\alpha \mapsto 1$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \mapsto w'^2/(2(T_1+T_2)w'+1) \in F^\times$, $\sigma_0 \mapsto 1$. The action of Ω_1 on the variable w' is easily seen to be $\sigma_\alpha(w') = w'$, $\sigma_0(w') = -w'/(2(T_1+T_2)w'+1)$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(w') = w'/(2(T_1+T_2)w'+1)$. Also $dw/w^2 = dw'/w'^2$. It follows that the first term of (5.16) is $\frac{|\lambda|}{2} \int \frac{d\xi}{|\xi|} \int \eta_{E'}(e'_\sigma)\frac{dw'}{|w'|^2}$ which is independent of $T_1 - T_2$ because the action of Ω_1 on w' and the cocycle e'_σ are independent of $T_1 - T_2$. Moreover the action of σ_α on w', $(T_1 + T_2)$ and $2(T_1 + T_2)w' + 1$ is trivial, so the data defining the principal value integral is the same for $W_T = \langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ as for $W_T = \langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta,\ \sigma_\alpha\rangle$ and $\langle\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$, and the same for $W_T = \langle\sigma_\alpha\rangle$ as for $W_T = \{1\}$.

To prove that the second term of (5.16) is independent of T_1+T_2 we return to the coordinates (w, ξ'). Note that kw_B/w lies in the quadratic extension E'' which is invariant by $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$, $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$ where k is a constant chosen to satisfy $\sigma_\alpha(k) = k$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0(k) = -k$. Also if σ is a nontrivial element of $\mathrm{Gal}(E''/F)$ $\quad \sigma(kw_B/w)(kw_B/w) = [\sigma(k)k]w_B w_D/w^2$. Thus up to a constant $w_B w_D/w^2$ modulo the norms of E' is trivial or nontrivial according as $w_B w_D/w^2$ modulo the norms of E. The result is that $\kappa(t_\sigma) = \eta_{E'}(e'_\sigma) = \eta_E(e'_\sigma)$, and $\kappa(t_\sigma)\eta_E(a_\sigma) = \eta_E(e'_\sigma a_\sigma) = \eta_E(e'^{-1}_\sigma a_\sigma)$ $\quad$ (η_E has order 2) $= \eta_E(\frac{w^2}{w_B w_D} \cdot \frac{w_B w_D}{w_A \varsigma}) = \eta_E(\frac{w^2}{w_A})$. Special argument is required if E' or $E'' = F$ but it is easy to check that the conclusion $\kappa(t_\sigma)\eta_E(a_\sigma) = \eta_E(w^2/w_A)$

still holds. Thus $\kappa(t_\sigma)\eta_E(a_\sigma)$ depends only on T_1-T_2 (through $w_A=2(T_1-T_2)w+1$) and not on T_1+T_2. Furthermore an examination of (5.13) shows that the action of Ω_1 on w and w_A depends only on T_1-T_2 and not T_1+T_2. Moreover the action of $\sigma_\beta\sigma_\alpha\sigma_\beta$ on w, T_1-T_2 and w_A is trivial, so the data defining the principal value integral is the same for $W_T=\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ as for $W_T=\langle\sigma_\beta\sigma_\alpha\sigma_\beta,\ \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ and $W_T=\langle\sigma_\alpha\rangle$, and the same for $W_T=\langle\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ as for $W_T=\{1\}$. Consequently, the second term of (5.13) depends only on T_1-T_2 and not T_1+T_2. We conclude that the germs are given by the entries of (5.1).

Compatibility of (2.8) for various pairs (T,κ) is known in the special case that T lies in a parabolic subgroup of G. Since we have expressed the subregular germ in terms of the germs associated to T lying in a parabolic subgroup compatiblity for all pairs (T,κ) associated to $SO(4)$ follows.

The following proposition will complete our study of the subregular germs.

PROPOSITION 5.25. *(1) Let (T,κ) be a pair associated to the endoscopic group $SO(4)$. Then the germ on G for the pair (T,κ) and the distinguished subregular class breaks into a sum of two terms. The first depends on T_1+T_2 but not T_1-T_2. The second term depends on T_1-T_2 but not T_1+T_2.*

(2) These germs are given as in (5.1) and the transfer (2.8) to the endoscopic group $SO(4)$ is valid for these germs.

PROOF: In the analysis of the distinguished subregular class (5.2) does not apply so that there are two divisors to consider. We select B_0, B_∞ as before, noting that B_0, B_∞ are now over F so that the group A is trivial (σ_0 does not occur). Recall that B_0 was selected to lie in a line ℓ_β of type β in the Dynkin curve of u_0 and that the points $(u_0,(B(W)))$ of $E_\beta(u)$ are such that $B(W)\in\ell_\beta$ for all W or $B(W)\in\ell'_\beta$ for all W. Let $E_\beta(\ell_\beta)$ or $E_\beta(\ell_\beta,u_0)$ denote the points of $E_\beta(u_0)$ for which the first condition holds.

The function $\kappa(t_\sigma)$ is seen to be identically 1 on $E_\alpha(u_0)$ and $E_\beta(u_0)$ as follows. By definition t_σ is given by (3.15)

$$\sigma_\alpha \ \mapsto\ (1/x(\alpha)^{\alpha^\vee}$$

$$\sigma_\beta \ \mapsto\ (1/x(\beta))^{\beta^\vee}$$

Since H is split, κ evaluated on the cocycle $\sigma_\alpha\mapsto 1$, $\sigma_\beta\mapsto x^{\beta^\vee}$, $(x\in F^\times)$ is trivial (5.14). Thus we may take t_σ to be given by

$$\sigma_\alpha \mapsto (1/x(\alpha))^{\alpha^\vee}$$
$$\sigma_\beta \mapsto (\lambda/x(\beta))^{\beta^\vee} = z(\beta)^{\beta^\vee} = \left(\frac{x(\alpha)w(\gamma)}{x(\gamma)}\right)^{\beta^\vee}.$$

We adjust this by the coboundary $c_\sigma = \sigma(t)t^{-1}$ $t = (1, x(\alpha))$ to obtain $t_\sigma c_\sigma$ given by $\sigma_\alpha \mapsto 1$, $\sigma_\beta \mapsto w/(w_B x(\alpha)x(\gamma))$. By (5.10) together with the fact that $w(\delta) = 0$ on $E_\alpha(u_0) \cap Y_U$ and $E_\beta(u_0) \cap Y_U$ we see that $w/(w_B x(\alpha)x(\gamma)) \in F^\times$. Thus by (5.14) $\kappa(t_\sigma)$ is identically 1 on the open patches $E_\alpha(u_0) \cap Y_U$ and $E_\beta(u_0) \cap Y_U$ and so identically 1 everywhere: $\kappa(t_\sigma) \equiv 1$.

We turn to the form on E_β. Recall that E_β is given by the equations $x(\beta) = z(\alpha) = w(\delta) = x(\alpha)w(\gamma) - z(\beta)x(\gamma) = 0$. For $x(\gamma) \neq 0$ we use coordinates given on the patch $x(\gamma) \neq 0$ in the proof of (4.4). The form ω_{E_β} on E_β is given by

$$\begin{aligned}\omega_{E_\beta} &= Res_{E_\beta}(\omega_Y/\lambda^2) = Res\left(\frac{d\lambda}{\lambda^2}dx(\alpha)dx(\beta)dx(\gamma)dx(\delta)dv\right) \\ &= x(\gamma)\frac{dw(\gamma)}{w(\gamma)^2}\frac{dx(\alpha)}{x(\alpha)}dx(\gamma)dx(\delta)dv.\end{aligned} \tag{5.26}$$

We take an invariant measure on the subregular class O to be

$$\omega_O = dx(\alpha)\,dx(\gamma)\,dx(\delta)\,dv(\alpha)\,dv(\gamma)\,dv(\delta)$$

where we use coordinates $x(\alpha), x(\gamma), x(\delta), v(\alpha), v(\gamma)$, on an open set of the double cover of O. Set

$$m(w,x,y,z) = \begin{pmatrix} 1 & x & y & z \\ & 1 & w & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in Sp(4),\ J_O = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Then $(x(\alpha), x(\gamma), v(\alpha), v(\beta), v(\delta))$ are coordinates for the element $m(0, x(\alpha),\ x(\gamma),\ x(\delta))^{\mathbf{v}_\beta}$, where $\mathbf{v}_\beta = J_O^{-1} m(0, v(\alpha), v(\gamma), v(\delta)) J_O$.

We take the quotient $\omega_{E_\beta}/\omega_O$ to obtain the form on $E_\beta(\ell_\beta)$

$$\frac{x(\gamma)}{x(\alpha)}\frac{dw(\gamma)}{w(\gamma)^2}d\xi_\beta \qquad \text{where } \xi_\beta \text{ is given as follows.} \tag{5.27}$$

If $(u, B_0^{nw})^v \in E_\beta(\ell_\beta)$ then $B_0^v \in \ell_\beta$, $B_0 \in \ell_\beta$ which implies that $v \in P_\beta \cap N_\infty$ so that $v = exp(-\xi_\beta\ X_{-\beta})$. Also recall that u_0 has the form (5.5) $u_0 = m(0,0,x,0)$ provided B_0 is taken to be upper triangular and B_∞ lower triangular. Thus $u_0^{v^{-1}} = u = m(0, \tilde{\xi}, x, 0)$ so that on $E_\beta(\ell_\beta)$ $\dfrac{x(\gamma)}{x(\alpha)} = \dfrac{1}{\xi_\beta}$.

Finally we must normalize the form of (5.27) so that it is compatible with form on $E_\alpha(u_0)$. The compatibility condition is

$$Res_{E_\beta}\omega_{E_\alpha} = Res_{E_\alpha}\omega_{E_\beta}$$

and provided the forms on $E_\alpha(u_0)$ and $E_\beta(\ell_\beta, u_0)$ are defined with respect to the same invariant form ω_0 on O

$$Res_{E_\beta(\ell_\beta)}\omega_{E_\alpha(u_0)} = Res_{E_\alpha(u_0)}\omega_{E_\beta(\ell_\beta)}.$$

This condition forces the normalization

$$\omega_{E_\beta(\ell_\beta)} = \frac{dw(\gamma)}{w(\gamma)^2}\,\frac{d\xi_\beta}{\xi_\beta} \tag{5.28}$$

for then $Res_{E_\alpha(u_0)}\omega_{E_\beta(\ell_\beta)} = \dfrac{dw(\gamma)}{w(\gamma)^2}$.

Write w_β for the restriction of $w(\gamma)$ to $E(\ell_\beta)$. Then by (5.10)

(5.29)

	$\underline{1}$	$\underline{\sigma_\alpha}$	$\underline{\sigma_\beta\sigma_\alpha\sigma_\beta}$	$\underline{\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta}$
$\sigma(\xi)$	ξ	$w_A\xi$	$w_D\xi/w_B$	$w_Aw_D\xi/w_B$
$\sigma(w)$	w	w/w_A	w	w/w_A
$\sigma(w_A)$	w_A	$1/w_A$	w_A	$1/w_A$
$\sigma(w_B)$	w_B	w_B/w_A	w_D	w_D/w_A
$\sigma(w_D)$	w_D	w_D/w_A	w_B	w_B/w_A
$\sigma(\xi_\beta)$	ξ_β	ξ_β	$w_{\beta,B}\xi_\beta/w_{\beta,D}$	$w_{\beta,B}\xi_\beta/w_{\beta,D}$
$\sigma(w_\beta)$	w_β	$w_\beta/w_{\beta,A}$	w_β	$w/w_{\beta,A}$
$\sigma(w_{\beta,A})$	$w_{\beta,A}$	$1/w_{\beta,A}$	$w_{\beta,A}$	$1/w_{\beta,A}$
$\sigma(w_{\beta,B})$	$w_{\beta,B}$	$w_{\beta,B}/w_{\beta,A}$	$w_{\beta,D}$	$w_{\beta,D}/w_{\beta,A}$
$\sigma(w_{\beta,D})$	$w_{\beta,D}$	$w_{\beta,D}/w_{\beta,A}$	$w_{\beta,B}$	$w_{\beta,B}/w_{\beta,A}$

where $w_{\beta,A} = 2(T_1 - T_2)w_\beta + 1, w_{\beta,B} = -2T_2w_\beta + 1$, and $w_{\beta,D} = 2T_1w_\beta + 1$.

We must also consider coordinates patches with $B_0, B_\infty/F$ such that $B_0^{\sigma_\beta}$ is the point of intersection of ℓ_β and ℓ_α (resp. ℓ'_β and ℓ_α). On this patch $z_c =^{def} z(\beta)$ and $\xi_c =^{def} v(\beta)$ serve as coordinates on $E(\ell_\beta, u_0)$ (resp. z'_c, ξ'_c on $E(\ell'_\beta, u_0)$).

To distinguish coordinates on various patches we add subscript α's to coordinates on $E(\ell_\alpha, u_0)$ and subscript β's to coordinates on $E(\ell_\beta, u_0)$ (resp. $E(\ell'_\beta, u_0)$). On $E(\ell_\beta, u_0)$ we have

(5.30)

$$\begin{aligned} w_\beta &= z_c/(\xi_c + 2T_2 z_c), & \xi_\beta &= -1/\xi_c \\ z_c &= -w_\beta/\xi_\beta(1-2T_2 w_\beta), & \xi_c &= -1/\xi_\beta. \end{aligned}$$

On $E(\ell_\alpha, u_0)$ we have

(5.31)

$$\begin{aligned} \xi_\alpha &= -1/\xi'_\alpha, & w_\alpha &= -w'_\alpha/(2(T_1-T_2)w'_\alpha + 1) \\ \xi'_\alpha &= -1/\xi_\alpha, & w'_\alpha &= -w_\alpha/(2(T_1-T_2)w_\alpha + 1). \end{aligned}$$

On $E(\ell'_\beta, u_0)$ we have

(5.32)

$$\begin{aligned} w'_\beta &= z'_c/(\xi'_c + 2T_2 z'_c), & \xi'_\beta &= -1/\xi'_c \\ z'_c &= -w'_\beta/\xi'_\beta(1-2T_2 w'_\beta), & \xi'_c &= -1/\xi'_c. \end{aligned}$$

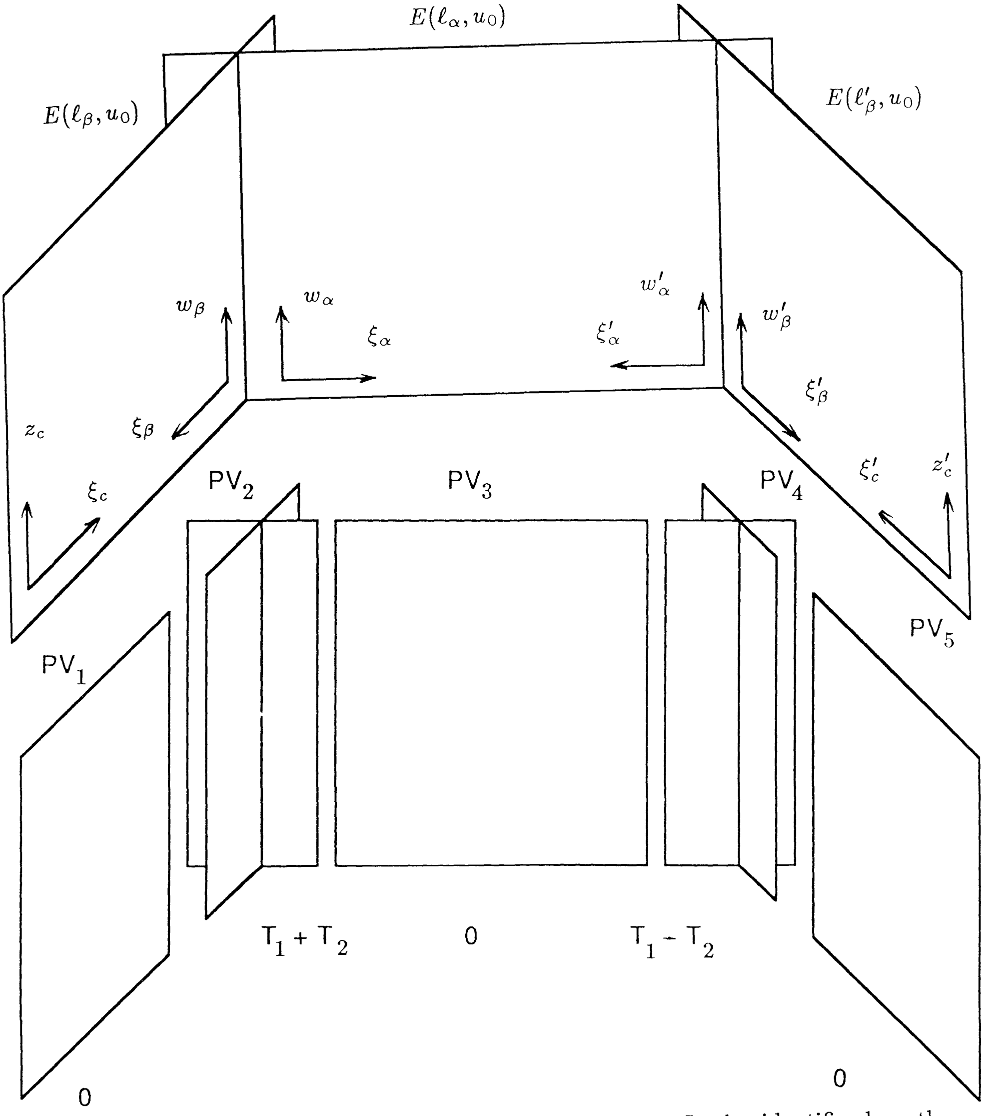
The form on $E(\ell_\beta, u_0)$ is $-dz_c d\xi_c/z_c^2 = dw_\beta d\xi_\beta/w_\beta^2\xi_\beta$ and the form on $E(\ell'_\beta, u_0)$ is $-dz'_c d\xi'_c/z'^2_c = dw'_\beta d\xi'_\beta/w'^2_\beta \xi'_\beta$.

Truncate near $E(\ell_\beta, u_0) \cap E(\ell_\alpha, u_0)$ in $E(\ell_\beta, u_0)$ (The ℓ_α-pole) by

$$\left| \frac{-2T_2 w_\beta + 1}{(T_1 - T_2)w_\beta + 1}\xi_\beta \right| \geq q^{-m}.$$

By (5.29) $\dfrac{-2T_2 w_\beta + 1}{(T_1 - T_2)w_\beta + 1}\xi_\beta$ is a variable over F. Since by (5.30)

$$\frac{1}{\xi_c + (T_1 + T_2)z_c} = -\frac{(-2T_2 w_\beta + 1)\xi_\beta}{(T_1 - T_2)w_\beta + 1}.$$



This diagram should help to clarify coordinates patches. It also identifies how the principal value integral will be broken into five pieces by truncating the integrals near the poles.

this region is the same as $|\xi_c + (T_1 + T_2)z_c| \leq q^m$. Let $\overline{\xi}_c = \xi_c + (T_1 + T_2)z_c$. Then z_c and $\overline{\xi}_c$ are variables over F and

$$PV_1 = \int_{|\overline{\xi}_c| \leq q^m} d\overline{\xi}_c \int_{\mathbf{P}^1} \frac{dz_c}{|z_c^2|} = 0.$$

Truncate in $E(\ell'_\beta, u_0)$ near the ℓ'_α-pole by $\left|\dfrac{-2T_2w'_\beta+1}{(T_1-T_2)w'_\beta+1}\xi'_\beta\right| \geq q^{-m}$. Then similarly $PV_5 = 0$. Truncate near the ℓ_β-pole in $E(\ell_\alpha, u_0)$ by $|(2T_1w_\alpha+1)\xi_\alpha| \geq q^{-m}$ and near the ℓ'_β-pole in $E(\ell_\alpha, u_0)$ by $\left|\dfrac{(2(T_1-T_2)w_\alpha+1)}{(-2T_2w'_\alpha+1)}\xi'_\alpha\right| \geq q^{-m}$. $(2T_1w_\alpha+1)\xi_\alpha$ and $\dfrac{(2(T_1-T_2)w'_\alpha+1)}{(-2T_2w'_\alpha+1)}\xi'_\alpha$ are variables over F. Then since $\dfrac{1}{(2T_1w_\alpha+1)\xi_\alpha} = \dfrac{(-2(T_1-T_2)w'_\alpha+1)}{(-2T_2w'_\alpha+1)}\xi'_\alpha$ let $\overline{\xi}_\alpha = (2T_1w_\alpha+1)\xi_\alpha$.

$$PV_3 = \int_{q^{-m}\leq|\overline{\xi}_\alpha|\leq q^m} \frac{d\overline{\xi}_\alpha}{|\overline{\xi}_\alpha|}\int_{\mathbf{P}^1}\frac{dw_\alpha}{|w_\alpha|^2} = 0.$$

The contribution PV_4 to the germ at the pole $\ell'_\beta \cap \ell_\alpha$ is given by (1.3). The function M is given by $m(\alpha_0) + 2m$ where α_0 is defined by:

$$\alpha_0 = \lim_{\lambda\to 0}\frac{\lambda}{\overline{\xi}'_\beta\overline{\xi}'_\alpha} = \lim\left\{\frac{\dfrac{x(\alpha)x(\beta)w(\gamma)}{x(\gamma)}}{\left[\dfrac{2(T_1-T_2)w'_\alpha+1}{-2T_2w'_\alpha+1}\right]\left[\dfrac{-2T_2w'_\beta+1}{(T_1-T_2)w'_\beta+1}\right]\xi'_\alpha\xi'_\beta}\right\}.$$

Here $\overline{\xi}'_\alpha$ and $\overline{\xi}'_\beta$ are variables over F. Their definitions – if not clear – may be read off from the denominator of this limit. On $E_\alpha(u_0)$ $x(\gamma)/x(\beta) = 1/2\xi'_\alpha$ (5.8), on $E(\ell'_\beta, u_0)$ $x(\gamma)/x(\alpha) = 1/\xi'_\beta$ (5.27) and $w'_\alpha = w'_\beta = w(\gamma)$ on their intersection so that

$$\alpha_0 = \lim_{\lambda\to 0}\frac{2x(\gamma)w(\gamma)((T_1-T_2)w(\gamma)+1)}{(2(T_1-T_2)w(\gamma)+1)} = \frac{2x(\gamma)w_3}{(1-(T_1-T_2)^2w_3^2)}$$

where $w_3 = w'_\alpha/((T_1-T_2)w'_\alpha+1)$ is a coordinate over F. Thus M and hence PV_4 depend only on $T_1 - T_2$.

Similarly the factor α at the $(\ell_\beta\cap\ell_\alpha)$-pole is up to inessential factors is

$$\frac{w}{(\dfrac{-2T_2w+1}{(T_1-T_2)w+1})(\dfrac{2T_1w+1}{-2T_2w+1})} = \frac{w_4}{(1-(T_1+T_2)^2w_4^2)}$$

where $w_4 = \dfrac{w}{(T_1-T_2)w+1}$ is a variable over F, so that PV_2 depends only on T_1+T_2. As in the proof of (5.23) we see that PV_2 is expressed in terms of the corresponding terms PV_2 associated to a Cartan subgroup T' contained in a parabolic subgroup ($W_{T'} = \langle\sigma_\alpha\rangle$ or $\{1\}$) for which compatibility is known. The term PV_4 is analyzed similarly. Hence the proposition follows.

§6. THE TRANSFER OF 2-REGULAR GERMS OF GSP(4)

There are no 2-regular elements in G(F) if G is not split. Consequently, we assume in this section that G is split. We also take G = Sp(4) but since we only prove the transfer of germs (2.8) for GSp(4) we assume that the endoscopic groups of G are cuspidal and split. This leaves the group H = SO(4). We may make this assumption by the remark of (2.5) and the fact that all the endoscopic groups of GSp(4) are split. (They are split because the derived group of the dual of GSp(4) is simply connected.)

Recall that all unipotent 2-regular elements are conjugate by $PSp(4,F)$ or equivalently by $GSp(4,F)$. We have seen that there is one divisor E_2 of the Igusa data which contributes to the 2-regular germ. On the patch Y_U, E_2 is described by the equation $z(\alpha) = x(\alpha) = x(\beta) = x(\gamma) = 0$. On the smaller patch where $x(\delta), w(\gamma) \neq 0$ the equations become

$$
\begin{aligned}
(6.1) \qquad \lambda &= x(\gamma)^2 w(\delta) z(\beta)/(w(\gamma)^2 x(\delta)) \\
x(\alpha) &= z(\beta)x(\gamma)/w(\gamma) \\
x(\beta) &= x(\gamma)^2 w(\delta)/(w(\gamma)^2 x(\delta)) \\
z(\alpha) &= \frac{x(\gamma)w(\delta)}{w(\gamma)x(\delta)}
\end{aligned}
$$

and E_2 is described simply by $x(\gamma) = 0$. Coordinates on E_2 on this patch are $x(\delta)$, $w(\gamma)$, $w(\delta)$, $z(\beta)$, $v(\alpha)$, $v(\beta)$, $v(\gamma)$, $v(\delta)$.

By [Sp, p. 148] $\mathcal{B}_{u_0}$ the 2-dimensional variety of Borel subgroups containing a given 2-regular element u_0, contains a unique projective line ℓ_β of type β and $\mathcal{B}_{u_0}$ is a union of lines of type α which intersect the line of type β. We consider the variety $E_2(u_0)$ of all points p of E_2 above $u_0 = \pi(p) \in G$. We fix a Borel subgroup B_0 in the line ℓ_β (whose points are Borel subgroups) and select any B_∞ opposite B_0. Then by the choice of B_0 $x_\alpha(u_0) = x_\beta(u_0) = x_\gamma(u_0) = 0$. Also if in the notation of (3.12) we have $(u, B_0^{nw})^v \in E_2(u_0)$, then $x_\alpha(u) = x_\beta(u) = x_\gamma(u) = 0$ and $u^v = u_0$. This forces $v = exp(\xi x_{-\beta})$ for some ξ. It follows that $x(\delta), v(\alpha), v(\gamma), v(\delta)$ serve as coordinates on an open set of the conjugacy class O_{u_0} of u_0 while $z =^{def} z(\beta)$, $\xi =^{def} v(\beta)$, $w =^{def} w(\gamma)$ and $\tilde{w} =^{def} w(\delta)$ serve as coordinates on an open set of the fibre $E_2(u_0)$. We also set

$$
\begin{aligned}
(6.2) \qquad & w_A = 2(T_1 - T_2)w + 1, \quad w_B = -2T_2 w + 1, \quad w_D = 2T_1 w + 1 \\
& \ell = \tilde{w}/w_B
\end{aligned}
$$

The differential form on E_2 is $\omega_{E_2} =^{def} Residue_{E_2}(\omega_Y/\lambda^\beta)$ $\quad \beta = b(E_2)/a(E_2) = 3$. Using the coordinates of (6.1) we obtain

$$\omega_Y/\lambda^3 = -x(\delta)\frac{dz}{z^2}\frac{d\tilde{w}}{\tilde{w}^2}dw\frac{dx(\gamma)}{x(\gamma)}dx(\delta)d\xi dv(\alpha)dv(\gamma)dv(\delta)$$

and

$$\omega_{E_2} = -x(\delta)\frac{dz}{z^2}\frac{d\tilde{w}}{\tilde{w}^2}dw\ dx(\delta)\ d\xi\ dv(\alpha)\ dv(\gamma)\ dv(\delta). \tag{6.3}$$

We separate this form into two parts: an invariant form on the conjugacy class O_{u_0} and a form on the fibre $E_2(u_0)$. $\omega_0 = -x(\delta)dx(\delta)dv(\alpha)dv(\gamma)dv(\delta)$ is easily seen to be an invariant form on O_{u_0} and $\omega_{E_2(u_0)} = \frac{dz}{z^2}\ \frac{d\tilde{w}}{\tilde{w}^2}dwd\xi$ is then the form on the fibre $E_2(u_0)$.

Next we consider the action of the group $W \times A$ on the variables. By our choice of Borel subgroup B_0, as with the subregular germs, A has two elements $\{1, \sigma_0\}$.

Equations (5.10) give the first two rows of (6.4)

(6.4)

	$\underline{w}$	$\underline{\ell}$	$\underline{z}$	$\underline{\xi}$	$\underline{T_1}$	$\underline{T_2}$
σ_α	$\overline{w}/\overline{w}_A$	$\ell/(1+f_T\ell)$	$z/\overline{w}_A$	ξ	T_2	T_1
σ_β	w/w_B	ℓ	z	$2T_2z+\xi$	T_1	$-T_2$
σ_0	$(\xi w+z)/(\xi+2T_2z)$	ℓ	$z/(\xi(2T_2z+\xi))$	$-1/\xi$	T_1	T_2

We have used the abbreviations $\ell = \tilde{w}/w_B$, $\overline{w} = w + (T_1 - T_2)\tilde{w}$, $\overline{w}_A = w_A + (T_1 - T_2)^2\tilde{w}$. The last row of (6.4) as well as the relation $\sigma_0(x(\alpha))/x(\alpha) = c\xi^{-1}$ (for some immaterial constant c depending on the representative A_σ in the normalizer), used below in the calculation of c_σ, is found by the method described at the end of (3.13). The cocycle t_σ is given by $\sigma_\alpha \mapsto (1/x(\alpha))^{\alpha^\vee}$, $\sigma_\beta \mapsto (1/x(\beta))^{\beta^\vee}$, $\sigma_0 \mapsto (\xi_\beta)^{\beta^\vee}$. We adjust t_σ by $c_\sigma = \sigma(t)t^{-1}$, $\quad t = (1, x(\alpha))$

c_σ : $\quad \sigma_\alpha \mapsto (x(\alpha))^{\alpha^\vee}$, $\sigma_\beta \mapsto (w_Bx(\alpha)^2)^{-\beta^\vee}$, $\sigma_0 \mapsto \xi^{-\beta^\vee}$

$c_\sigma t_\sigma$: $\quad \sigma_\alpha \mapsto 1$, $\sigma_\beta \mapsto (w_Bx(\alpha)^2x(\beta))^{-\beta^\vee}$, $\sigma_0 \mapsto 1$.

κ is trivial on d_σ: $\sigma_\alpha \mapsto 1$, $\sigma_\beta \mapsto (\lambda^2)^{\beta^\vee}$, $\sigma_0 \mapsto 1$. So we may consider

$$c_\sigma t_\sigma d_\sigma \;=\; \sigma_\alpha \mapsto 1, \quad \sigma_0 \mapsto 1, \quad \sigma_\beta \mapsto (\frac{z(\alpha)^2}{w_Bx(\beta)})^{\beta^\vee} = \ell^{\beta^\vee}. \tag{6.5}$$

$\kappa(t_\sigma) \;=\; \kappa(t''_\sigma)$ depends only on ℓ.

We recall the fact that $a(E_2) \;=\; 2$ implies that there is a term of the asymptotic expansion $F_1(\theta, 3)$ for every character θ in $F^\times$ of order 2.

PROPOSITION 6.6. *$F_1(\theta, 3) = 0$ if θ is non-trivial.*

PROOF: Let T_0 be a split Cartan subgroup. Then the action of G on Y_Γ gives an action of $T_0 \subseteq G$ on Y_Γ. To make this explicit select two Borel subgroups B_0, B_∞ over F with $B_0 \cap B_\infty = T_0$. Then the action of $t_0 \in T_0(\overline{F})$ is given on $Y_1(B_\infty, B_0, \Sigma)$ by

$$(b, B_0^{nw})^v \;\rightarrow\; (b, B_0^{nw})^{vt_0}.$$

In particular it acts on the fibres $\varphi^{-1}(\lambda)$ in Y_Γ. Supposing $t_0 \in C_G(u_0)(\overline{F})$, it is not difficult to see that the action on points $E_2(u_0)$ above $u_0 \in G(F)$ in E_2 is given by

$$\begin{aligned} z &\longmapsto sz \\ \xi &\longmapsto s\xi \\ w &\longmapsto w \\ \ell &\longmapsto \ell \end{aligned}$$

where $s = \beta(t_0)$. By (6.4) this morphism from $E_2(u_0)$ to $E_2(u_0)$ is defined over F provided $s \in F^\times$. Call this morphism φ_s.

The morphism φ_s is easily seen to carry the form

$$\omega_{E_2(u_0)} \;=\; \frac{dz}{z^2}\,\frac{d\tilde{w}}{\tilde{w}^2}\,dw\;d\xi \quad \text{to itself.}$$

The action of φ_s on $m_{\theta,E}$ is also easily calculated. By (6.4) $\sigma(x(\gamma))x(\gamma)^{-1}$ for $\sigma \in Gal(\overline{F}/F)$ evaluated on E_2 is a function of w, ℓ and the tangent direction of Γ but not of ξ or z. By Hilbert's 90th we write $\widetilde{x(\gamma)} = x(\gamma)a$ with a a function of w, ℓ and the tangent direction, and $\widetilde{x(\gamma)}$ a variable over F. Then

$$\lambda \;=\; \frac{x(\gamma)^2\tilde{w}z}{w^2x(\delta)} \;=\; \frac{\widetilde{x(\gamma)}^2\,\tilde{w}z}{a^2w^2x(\delta)}$$

and $\theta(\lambda) = \theta(\frac{\tilde{w}z}{a^2w^2x(\delta)})$. Thus

$$m_{\theta,E} \overset{def}{=} \lim_{\lambda\to 0}\frac{\kappa(t_\sigma)}{\theta(\lambda)} \;=\; \frac{\kappa(t_\sigma)}{\theta(\frac{\tilde{w}z}{a^2w^2x(\delta)})}.$$

t_σ depends only on ℓ so that φ_s carries $m_{\theta,E}$ to

$$\frac{\kappa(t_\sigma)}{\theta(\frac{\tilde{w}\;z\;s}{a^2w^2x(\delta)})} \;=\; \frac{m_{\theta,E}}{\theta(s)}.$$

A change of coordinates does not change a principal value integral. Changing coordinates on $E_2(u_0)$ by the automorphism φ_s gives

$$F_1(\theta,3) = \left[\int_{E_2(u_0)} m_{\theta,E}|\omega_{E_2(u_0)}|\right] \mu_O^H(f) \;=\; \frac{1}{\theta(s)}\left[\int m_{\theta,E}|\omega_{E_2(u_0)}|\right] \mu_O^H(f).$$

So that if $\theta(s) \neq 1$ for some $s \in F^\times$ we must have $F_1(\theta,3) \;=\; 0$.

We turn to the Cartan subgroups associated to the endoscopic group SO(4). We define a birational map from $E_2(u_0)$ to $(\mathbf{P}^1)^4$ by

(6.7)
$$\begin{aligned}
w_2 &= \frac{-T_1 w_B + \ell_1}{T_2 w_B} & \ell_1 &\stackrel{def}{=} (1+T_2(T_1+T_2)\ell)(T_1+T_2)\\
\ell_2 &= \frac{f_T\ell}{f_T\ell+2} & w_B &\stackrel{def}{=} (-2T_2 w+1)\\
\xi_1 &= \xi & f_T &\stackrel{def}{=} (T_2^2 - T_1^2)\\
\xi_2 &= \xi + \rho z & \rho &\stackrel{def}{=} \frac{2T_2\ell_1}{-T_1 w_B + \ell_1}
\end{aligned}$$

where $w_2, \ell_2, \xi_1, \xi_2$ are coordinates on $(\mathbf{A}^1)^4 \subseteq (\mathbf{P}^1)^4$. The map is birational, for it may be inverted by inverting the relations for ℓ_2, ξ_1, w_2 and ξ_2 in that order. The form $\omega_{E_2(u_0)}$ in these coordinates is found to be

(6.8)
$$\frac{(T_2^2 - T_1^2)}{2}\,\frac{d\ell_2}{\ell_2^2}\,\frac{dw_2}{w_2}\,\frac{d\xi_1 d\xi_2}{(\xi_1-\xi_2)^2} \;=\; \omega_{E_2(u_0)}$$

Transferring the action of $W \times A$ via (6.7) to the coordinates $w_2, \ell_2, \xi_1, \xi_2$ gives:

(6.9)

	$\underline{1}$	$\underline{\sigma_\alpha}$	$\underline{\sigma_\beta\sigma_\alpha\sigma_\beta}$	$\underline{\sigma_0}$
w_2	w_2	w_2/f^2	$1/w_2$	$(\xi_2/\xi_1)w_2$
ℓ_2	ℓ_2	$-\ell_2$	$-\ell_2$	ℓ_2
ξ_1	ξ_1	ξ_1	ξ_2	$-1/\xi_1$
ξ_2	ξ_2	ξ_2	ξ_1	$-1/\xi_2$

where $f = \left(\dfrac{1+\ell_2}{1-\ell_2}\right)$.

t''_σ is given by $\sigma_0 \;\to\; 1$, $\sigma_\alpha \;\to\; 1$, $\sigma_\beta \;\to\; \ell^{\beta^\vee} \;=\; (\dfrac{2\ell_2}{f_T}\,\dfrac{1}{1-\ell_2})^{\beta^\vee}$. The factor $2\ell_2/f_T$ lies in $F^\times$ so that $\kappa(t_\sigma) = \kappa(t'_\sigma)$ where t'_σ is defined by $\sigma_0 \to 1, \sigma_\alpha \to 1$, $\sigma_\beta \to (1-\ell_2)^{-\beta^\vee}$. This implies that $\sigma_\beta\sigma_\alpha\sigma_\beta \to ((1+\ell_2)^{-1},(1-\ell_2)^{-1},1-\ell_2,1+\ell_2)$ so that as in the proof of (5.23) $\kappa(t_\sigma)=\eta_{E'}(1-\ell_2^2)$ where $E'=Inv(\sigma_0,\sigma_\alpha)$ the field invariant by σ_α and σ_0.

We may always choose the homomorphism $\mathrm{Gal}(\overline{F}/F) \to A$ so that $\xi_1,\xi_2=0,\infty$ are not rational points. Thus $\xi_2/\xi_1 \neq 0,\infty$ and is well defined. Assume that such a choice is made. Also if $\mathrm{Im}((\mathrm{Gal}(\overline{F}/F) \subseteq W) = \langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ then $\ell_2=\pm 1$, $f=0,\infty$ are F-rational points so that the action of $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ on w_2 for $f=0,\infty$ is not defined. But this has an effect because the integrals are principal value integrals.

The following lemma is proved in section 7.

LEMMA 6.10. *The principal value integral $\int_{E_2(u_0)} m_{\kappa,E}\,|\omega_{E_2}(u_0)|$ is preserved under the birational map (6.7).*

COROLLARY 1. *The 2-regular germ is zero if κ is trivial and $W_T \subseteq \langle 1,\sigma_\alpha,\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$.*

PROOF: For $s\in F^\times$ the automorphism of $E_2(u_0)$ sending w_2 to w_2, ℓ_2 to $s\ell_2$, ξ_1 to ξ_1 and ξ_2 to ξ_2 is defined over F. It takes the form $\omega_{E_2(u_0)}$ to $\omega_{E_2(u_0)}/s$ so that changing coordinates using this automorphism

$$\int_{E_2(u_0)} |\omega_{E_2(u_0)}| = \frac{1}{|s|}\int_{E_2(u_0)} |\omega_{E_2(u_0)}| (=0 \quad \text{if } |s|\neq 1).$$

COROLLARY 2. *The 2-regular germs have the form $|T_1^2-T_2^2|c(T,\kappa)$ for some constants $c(T,\kappa)$ depending on (T,κ).*

PROOF: The data (6.9) is independent of T_1,T_2 and $|\omega_{E_2(u_0)}| = |T_1^2-T_2^2|\;|\omega'|$ where ω' is a form independent of T_1,T_2.

COROLLARY 3. *The transfer of orbital integrals in (2.8) holds for a function f^H independent of (T,κ) associated to $H=SO(4)$.*

PROOF: To prove the compatiblity of functions f^H for various choices of (T,κ) associated to $SO(4)$ we reason as follows. We find it convenient to pass to the adjoint group G/Z, $Z=Z(G)$ in which T/Z becomes a product of two tori. Germs are not affected (2.5) but we must replace $SO(4)$ by the product $H=PSL(2)\times PSL(2)$. We write $T/Z=S_1\times S_2$ where the tangent direction in S_1 is T_1-T_2 and T_1+T_2 in S_2. Fix a function $f\in C_c^\infty(G)$ such that $\mu_O^H(f)=0$ if O is not 2-regular. Using transfer factors $\Delta_H^{T,\kappa}$ we fix normalized subregular germs A_{S_1},B_{S_2} on H. A_{S_1} are functions on S_1 of the form $\lambda|T_1-T_2|a_{S_1}$, a_{S_1} a constant independent of S_2 (see [LS]). Similarly $B_{S_2}=\lambda|T_1+T_2|b_{S_2}$. By (2.9)

$$\Delta_G^{T,\kappa}\Phi_G^{T,\kappa}=\lambda^2|T_1^2-T_2^2|c(T,\kappa)=\lambda|T_1-T_2|a_{S_1}\mu_{O'}^{H'}(f_{qs}).$$

It was argued in (2.9) that f_{qs} is independent of $T_1 - T_2$. The choice of f_{qs} may also be made independently of S_1. To see this we use the explicit formula for f^M in [R] and note that the choice of f_{qs} in the definition of QSR (2.10) may be chosen independently of S_1. Thus we conclude that $\mu_{0'}^{H'}(f_{qs}) = \lambda|T_1 + T_2|b'_{S_2}$ or $c(T,\kappa) = a_{S_1}b'_{S_2}$ for some constants b'_{S_2}. Reversing the roles of S_1 and S_2 we have $c(T,\kappa) = a'_{S_1}b_{S_2}$ for constants a'_{S_1}. We conclude that $c(T,\kappa) = \alpha_1 a_{S_1}b_{S_2}$ for some constant α_1. Thus the 2-regular germ is $\alpha_1\lambda^2|T_1^2 - T_2^2|a_{S_1}b_{S_2} = \alpha_1 A_{S_1}B_{S_2}$ which up to scalar α_1 is the 2-regular germ on H.

§7 Spurious Divisors and some technical details

This final section contains the remaining details of the proof of the transfer of orbital integrals on GSp(4). It is shown that the construction in the preceding sections satisfies the conditions of (1.1), none of the divisors outside of Y_U contributes to the germ expansion, any amount of blowing up along subvarieties of divisors outside of Y_U has no effect on the subregular germs, and the birational map $E_2(u_0) \to (\mathbf{P}^1)^4$ of (6.7) preserves principal value integrals.

To obtain coordinates outside Y_U we construct the variety Y_Γ from scratch. The variety of stars is defined on an open patch $S(B_\infty, B_0)$ by

(7.1)
$$\epsilon_{-\alpha}(s_4)\epsilon_{-\beta}(r_4)\epsilon_{-\alpha}(s_3)\epsilon_{-\beta}(r_3)\epsilon_{-\alpha}(s_2)\epsilon_{-\beta}(r_2)\epsilon_{-\alpha}(s_1)\epsilon_{-\beta}(r_1) = 1.$$

where $\epsilon_{-\alpha}(s) = \begin{pmatrix} 1 & & & \\ s & 1 & & \\ & & 1 & \\ & & -s & 1 \end{pmatrix}$ and $\epsilon_{-\beta}(r) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & r & 1 & \\ & & & 1 \end{pmatrix}$.

Multiplying out the relation (7.1) we obtain

(7.2)
$$\begin{aligned} &s_1 + s_2 + s_3 + s_4 = 0 \\ &r_1 + r_2 + r_3 + r_4 = 0 \\ &r_2 s_1 + r_3(s_1 + s_2) + r_4(s_1 + s_2 + s_3) = 0 \\ &r_2 s_1^2 + r_3(s_1 + s_2)^2 + r_4(s_1 + s_2 + s_3)^2 = 0 \end{aligned}$$

There is a $\mathbf{G}_m \times \mathbf{G}_m$ action on $S(B_\infty, B_0)$ given by $(s_i, r_i) \mapsto (ss_i, rr_i)$. There is an action of the cyclic group of order four given by $(s_i, r_i) \mapsto (s_{i+1}, r_{i+1})$ with indices read modulo 4 and an action of an involution given by $(s_i, r_i) \mapsto (-s_{1-i}, -r_{2-i})$. They combine to give an action of the Weyl group on $S(B_\infty, B_0)$ which commutes with the $\mathbf{G}_m \times \mathbf{G}_m$ action.

We note for future reference that the equation (7.2) imply

(7.3) $\qquad r_2s_1s_2 = -r_4s_4s_3$ plus identities obtained by permuting the indices cyclically.

There is a proper morphism $S_1 \xrightarrow{\varphi} S$ where $S_1(B_\infty, B_0) = \varphi^{-1}S(B_\infty, B_0)$ is covered by coordinate patches $U(m,n)$ $\quad m,n \in \{1,2,3,4\}$. The coordinates and relations on $U(m,n)$ are

$$
\begin{aligned}
& r, s, R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4 \qquad \text{satisfying the relations} \\
& R_m = S_n = 1 \\
& S_1 + S_2 + S_3 + S_4 = 0 \\
& R_1 + R_2 + R_3 + R_4 = 0 \\
& R_2S_1 + R_3(S_1+S_2) + R_4(S_1+S_2+S_3) = 0 \\
& R_2S_1^2 + R_3(S_1+S_2)^2 + R_4(S_1+S_2+S_3)^2 = 0
\end{aligned} \tag{7.4}
$$

The morphism φ from $U(m,n)$ to $S(B_\infty, B_0)$ is given by $(S_i, R_j, s, r) \mapsto (sS_i, rR_j)$. It is proved in [H] that φ is a proper map from S_1 to S.

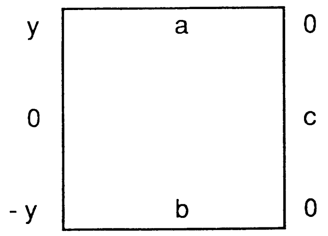
LEMMA 7.5. *(a) S_1 is covered by patches isomorphic to $U(1,1)$*

(b) S_1 is nonsingular.

PROOF: (a) The patches $S(B_\infty, B_0)$ as (B_∞, B_0) vary are all isomorphic. So we may fix (B_∞, B_0).

Let $Z = \{R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4\}$ and for $p \in U(m,n)$ set $Z_p = \{z \in Z \mid z(p) = 0\}$. The possibilities for Z_p are calculated in [H]. Up to a symmetry of the Weyl group acting on the indices they are

(7.6) $\qquad a + b + c = 0, y \neq 0.\quad$ If $a = 0$ then $bc \neq 0$.



(7.7) $$a+b+c=0 \quad x \neq 0. \text{ If } b=0 \text{ then } ac \neq 0.$$

$$\begin{array}{c|c|c} a & 0 & c \\ 0 & & -x \\ b & x & 0 \end{array}$$

(7.8) $$x \neq 0. \text{ If } y=0 \text{ then } z \neq 0.$$

$$\begin{array}{c|c|c} y & x & z \\ 0 & & 0 \\ -y & -x & -z \end{array}$$

(7.9) $$y \neq 0. \text{ If } x=0 \text{ then } z \neq 0.$$

$$\begin{array}{c|c|c} y & x & 0 \\ z & & -x \\ 0 & -z & -y \end{array}$$

where the variables are arranged around the squares as follows:

$$\begin{array}{c|c|c} S_2 & R_2 & S_1 \\ R_3 & & R_1 \\ S_3 & R_4 & S_4 \end{array}$$

Notice that (7.3) now states that the product of variables along an edge is the negative of the product of variables along the opposite edge.

By examining (7.6)-(7.9) it is evident that there are always two adjacent variables on the square that are non-zero. By the action of the Weyl group, we may take the adjacent variables to be R_1, S_1. Such a point is contained in a patch isomorphic to $U(1,1)$. This proves (a).

(b) By (a) it is enough to prove that $U(1,1)$ is nonsingular. We consider two patches A and B.

(A) Let $x = S_4$, $y = S_3R_4$. Then inverting the relations (7.2) we find

$$\begin{array}{ll} S_1 = 1 & R_1 = 1 \\ S_2 = xy + y - 1 & R_2 = -xy/(xy + y - 1) \\ S_3 = -x - y - xy & R_3 = x/((xy + y + x)(xy + y - 1)) \\ S_4 = x & R_4 = -y/(xy + y + x) \end{array} \tag{7.10}$$

We see that x and y are local coordinates, unless $(x,y) = (0,1)$ or $(0,0)$. (Unless the numerator and the denominator of R_2, R_3, or R_4 simultaneously vanish, x and y describe a point in $S_1(B_\infty, B_0)$ although not necessarily a point in $U(1,1)$). By (7.10) it follows that

$$t \stackrel{def}{=} y/x = \frac{S_3R_4}{S_4} = \frac{-R_4}{S_2R_3} = \frac{R_2}{R_3S_3S_4}. \tag{7.11}$$

Near $(x,y) = (0,0)$ we may select coordinates (x,t) or $(y,1/t)$ unless $R_2 = R_4 = S_4 = 0$, $R_1 = S_1 = 1$, $S_2R_3 = 0$. But the list of possible patterns Z_p in (7.6)-(7.9) shows that this never occurs. We conclude that patch (A) (together with the variants (x,t) or $(y,1/t)$) covers points such that $(x,y) \neq (0,1)$, i.e., $S_4 = 0, S_3R_4 = 1$.

(B) Let $S_1 = R_1 = 1$, $a = S_2$, $b = R_3$. Then inverting the relations (7.4) we find

$$\begin{array}{ll} S_1 = 1 & R_1 = 1 \\ S_2 = a & R_2 = -b(1+a)^2/(1+ba^2) \\ S_3 = (1+a)/(ab-1) & R_3 = b \\ S_4 = -ab(1+a)/(ab-1) & R_4 = -(ab-1)^2/(1+ba^2) \end{array} \tag{7.12}$$

We see a and b are local coordinates unless $ab = 1$ and $1 + a = 0$, i.e., $a = -1$, $b = -1$ $(S_2 = -1, S_1 = 1, R_3 = -1)$.

Points in the complement of both patches must satisfy $S_4 = 0; R_1, S_1, S_2, R_3, S_3, R_4 \neq 0$. This is impossible by the list of possible patterns (7.6)-(7.9).

Next we turn to a calculation of $t^{-1}n^{-1}t\ n \in N$ for $n \in N, t \in T$. Using the notation following (5.26) we let

$$\begin{array}{lll} n = m(n_\beta, n_\alpha, n_\gamma, n_\delta) & (n'_\gamma = n_\gamma - n_\alpha n_\beta) & \\ y = m(y_\beta, y_\alpha, y_\gamma, y_\delta) & (y'_\gamma = y_\gamma - y_\alpha y_\beta), & n, y \in N \\ t = \operatorname{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \in T. & & \end{array}$$

LEMMA 7.13.

$$
\begin{aligned}
y_\alpha &= (1 - t_2 t_1^{-1}) n_\alpha \\
y_\beta &= (1 - t_2^{-2}) n_\beta \\
y_\gamma &= (1 - t_2^{-1} t_1^{-1}) n_\gamma + (t_2^{-1} t_1^{-1} - t_2 t_1^{-1}) n_\alpha n_\beta \\
y'_\gamma &= (1 - t_2^{-1} t_1^{-1}) n_\gamma + (t_2^{-2} - 1) n_\alpha n_\beta \\
y_\delta &= (1 - t_1^{-2}) n_\delta + (t_2^{-1} t_1^{-1} - t_2 t_1^{-1}) n_\alpha n'_\gamma
\end{aligned}
$$

provided $y = t^{-1} n^{-1} t\, n$.

PROOF: Elementary matrix computation.

Identify B_0 with the group of upper triangular matrices and fix an ordering on the positive roots a choice of root vectors by the condition

$$\prod_{\gamma'} exp(x(\gamma') X_{\gamma'}) = m(x(\beta), x(\alpha), x(\gamma), x(\delta)).$$

By definition (3.12) $w(\alpha) = w(\beta) = 1$, $w(\gamma) = \lambda y_\gamma / y_\alpha y_\beta$ and $w(\delta) = \lambda^2 y_\delta / y_\alpha^2 y_\beta$ so that with $\alpha = t_1 t_2^{-1}, \beta = t_2^2, \gamma = t_1 t_2, \delta = t_1^2$ we have:
(7.14)

$$
\begin{aligned}
w(\gamma) &= \frac{(1-\gamma^{-1})\lambda}{(1-\alpha^{-1})(1-\beta^{-1})} \left(\frac{n_\gamma}{n_\alpha n_\beta}\right) + \frac{(\gamma^{-1} - \alpha^{-1})\lambda}{(1-\alpha^{-1})(1-\beta^{-1})} \\
w(\delta) &= \frac{\lambda^2 (1-\delta^{-1})}{(1-\alpha^{-1})^2 (1-\beta^{-1})} \left(\frac{n_\delta}{n_\alpha^2 n_\beta}\right) + \frac{\lambda^2 (\gamma^{-1} - \alpha^{-1})}{(1-\alpha^{-1})^2 (1-\beta^{-1})} \left(\frac{n'_\gamma}{n_\alpha n_\beta}\right).
\end{aligned}
$$

To complete the description of $w(\gamma)$ and $w(\delta)$ it is necessary to compute $n_\gamma/(n_\alpha n_\beta))$ and $n_\delta/(n_\alpha^2 n_\beta)$ in terms of coordinates on the star variety. The functions $n_\alpha, n_\beta, n_\gamma$ and n_δ are determined by the relation $\mathbf{B} n_W n^{-1} \in \mathbf{B}\omega$ for all $\omega \in W$ (the Weyl group) with $W = W(\omega)$ (a Weyl chamber).

LEMMA 7.15. $n'_\gamma / n_\alpha n_\beta = S_4/S_1$, $n_\gamma/(n_\alpha n_\beta) = (S_4/S_1) + 1$, $n_\delta/(n_\alpha^2 n_\beta) = (R_1 S_4)/(R_4 S_3)$, $z(\alpha) = -s_4 \lambda/(1-\alpha^{-1})$, $z(\beta) = r_1 \lambda/(1-\beta^{-1})$.

PROOF: The final two relations are found in both [L] and [H]. By $n_W n^{-1} \in \mathbf{B}\omega$ we have (by writing out these matrices)

$n_\alpha = -1/s_4, n_\beta = 1/r_1, n'_\gamma = -n_\beta/s_1 = -1/r_1 s_1$. $n_\gamma = n'_\gamma + n_\alpha n_\beta = -1/r_1 s_1 - 1/r_1 s_4$, $r_4 s_4 n_\delta = n_\alpha + r_4 n_\gamma = -1/s_4 - r_4/r_1 s_1 - r_4/r_1 s_4 = -(r_1 s_1 + s_4 r_4 + r_4 s_1)/r_1 s_1 s_4 =$

$-r_3s_2/(r_1s_1s_4) = -r_3s_2/(-s_2r_3s_3) = 1/s_3$, $n_\delta = 1/(s_3r_4s_4) = -1/(r_2s_1s_2)$. We have made use of (7.2) and (7.3). This completes the proof.

Now combining (7.14) and (7.15) with $A = (1-\alpha^{-1})/\lambda$, $B = (1-\beta^{-1})/\lambda$, $C = (1-\gamma^{-1})/\lambda$, $D = (1-\delta^{-1})/\lambda$ we find

(7.16)

$$\begin{aligned} w(\gamma) &= \frac{C}{AB}\left(\frac{s_4}{s_1}+1\right)+\frac{(A-C)}{AB} \\ &= \frac{C}{AB}\frac{s_4}{s_1}+\frac{1}{B} \\ w(\delta) &= \frac{D}{A^2B}\frac{(r_1s_4)}{(r_4s_3)}+\frac{(A-C)}{(A^2B)}\frac{s_4}{s_1} \\ &= \frac{(s_4r_1)}{(r_4s_3)}\left(\frac{D}{A^2B}\frac{r_1s_1}{r_1s_1}+\frac{(A-C)}{A^2B}\frac{r_4s_3}{r_1s_1}\right). \end{aligned}$$

We are now in a position to identify all divisors and calculate their associated constants $a(E)$, $b(E)$, $\beta(E)$. By (7.5) we may assume that $R_1 = S_1 = 1$.

Begin with the assumption that $g =^{def} \left(\frac{D}{A^2B} + \frac{A-C}{A^2B}R_4S_3\right) \neq 0$ at p, a point on the divisor E. We consider several patches. We begin by using (7.15), (7.16) to rewrite the equations

$$\begin{aligned} \lambda w(\delta) &= z(\alpha)^2 z(\beta) x(\delta) \\ x(\alpha) &= \frac{\lambda}{z(\alpha)} \\ x(\beta) &= \frac{\lambda}{z(\beta)} \\ x(\gamma) &= \frac{\lambda w(\gamma)}{z(\alpha)z(\beta)} \end{aligned}$$

in the form.

(7.17)

$$\begin{aligned} \lambda &= \frac{R_4S_3S_4s^2rx(\delta)}{A^2Bg} \\ x(\alpha) &= \frac{-S_3R_4rx(\delta)s}{ABg} \\ x(\beta) &= \frac{S_3R_4S_4s^2x(\delta)}{A^2g} \\ x(\gamma) &= \frac{-S_3R_4sx(\delta)}{A^2g}\left(\frac{C}{AB}S_4+\frac{1}{B}\right) \end{aligned}$$

($g \neq 0$, Patch A $(x, y) \neq (0,0)$ $\quad x = S_4, S_3R_4 = y$)

$$\lambda = \frac{yxs^2rx(\delta)}{(A^2Bg)} \tag{7.18}$$

The form $\omega = d\lambda \ dx(\alpha) \ dx(\beta) \ dx(\gamma) \ dx(\delta) \ dv$ in these coordinates becomes (we may treat A, B, C, D as constants as $\lambda \to 0$)

$$\begin{aligned}\omega &= d\lambda \ d(\frac{\lambda A}{-S_4 s}) \ d(\frac{\lambda B}{r}) \ d(\frac{\lambda w(\gamma)AB}{-S_4 sr}) \ dx(\delta) \ dv \\ &= \frac{\lambda^3 x(\delta)}{g^2} \ \frac{CD}{A^3 B} dy \frac{ds}{s} \frac{dr}{r^2} \frac{dx}{x} dx(\delta) dv \end{aligned} \tag{7.19}$$

We obtain the divisors on this patch

E_a defined by $x = 0$; $\quad a(E_a) = 1, \quad \beta(E_a) = 3, \ S_4 = R_2 = R_3 = 0,$

$$x(\beta) = 0, \quad x(\alpha) = \frac{-yrsx(\delta)}{ABg},$$
$$x(\gamma) = \frac{-ysx(\delta)}{A^2 g}(\frac{C}{AB}S_4 + \frac{1}{B})$$

E_b defined by $y = 0$; $\quad a(E_b) = 1, \quad \beta(E_b) = 4, \quad R_2 = R_4 = 0, \ x(\alpha) = x(\beta) = x(\gamma) = 0.$

E_2 defined by $s = 0$; $\quad a(E_2) = 2, \quad \beta(E_2) = 3, \ x(\alpha) = x(\beta) = x(\gamma) = 0.$

E_α defined by $r = 0$; $\quad a(E_\alpha) = 1, \quad \beta(E_\alpha) = 2, \ x(\alpha) = 0$, etc.

E_{id} defined by $x(\delta) = 0$; $\quad x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0.$

Now we drop the assumption that $(x, y) \neq (0,0)$ and introduce the assumptions $g \neq 0$, Patch A, coordinates $t = y/x, x \quad (y = tx)$.

(7.18) and (7.19) become

$$\lambda = \frac{tx^2s^2rx(\delta)}{A^2Bg}$$
$$\omega = \frac{\lambda^3 x(\delta)CD}{A^3B} dt \frac{ds}{s} \frac{dr}{r^2} dx \ dx(\delta) \ dv$$

We have previously considered the divisors given by $t = 0, \quad s = 0, \quad r = 0, \quad x(\delta) = 0$. We obtain the new divisor given by $x = 0$.

E_c given by $x = 0; \quad a(E_c) = 2, \quad \beta(E_c) = 7/2$

Now we drop our previous assumptions and introduce the assumptions $g \neq 0$, Patch A, coordinates $u = x/y, y \quad (x = uy)$.

Equation (7.18) becomes

$$\lambda = \frac{uy^2 s^2 r x(\delta)}{A^2 B g}.$$

It is easy to see that all of these divisors meet a patch previously considered.

Again we drop our previous assumptions and assume that we are on patch B with coordinates a, b and that $g \neq 0$. We have by (7.10) and (7.12)

$$x = \frac{-ab(1+a)}{ab-1} \qquad y = \frac{(1+a)(1-ab)}{1+ba^2}.$$

Since $a = S_2$ and $b = R_3$. If $ab = 1$ then $S_2 R_3 = 1$. But $1 = S_2 R_3 = \dfrac{x}{xy+y+x}$ so that $(x,y) \neq (0,1)$. Likewise if $1 + ba^2 = 0$ then $S_2^2 R_3 = -1$. But $-1 = S_2^2 R_3 = \dfrac{x(xy+y-1)}{xy+y+x}$ so that again $(x,y) \neq (0,1)$. Since we have already investigated the divisors for $g \neq 0$ $(x,y) \neq (0,1)$ we may assume $ab - 1 \neq 0$ and $1 + ba^2 \neq 0$. (7.18) and (7.19) become

$$\lambda = \frac{ab(1+a)^2 s^2 r x(\delta)}{(1+ba^2)(A^2 B g)}$$
$$\omega = \lambda^3 x(\delta) \frac{(1-ab)}{ab(1+a)} d\left[\frac{(1+a)(1-ab)}{1+ba^2}\right] \frac{ds}{s}\frac{dr}{r^2} d\left[\frac{ab(1+a)}{1-ab}\right] dx(\delta)\ dv.$$

The three divisors $s = 0$, $r = 0$, and $x(\delta) = 0$ have already been considered. If $(1+a) = 0$ then $R_2 = S_3 = S_4 = 0$. Thus $(x,y) = (0,0)$ and we find that $(1+a) = 0$ defines the divisor E_c considered above. If $b = 0$ then $S_4 = R_3 = R_2 = 0$. $(x,y) = (0, -1-a)$ and we find that

$b = 0$ defines the divisor E_a. Finally if $a = 0$ then $S_2 = S_4 = 0$ and $(x, y) = (0, 1)$. Call this divisor E_d. We have $a(E_d) = 1$. Moreover, writing $(1+a)(1-ab)/(1+ba^2) = am_1 + 1$, $b(1+a)/(1-ab) = m_2$ we have

$$d\left[\frac{(1+a)(1-ab)}{1+ba^2}\right] d\left[\frac{ab(1+a)}{1-ab}\right] = d(am_1)\, d(am_2) = \frac{\partial(am_1, am_2)}{\partial(a,b)} da\, db.$$

Clearly $a^{-1}\ \partial(am_1, am_2)/\partial(a,b)$ is regular at the generic point of E_d. Consequently $\beta(E_d) > 3$. At this point we drop the assumption that we are on patch B with $g \neq 0$ with coordinates a, b.

If $g = 0$ at $p \in E$ then by the definition of g we have $S_3R_4 \neq 0$. Also the assumption that the tangent direction is regular implies that $S_3R_4 \neq 1$, or that $(x,y) \neq (0,1), (0,0)$. Thus we are on Patch A. Also the assumption that E does not meet Y_U of (3.7) together with $S_3R_4 \neq 0$, $S_3R_4 \neq 1$, $S_1 = R_1 = 1$ implies that $S_4 = 0$ at p. In fact, using (7.3) twice we have $0 = S_1S_2^2S_3S_4R_1R_2R_3R_4 = -S_1^2S_2S_4^2R_1^2R_2R_4 = (S_1R_1^2S_3R_4^2)S_4^3$ hence $S_4 = 0$ $(S_1, R_1, S_3, R_4 \neq 0)$. Since $w(\gamma) = \frac{C}{AB}S_4 + \frac{1}{B}$ we have that $w(\gamma) \neq 0$ at p. The relations (3.7)

$$\lambda = \frac{z(\alpha)z(\beta)x(\gamma)}{w(\gamma)}, \quad x(\alpha) = \frac{z(\beta)x(\gamma)}{w(\gamma)}, \quad x(\beta) = \frac{z(\alpha)x(\gamma)}{w(\gamma)}, \quad x(\gamma)w(\delta) = w(\gamma)z(\alpha)x(\delta)$$

become using (7.15) and (7.16)

$$\lambda = \frac{-xsrx(\gamma)}{wAB}, \quad x(\alpha) = \frac{rx(\gamma)}{wB}, \quad x(\beta) = \frac{-xsx(\gamma)}{wA}, \quad x(\gamma)g = \frac{-ywsx(\delta)}{A}.$$

If $x(\gamma) \neq 0$, then this last equation, together with the definition of g, yields

$$\frac{1}{y} = \frac{BA^2}{D}\left[\frac{-wsx(\delta)}{Ax(\gamma)} - \left(\frac{(A-C)}{A^2B}\right)\right].$$

This is non-zero on patch A so that y is given by the reciprocal of the right hand side of this equation. If $x(\delta) \neq 0$, then we obtain

$$s = \frac{-Ax(\gamma)g}{ywx(\delta)}.$$

If $x(\gamma) = x(\delta) = 0$ then $x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0$ and we obtain the divisor E_{id} which is irrelevant for our study of subregular and 2-regular germs. The only possible new divisor on this patch is that given by $x = 0$. But $x = 0$ implies $x(\beta) = 0$, and by (7.6)-(7.9), $R_2 = R_3 = S_4 = 0$. We recognize this as the divisor E_a.

We summarize the results in the first part of the following proposition.

PROPOSITION 7.20. *Let V be the open subvariety of elements in $Y_1(B_\infty)$ which do not lie over the identity element. Then*

(a) V is non-singular,

(b) divisors have normal crossings in V,

(c) divisors which do not meet $Y_U \cap V$ make no contribution to the germ expansion of κ-orbital integrals for κ nontrivial,

(d) If E is a divisor which does not meet Y_U, $\beta(E) \leq 3$, and T is an elliptic Cartan subgroup then E has no F-rational points.

PROOF: (a) and (b) have been verified on each patch.

(d) Up to conjugacy by the Weyl group the only divisors not meeting Y_U are:

$$\begin{array}{lll} E_a & \beta(E_a) - 1 & = 2 \\ E_b & \beta(E_b) - 1 & = 3 \\ E_c & \beta(E_c) - 1 & = 5/2 \\ E_d & \beta(E_d) - 1 & > 2. \end{array}$$

The condition that $\beta(E) \leq 3$ forces $E = E_a$ on which $S_4 = R_2 = R_3 = 0$. The corresponding pattern is (7.7). Let p be an F-rational point of a divisor under the action defined in (3.12). It is clear by (3.12) that if $z_1(W, \alpha_1) = 0$ at p for some W and simple root α_1 then $z_1(\sigma_T^{-1}W, \alpha_1) = 0$ at p. In other words the set Z_p of (7.5) is invariant under the Weyl group elements $\sigma_T \in W_T$. The symmetries of pattern (7.7) fix a chamber wall, hence the elements σ_T of W_T do as well. This chamber wall corresponds to a proper parabolic subgroup over F and W_T may be identified with a subgroup of the Weyl group of P. It follows that T is stably conjugate to a Cartan subgroup of P and hence that T is not elliptic.

(c) The germ expansion, excluding the germ of the identity element, has terms of homogeneity $0, 1$, and 2. A divisor E contributes a term of homogeneity $\beta(E) - 1$ (1.1). The only possiblity then is E_a and homogeneity $\beta(E_a) - 1 = 2$. But we have seen that if E_a has F-rational points T is not elliptic. If T is not elliptic, the germ expansion reduces to an expansion on a Levi factor for which the terms have homogeneity 0 and 1 but not 2. The proposition follows.

PROPOSITION 7.21. *The morphism defined by (6.7) preserves principal value integrals.*

PROOF: First we remark that the morphism extends to $E_2(u_0) \to \mathbf{P}^1 \times \mathbf{P}^1$ (ξ_1, ξ_2) with ξ_1, ξ_2 defined by (6.7). In fact, this morphism has a simple geometric interpretation. Let $(b, (B(W))) = (b_0, B_0^{nw})^v$. By the discussion following (6.1) $n_W v = exp(z_W X_{-\beta})$ for appropriate choices of z_W for points of $E_2(u_0)$. ξ_1 and ξ_2 have been chosen so that $\xi_1 = z_{W_+}$ and $\xi_2 = z_{W_-}$ where

W_- is the Weyl chamber opposite W_+. Thus $E_2(u_0) \to \mathbf{P}^1 \times \mathbf{P}^1$ (ξ_1, ξ_2) may be identified with the morphism $E_2(u_0) \to \mathbf{P}^1 \times \mathbf{P}^1 = B_0 \backslash P_\beta \times B_0 \setminus P_\beta$, $(b, (B(W))) \mapsto (B(W_+), B(W_-))$. From the definition of $\xi(= \xi_1)$ following (6.1) it is clear that $\xi_1 = z_{W_+}$. But $\xi_2 = \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(\xi_1) = \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(z_{W_+}) = z_{W_-}$ by (6.9) and the fact that $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(B(W_+)) = B(W_-)$.

Next we prove that the birational map $E_2(u_0) \to \mathbf{P}^1(\ell_2)$ extends to $Y_1(B_\infty, B_0) \cap E_2(u_0)$. Up to a linear fractional transformation (6.7), (7.16) ℓ_2 is equal to $(S_1R_1)/(R_4S_3)$. By (7.3)

$$\frac{S_1R_1}{R_4S_3} = \frac{R_3S_3}{R_2S_1} = \frac{-R_1S_4}{R_2S_2} = \frac{-R_3S_2}{R_4S_4}.$$

The morphism does not extend only if all the numerators and denominators in (7.22) simulataneously vanish. But if $S_i \neq 0$ then $R_i, R_{i+1} = 0$ which implies that $R_{i+2}, R_{i+3} \neq 0$ so that $S_{i+1} = S_{i+2} = S_{i+3} = 0$ contradicting $S_1 + S_2 + S_3 + S_4 = 0$. Thus the map extends.

Next we check that the birational map $E_2(u_0) \to \mathbf{P}^1$ (w_B) extends to $Y_1(B_\infty) \cap E_2(u_0)$. Up to a scalar w_B equals (7.16), (7.3) $S_4/S_1 = -R_2S_2/(R_4S_3)$. By the patterns of (7.6)-(7.9) it is not possible for $S_4 = S_1 = R_2S_2 = R_4S_3 = 0$ so that the map extends.

By considering the zero patterns of (7.6)-(7.9) and assuming that T is elliptic (T is elliptic when the 2-regular germ is non-zero) we may assume that we find ourselves at an F-rational point so that one of the following holds:

$$\begin{array}{lll}
(i) & S_i \neq 0 \quad R_i \neq 0 & \forall i \\
(ii) & S_1 = S_3 = 0 & S_i, R_j \neq 0 \text{ otherwise} \\
(iii) & S_2 = S_4 = 0 & S_i, R_j \neq 0 \text{ otherwise} \\
(iv) & R_1 = R_3 = 0 & S_i, R_j \neq 0 \text{ otherwise} \\
(v) & R_2 = R_4 = 0 & S_i, R_j \neq 0 \text{ otherwise}
\end{array}$$

Next we check that the birational map $E_2(u_0) \to \mathbf{P}^1$ (w_2) given by (6.7) extends to $E_2(u_0) \cap Y_1(B_\infty)$. Up to a linear fractional transformation it is enough to consider ℓ_1/w_B or even

$$q \stackrel{def}{=} (1 - \frac{s_1r_1}{r_4s_3})/(\frac{s_4}{s_1}).$$

On the patch (B) of (7.12) q is calculated to be $q = \dfrac{1-b-2ab}{b(1+a)^2}$. On patch B $\quad (a,b) \neq (-1,-1)$ so that this gives a well-defined point in $\mathbf{P}^1$. If $w_B = 0$, then $S_4 = 0$ so that we fall into case (iii). But in case (iii) $R_1 = S_1 = 1$, $S_2 \neq -1$ so that points of case (iii) lie in patch B. This case has already been treated. If $w_B = \infty$ then $S_1 = 0$ so that we fall into case (ii). But in case (ii)

$$\left(1 - \frac{s_1r_1}{r_4s_3}\right) = \left(1 + \frac{R_1S_4}{R_2S_2}\right)$$

is finite, so that the map to $\mathbf{P}^1$ extends to such points. Finally, if $w_B \neq 0, \infty$ it is easy to see that the map extends.

We consider coordinate transformations from one chart to another. We compare coordinates for the pair (B_∞, B_0) with those for the pair $(B'_\infty, B_0) = (B^n_\infty, B_0)$ with $n \in N_0$. We have

$$(b, B_0^{n_W})^v = (b'', B_0^{n''_W})^{v''n} \tag{7.23}$$

where $n_W, v \in N_\infty$, n''^n_W, $v''^n \in B^n_\infty$, $b''^n \in B_0$. Write $vn^{-1} = b_s v''$. Then $b^{b_s} = b''$. Write $b_s = sy$, $s = diag(s_1, s_2, s_2^{-1}, s_1^{-1})$ $y = m(y_\beta, y_\alpha, y_\gamma, y_\delta)$. Then

(7.24)

$$\frac{x(\alpha)''}{x(\alpha)} = s_2 s_1^{-1}, \quad \frac{x(\beta)''}{x(\beta)} = \frac{2T_2 z y_\beta s_2^2 + 1}{s_2^2}, \quad \frac{x(\gamma)''}{x(\gamma)} = \frac{s_2^2 y_\beta z + w}{s_1 s_2 w}, \quad \frac{x(\delta)''}{x(\delta)} = s_1^{-2}$$

$$w''_B = \frac{w_B}{(2T_2 z s_2^2 y_\beta + 1)}, \quad \tilde{w}'' = \frac{\tilde{w}}{(2T_2 z s_2^2 y_\beta + 1)}, \quad \ell'' = \ell$$

where doubly primed objects are the corresponding variables on the new coordinate patch. Also

$$z'' = \frac{s_2^2 z}{2T_2 z s_2^2 y_\beta + 1}.$$

We remark that this is the most general coordinate transformation that need be considered. For let B_∞, B'_∞ be any two Borel subgroups. If there are any points of $E_2(u_0)$ on $Y_1(B_\infty)$, $Y_1(B'_\infty)$, say $(p, (B(W)))$ and $(p', (B(W)'))$, then $B(W_+) \in \ell_\beta$, $B(W_+)' \in \ell_\beta$ the line of type β in the variety of Borel subgroups contain u_0. Since the set of Borels opposite B_∞ is open and $B(W_+) \in \ell_\beta$ and open sets of the Borels in ℓ_β are opposite B_∞. Likewise for B'_∞. Thus there is a $B_0 \in \ell_\beta$ opposite both B_∞ and B'_∞, so that $B'_\infty = B^n_\infty$ for $n \in N_0 \subseteq B_0$.

We have

$$\begin{aligned} b_s v'' = vn^{-1} &= exp(\xi X_{-\beta}) exp(-u X_\beta) \quad mod\ N_\beta \\ &= \begin{pmatrix} 1/(1-\xi u) & -u \\ 0 & 1-\xi u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi/(1-\xi u) & 1 \end{pmatrix} mod\ N_\beta \end{aligned}$$

So that $s_1 = 1$, $s_2 = 1/(1-\xi u)$, $v'' = exp(\xi'' X_{-\beta})$, $\xi'' = \xi/(1-\xi u)$, $s_2 y_\beta = -u$,

$$\frac{x(\alpha)''}{x(\alpha)} = \frac{1}{(1-\xi u)}, \quad \frac{x(\beta)''}{x(\beta)} = (1-\xi u)(1-\xi u - 2T_2 z u), \quad \frac{x(\gamma)''}{x(\gamma)} = -uz + (1-\xi u) w, \quad \frac{x(\delta)''}{x(\delta)} = 1.$$

$$w''_B = \frac{(1-\xi u) w_B}{(-2T_2 z u + 1 - \xi u)}, \quad \tilde{w}'' = \frac{(1-\xi u)\tilde{w}}{(1-\xi u - 2T_2 z u)} \quad \ell'' = \ell, \quad \ell''_1 = \ell_1, \quad \ell''_2 = \ell_2$$

$$z'' = \frac{z}{(1-\xi u - 2T_2 z u)(1-\xi u)}, \quad \frac{\ell_1''}{w_B''} = \frac{(1-(\xi+2T_2)u)}{(1-\xi u)} \frac{\ell_1}{w_B}.$$

Since $\ell_2'' = \ell_2$ the birational map $E_2(u_0) \to \mathbf{P}^1(\ell_2)$ extends. Selecting u so that $(1-\xi u) \neq 0$ $(1-(\xi+2T_2u) \neq 0)$ the birational map $E_2(u_0) \to \mathbf{P}^1(\ell_1/w_B)$ extends so that $E_2(u_0) \to \mathbf{P}^1(w_2)$ does as well.

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