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Shalika germs on $GSp(4)$


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This paper calculates all of the Shalika germs of the group $GSp(4)$ and its inner forms over a local $p$-adic field of characteristic zero. As a consequence we conclude that the conjectures of Langlands and Shelstad \cite{LS2}, relating linear combinations of germs on a reductive group $G$ to germs on the endoscopic groups of $G$, are valid for $G = GSp(4)$ or one of its inner forms. More generally, when $G = Sp(4)$ or one of its inner forms we show that the germs associated to the regular and subregular unipotent classes satisfy the conjectures of Langlands and Shelstad.

\section{A Review of Igusa Theory}

The approach to calculating Shalika germs used here was introduced by R.P. Langlands in \cite{L}. This paper fixes the group $GSp(4)$ and studies all of its germs. This should be contrasted with \cite{H} which studies a particular germ, that associated to the subregular unipotent classes, for all reductive groups. The essential ingredient is a theorem of Igusa on asymptotic expansions. For the sake of completeness we restate the theorem.
1. Let $X$ be a smooth variety over a curve $V (X \cong \Gamma)$ with $X, \Gamma$ and $\varphi$ defined over a local $p$-adic field $F$ of characteristic zero.

2. Suppose that there is a point $x_0$ on $\Gamma(F)$ such that $X$ is smooth outside $\varphi^{-1}(x_0)$ and $\varphi^{-1}(x_0)$ is a divisor with normal crossings with the property that any irreducible component of $\varphi^{-1}(x_0)$ with an $F$-rational point is defined over $F$. Let $\mathcal{E}$ be the set of irreducible components of $\varphi^{-1}(x_0)$ defined over $F$. Let $a(E)$ for $E \in \mathcal{E}$ be the multiplicity of $E$ in $\varphi^{-1}(x_0)$.

3. Let $\omega$ be a form of maximal degree on $X$ which is defined over the algebraic closure $\overline{F}$ of $F$ which is nonvanishing and regular outside $\varphi^{-1}(x_0)$. Write the divisor of $\omega$ as $D = \sum_{E \in \mathcal{E}} (b(E) - 1)E$ with $b(E) \in \mathbb{Z}$. We may ignore the term $D$ having no $F$-rational points.

4. Suppose that there is a torus $T$ over $F$ which is split by a Galois extension $E/F$ and rational functions $t_\sigma \in T(K_E)$ for $\sigma \in \text{Gal}(E/F)$ and $K_E$ the field of rational functions on $X \times \text{Spec}(F) \text{Spec}(E)$. Suppose that $t_\sigma$ defines a cohomology class $[t_\sigma]_p$ of $H^1(\text{Gal}(E/F), T(E))$ for $F$-rational points $p$ in a Zariski open set of $X$ and that $p \mapsto [t_\sigma]_p$ extends to a locally constant function on the $F$-rational points of $X \setminus \varphi^{-1}(x_0)$. For any character $\kappa$ of $H^1(\text{Gal}(E/F), T(E))$ and divisor $D \in \mathcal{E}$ we define a character $\kappa_D$ of $F^\times$ ($\kappa_D : F^\times \rightarrow \mathbb{C}^\times$) as follows. Pick local $p$-adic coordinates $\mu_1, \ldots, \mu_n$ over $F$ at $p_0 \in D(F)$ such that $\mu_1 = 0$ defines $D$ locally. Choose, if possible, $\kappa_D$ so that $\kappa(t_\sigma)/\kappa_D(\mu_1)$ extends to a function constant in a neighborhood of $p_0$. If $p_0$ lies in no other divisor of $\mathcal{E}$ then such a character exists and is independent of $p_0 \in D(F)$ and the choice of local coordinates.

5. If $\theta$ is a character of finite order of $F^\times$ and $\beta \in \mathbb{Q}$ let $\mathcal{E}(\theta, \beta)$ be the set of divisors in $\mathcal{E}$ such that $\theta^a(E) = \kappa_E$ and $b(E)/a(E) = \beta$. Let $e(\theta, \beta)$ be the maximum number of divisors of $\mathcal{E}(\theta, \beta)$ with non-empty intersection.

6. Let $f$ be a locally constant function on $X$ whose support is proper over $\Gamma$.

7. Normalize the valuation on $F$ by the additive Haar measure $dx$ so that $d(ax) = |a|dx$ and set $m(\lambda) = -\log q|\lambda|$ where $q$ is the order of the residue field of $F$. Extend, whenever necessary, $| \cdot |$ to extensions $E$ of $F$. Normalize the Haar measure $dx$ so that $\int_{|x| \leq 1} dx = 1$. Finally let $\lambda$ be a local $F$-parameter on $\Gamma$ such that $\lambda = 0$ defines the point $x_0$. 

196
**Proposition 1.1.** In the above situation, for $|\lambda|$ sufficiently small there is an expansion

$$F(\lambda) \overset{\text{def}}{=} \int_{\varphi^{-1}(\lambda)} \kappa(t_\sigma) f \frac{|\omega|}{|\varphi^{*}(d\lambda)|} = \sum \theta(\lambda)|\lambda|^{\beta-1} \sum_{r=1}^{e(\theta,\beta)} m(\lambda)^{-1} F_r(\theta, \beta).$$

The first sum runs over $(\theta, \beta)$ with $\mathcal{E}(\theta, \beta)$ nonempty. $F_r(\theta, \beta)$ is independent of $\lambda$ but depends on $r, \theta, \beta, f, \kappa$.

**Proof:** See [L] for a proof and details. The only difference in our presentation is that [L] incorporates $\kappa(t_\sigma)$ directly into the definition of $f$ which is not assumed to be locally constant.

We also need the explicit formula for $F_r(\theta, \beta)$ when $e(\theta, \beta) = 1, 2$. Begin with $e(\theta, \beta) = 1$. By construction $\kappa(t_\sigma)/\theta(\lambda)$ extends generically to $E \in \mathcal{E}$. Let $m_{\delta, E}$ be its restriction to $E$. By construction $\omega/(\lambda^{\beta-1} d\lambda)$ extends generically to $E$. Let $\omega_E$ be its restriction. Since $\beta$ is rational $\omega_E$ is defined up to a root of unity. Then we have

$$F_1(\theta, \beta) = \sum_{E \in \mathcal{E}(\theta, \beta)} \text{PV} \int_E m_{\delta, E} f |\omega_E|.$$

A principal value integral is required since $\omega_E$ may have poles and $m_{\delta, E}$ might not be locally constant. Now let $e(\theta, \beta) = 2$. We proceed as before but along the intersection of two divisors $E, E'$ in $\mathcal{E}(\theta, \beta)$ the forms $\omega_E$ and $\omega_{E'}$ have simple poles, and the principal value integrals $\text{PV} \int_E m_{\delta, E} f |\omega_E|$ diverge. The difficulty stems from the fact that the form $dx/x$ is scale invariant. The problem is overcome by truncating the integral on $E$ and $E'$ near the pole and adding a principal value integral on $E \cap E'$. We continue to write $F_1(\theta, \beta)$ as a sum of integrals over divisors but when $e(\theta, \beta) = 2$ the integrals must be regularized in this manner.

**Lemma 1.3.** Suppose that $\lambda = \alpha \mu_1 \ldots \mu_n$ in local $p$-adic $F$-coordinates on a patch $U$ containing $p \in D(F), D = E_1 \cap E_2$, that $\mu_i = 0$ defines $E_i$, $i = 1, 2$, $a_1 = a_2 = 1$, $b_1 = b_2$, and $E_i \in \mathcal{E}(\theta, \beta)$ $i = 1, 2$, for some $(\theta, \beta)$. Set

$$\omega_D = \text{Residue}_D \omega_{E_1} = \frac{\omega_{E_1}}{\frac{\partial \omega_{E_1}}{\partial \mu_2}} \bigg|_D = \text{Residue}_D \omega_{E_2}, \text{ and } m_{\theta, D} = m_{\theta, E_1} |_D = m_{\theta, E_2} |_D.$$

Suppose that $U$ is chosen small enough that $|\alpha|$ is constant on $U$. Suppose that the integrals on $E_i$ are truncated by $|\mu_i| \geq q^{-m_i}, i = 1, 2$, then the contribution to the term $F_1(\theta, \beta)$ from the pole $D$ is given locally by

$$(1 - \frac{1}{q}) \int_{U \cap D} m_{\theta, D} (1 - M) f |\omega_D|.$$

197
where \( M = m(\alpha) + a_1m_1 + a_2m_2 + \sum_{i>2} a_i m_i \).

PROOF: This is a special case of the general formula found in [L].

§2. BACKGROUND ON SHALIKA GERMS

Let \( G \) be a reductive group over a \( p \)-adic local field \( F \) of characteristic zero, and let \( T \) be a Cartan subgroup over \( F \). For every unipotent orbit \( O \) in \( G(F) \) let \( \mu_0 \) denote an invariant measure on \( O \). Let \( dg \) be an invariant measure on \( T(F) \setminus G(F) \). Shalika [Sh] has shown that there exist functions \( \Gamma_0(\gamma) \) called germs defined on the regular elements of \( T(F) \) for all unipotent classes \( O \) in \( G(F) \) such that for every \( f \in C_c^\infty(G) \), the space of locally constant functions of compact support on \( G(F) \), there is a neighborhood \( V_f \) of the identity element in \( T(F) \) in which the expansion

\[
\int_{T(F) \setminus G(F)} f(x^{-1} \gamma x) dg = \sum_0 \mu_0(f) \Gamma_0(x) \text{ holds for } \gamma \text{ regular in } V_f.
\]

If \( g \in (T \setminus G)(F) \), then \( \sigma(g)g^{-1} \) for \( \sigma \in Gal(\overline{F}/F) \) defines a cocycle of \( H^1(Gal(\overline{F}/F), T) \). Now let \( dg \) denote an invariant measure associated to an invariant form on \( T \setminus G \). Similarly normalize measures \( \mu_0 \) on unipotent classes belonging to the same stable conjugacy class by fixing an invariant form on the stable conjugacy class. For \( h \in G(\overline{F}) \) such that \( \sigma(h)h^{-1} \) gives a cocycle of \( Gal(\overline{F}/F) \) in \( Z \) the center of \( G \), define \( f_h \) by \( f_h(x) = f(h^{-1}xh) \). Let \( \kappa \) be a character on \( H^1(Gal(\overline{F}/F), T) \).

We form a \( \kappa \)-orbital integral and take its germ expansion:

\[
\Phi^{T,\kappa}(\gamma, f) \overset{def}{=} \int_{(T \setminus G)(F)} \kappa(\sigma(g)g^{-1}) f(g^{-1} \gamma g) dg = \sum_0 \mu_0(f) \Gamma_0^{T,\kappa}(\gamma) \quad \gamma \text{ in } V_f.
\]

Comparing the \( \kappa \)-orbital integrals of \( f \) and \( f_h \) it follows easily that

\[
(2.1) \quad \Gamma_0^{T,\kappa} = \Gamma_0^{T,\kappa} \kappa(\sigma(h)h^{-1}).
\]

The character \( \kappa \) restricted to \( H^1(Gal(\overline{F}/F), Z) \) depends only on the endoscopic group \( H \) defined by \( (T, \kappa) \) and not directly on \( T \) (see for instance [H]). For background on endoscopic groups see [L2]. We may therefore write for an adjoint conjugacy class \( O \) (that is, the inverse image of an \( F \)-orbit in the adjoint group)
The sum runs over all \( F \)-classes \( O' \) in the adjoint class, and \( h \) is determined by \( O' = O'_1 \) for a fixed \( F \)-class \( O_1 \) in \( O \). Using this fixed choice \( O_1 \subseteq O \) write \( \Gamma^T_{O_1} \) for \( \Gamma^T_{O_1} \). Then the germ expansion becomes

\[
\Phi^T_{\gamma, f} = \sum_{O \text{ adjoint}} \mu^H_O (f) \Gamma^T_{O_1} (\gamma).
\]

All ordinary orbital integrals may be recovered as linear combinations of \( \kappa \)-orbital integrals. One advantage of considering \( \kappa \)-orbital integrals instead of ordinary orbital integrals is that one is able to group \( F \)-classes belonging to the same adjoint class together in this way. In all that follows a germ is associated to an adjoint unipotent conjugacy class using the measures \( \mu^H_O \).

We will make use of the following results from Harish-Chandra and Rogawski. Say that an orbit \( O \) is \( r \)-regular if \( r = (\dim C_G(u) - \text{rank } G)/2 \) for \( u \in O \). If \( r = 0,1 \) the classes are also called regular and subregular respectively. Let \( Z_G^0 \) be the connected center of \( G \). For details on normalizations of measures and proofs we refer to [H-Ch] and [R].

**Proposition 2.4.**

1. If \( z \in Z_G^0 (F) \) and \( \gamma \in T(F) \) lie sufficiently close to the identity then
   \[
   \Gamma^T_{O_1} (z \gamma) = \Gamma^T_{O_1} (\gamma).
   \]

2. If \( X \in \text{Lie } G(F) \) is regular and \( \exp(X) \) is sufficiently close to the identity then
   \[
   \Gamma^T_{O_1} (\exp(t^2 X)) = |t|^{2(r - r_0)} \Gamma^T_{O_1} (\exp(X))
   \]
   for \( t \in F^\times \) sufficiently small for every \( r \)-regular class \( O, r_0 = (\dim G - \text{rank } G)/2 \).

3. Let \( M \) be the connected centralizer of a semisimple element \( \gamma_0 \) in \( T \). Then for every \( f \in C^\infty_c (G) \) there exists \( f^M \in C^\infty_c (M) \) such that
   \[
   \Phi^T_G (\gamma, f) = \Phi^T_M (\gamma, f^M)
   \]
   for regular elements \( \gamma \) in a sufficiently small neighborhood \( V_f \) of \( \gamma_0 \) in \( T(F) \).

(2.5) Statement (2.4.1) tells us that the centers are mostly irrelevant to the study of germs. By passing to the derived group and then to the simply connected cover we may assume that \( G \)
is simply connected or semi-simple. Notice that the function $\kappa(\sigma(g)g^{-1})$ on $(T \setminus G)(F)$ always pulls back to the simply connected cover $G_{sc}$ of the derived group and that $T_{sc} \setminus G_{sc} \rightarrow T \setminus G$ is an isomorphism over $F$. Also $(T_{sc}, \kappa_{sc})$ defines an endoscopic group $H'$ of $G_{sc}$ which differs from the endoscopic group $H$ of $G$ defined by $(T, \kappa)$ only by a central factor. The simply connected semi-simple group is more difficult to deal with than say the adjoint group because there are more endoscopic groups associated to the simply connected groups.

(2.6) By combining (2.4.3) with (2.4.1) writing $T = T_{1}Z_{M}$ where $T_{1} \subseteq M_{der}$ for sufficiently small $\gamma_{1} \in T_{1}(F)$, and sufficiently small $z_{1}, z_{0} \in Z_{M}^{0}(F)$ we have

$$\Phi_{G}^{T}(\gamma_{1}z_{1}z_{0}, f) = \Phi_{G}^{T}(\gamma_{1}z_{0}, f)$$

provided $G(\gamma_{0}) = M$. Thus we may consider the germs of the expansion of $\Phi_{G}^{T}(\gamma, f)$ near $z_{0}$ as functions on $T_{1}$ alone instead of $T$.

(2.7) Langlands and Shelstad [LS2] have defined transfer factors $\Delta_{G}^{T, \kappa}(\gamma)$ and have conjectured that for every $f \in C_{c}^{\infty}(G)$ there exists a function $f^{H} \in C_{c}^{\infty}(H)$ on the endoscopic group $H$ associated to $(T, \kappa)$ such that identifying Cartan subgroups in $H$ and $G$ we have

$$\Delta_{G}^{T, \kappa}\Phi_{T, \kappa}(\gamma, f) \Rightarrow \Delta_{H}^{T, st}\Phi_{T, st}(\gamma, f^{H})$$

for all $(T, \kappa)$ associated to $H$. The integral on the right is a stable orbital integral on $H$ (that is, the character $\kappa$ is trivial). We define $\Delta_{G}^{T, \kappa}$ for $\gamma$ regular and sufficiently close to 1 for $G$ quasi-split by the condition

$$\Delta_{G}^{T, \kappa}\Phi_{G}^{T, \kappa}(\gamma, f) = \mu_{0}^{H}(f)$$

for 0 the regular adjoint unipotent class and $f$ supported on regular elements of $G$. The factor $\Delta_{H}^{T, st}$ is defined similarly. Implicit in our definition of transfer factors is a choice of measure $\mu_{0}^{H}$. In [LS2] the factors $\Delta_{G}^{T, \kappa}$ and $\Delta_{H}^{T, st}$ are combined into a canonically defined single factor but they show the definition by (2.9) is equivalent to theirs (up to a scalar). For the reduction of this problem (2.8) to the problem of matching germ expansions of $\Delta_{G}^{T, \kappa}\Phi_{T, \kappa}$ and $\Delta_{H}^{T, st}\Phi_{T, st}$ near the identity element see [LS3].

Proposition (2.4) remains true with minor modifications when the transfer factor is included. From this point on we shift notation to let $\Gamma_{0}^{T, \kappa}$ be the germ of $\Delta_{G}^{T, \kappa}\Phi_{T, \kappa}$. By (2.9) we conclude that (2.4.1) holds for $G$ quasi-split. More generally, [LS2] shows that there is a character $\theta$ of $Z_{G}^{0}$
such that $\Delta(z\gamma) = \theta(z)\Delta(\gamma)$. By the explicit description of the transfer factor in [LS2] (2.4.2) holds with $\tau_0 = 0$. A version of (2.4.3) with transfer factors is proved in (2.11).

Statement (2.4.1) gives a necessary condition for the transfer of germs to an endoscopic group. Identifying the connected center $Z^0_H$ of $H$ with a subgroup $T \subseteq G$ it says that the germs of $\kappa$-orbital integrals on $G$ associated to $H$ should be invariant by $Z^0_H$.

The rest of this section shows, roughly speaking, that to prove (2.6) for a fixed $(T, \kappa)$ when $H$ is a product $H = H_1 \times H_2$ it is often enough to prove that the germs of $\Delta T, \kappa \Phi T, \kappa(\gamma, f)$ have a decomposition of the form $T^{T, \kappa} = \sum a_i b_i$ where $a_i$ are functions on $T \cap (H_1 \times \{1\})$ and $b_i$ are functions on $T \cap (\{1\} \times H_2)$. Since the germs on a product of groups equal the products of germs on the individual groups it is clear that such a decomposition is a necessary condition for (2.8) to hold. This section does not show that the choice of $f^H$ can be made independently of $(T, \kappa)$. This will be shown later in the special case $G = GSp(4)$.

For any reductive group $G$ let $G_{qs}$ denote a quasi-split inner form of $G$. It is unique up to an isomorphism over $F$. There is an injection of stable conjugacy classes of Cartan subgroups in $G$ to stable conjugacy classes of Cartan subgroups in $G_{qs}$. If $T$ is a Cartan subgroup over $F$ in $G$, write $T_{qs}$ for an image in $G_{qs}$ and for $\gamma \in T$ write $\gamma_{qs}$ for the corresponding element in $G_{qs}$. $T$ and $T_{qs}$ are then isomorphic over $F$ and this isomorphism may be used to identify characters on $H^1(Gal(F/F), T)$ and $H^1(Gal(\overline{F}/F), T_{qs})$.

If $S$ is a torus over $F$ in $G$ let $C(S)$ denote its centralizer in $G$.

**Definition 2.10.** The $\kappa$-orbital integrals on $G$ are said to have quasi-split reduction (QSR) if for every triple $(S, T, \kappa)$, $1 \neq S \subseteq T$, $S$ torus over $F$, $T$ Cartan subgroup over $F$, $\kappa$ character on $H^1(Gal(\overline{F}/F), T)$ and for every function $f \in C^\infty_c(C(S))$ there exists a function $f_{qs} \in C^\infty_c(C(S)_{qs})$ such that

$$\Phi_{C(S)}^{(T, \kappa)}(\gamma, f) = \Phi_{C(S)_{qs}}^{(T_{qs}, \kappa)}(\gamma_{qs}, f_{qs})$$

for every $\gamma \in T^1(F)$ (the regular elements of $T$). We note that this condition is independent of the choice of $T_{qs}$.

**Proposition 2.11.** Suppose that $G$ is quasi-split and the $\kappa$-orbital integrals of $G$ have quasi-split reduction. Let $(S, T, \kappa)$ be a triple as in the definition of QSR. Let $f \in C^\infty_c(G)$ and let $s \in S(F)$ be an element sufficiently close to the identity such that $C_G(s) = C(S)$. Then there exists a neighborhood $V$ (depending on $f$) of $s$ in $T(F)$ and a function $f_{qs}$ in $C^\infty_c(C(S)_{qs})$ (depending on $s$) such that
\[ \Delta_G^{\kappa} \Phi_G^{(T,\kappa)}(\gamma, f) = \Delta_C^{(s)\kappa} \Phi_C^{(T_{qs},\kappa)}(\gamma_{qs}, f_{qs}) \quad \text{for} \quad \gamma \in V \cap T'(F). \]

**Proof:** This is no more than (2.4.3) combined with the definition of QSR. A few details will make this clear.

Set \( M = C(S) \). The map \( i : T \to M \) gives \( i_* : H^1(\text{Gal}(\overline{F}/F), T) \to H^1(\text{Gal}(\overline{F}/F), M) \). If \( Tg \in (T \setminus G)(F) \) then the cocycle \( i_*(\sigma(g)g^{-1}) \), \( \sigma \in \text{Gal}(\overline{F}/F) \) belongs to \( H^1(\text{Gal}(\overline{F}/F), M) \). Let \( g_1, \ldots, g_r \in G(\overline{F}), Tg_1, \ldots, Tg_r \in (T \setminus G)(F) \) be such that \( i_*(\sigma(g_i)g_i^{-1}), i = 1, \ldots, r \) are representatives of the classes in \( H^1(\text{Gal}(\overline{F}/F), M) \) so obtained. Then \( M^{\gamma_i} \) is an inner form of \( M \). Set \( T_i = T^{\gamma_i}, M_i = M^{\gamma_i}, \gamma_i = \gamma^{\delta_i} \).

It is easy to check that \( (T \setminus G)(F) \) is a disjoint union of \( X_i = g_i((T_i \setminus M_i)(F))G(F), i = 1, \ldots, r \). The \( \kappa \)-orbital integral on \( G \) is an integral over \( (T \setminus G)(F) \) which breaks up as a sum of integrals over the \( X_i \). First we prove a version of the proposition for each \( X_i \), i.e., that there exists \( (f_i)_{qs} \in C_c^\infty(M_{qs}) \) such that

\[
\int_{(g_i^{-1}X_i)} \kappa_i(\sigma g^{-1})f(g^{-1}\gamma_i g)dg = \Phi_{M_{qs}}^{T_i,\kappa}(\gamma_i, (f_i)_{qs})
\]

where \( \kappa_i \) is the character on \( H^1(\text{Gal}(\overline{F}/F), T_i) \) obtained by identifying \( T \) and \( T_i \) by the isomorphism \( \gamma \mapsto \gamma_i \).

Let \( \epsilon_1, \ldots, \epsilon_r \) be the elements of the image of \( (T_i \setminus M_i)(F) \) in \( H^1(\text{Gal}(\overline{F}/F), T_i) \) and let \( m_1, \ldots, m_r \) be representatives in \( (T_i \setminus M_i)(F) \). Then dropping super and subscript i's on \( m_j, \epsilon_j, T_i \) we have, using the definition of \( X_i \)

\[
\int_{g_i^{-1}X_i} = \sum_j \kappa_i(\epsilon_j) \int_{T_i \setminus M_i(F) \setminus T_i(M_i(F))} f(g^{-1}\gamma_i m_j g)dg = \sum_j \kappa_i(\epsilon_j) \int_{T_i \setminus M_i(F) \setminus T_i(M_i(F))} f_i(m^{-1}\gamma_i m_j m)dm \quad \text{(by 2.4.3)}
\]

\[
= \Phi_{M_i}^{T_i,\kappa}(\gamma_i, f_i) = \Phi_{M_{qs}}^{T_i,\kappa}(\gamma_{qs}, (f_i)_{qs}) \quad \text{by the definition of QSR}.
\]

Thus combining the \( X_i \) and \( (f_i)_{qs} \)

\[
\Delta_G^{\kappa} \Phi_G^{(T,\kappa)}(\gamma, f) = \delta \Delta_{M_{qs}}^{\kappa} \Phi_{M_{qs}}^{T_i,\kappa}(\gamma_{qs}, f'_{qs}) = \Delta_{M_{qs}}^{T_i,\kappa} \Phi_{M_{qs}}^{T_i,\kappa}(\gamma_{qs}, \delta f'_{qs}), \delta = \Delta_T^{\kappa,\kappa} / \Delta_{M_{qs}}^{T_i,\kappa}.
\]

Fix \( s \) small enough so that (2.9) holds, then this equation becomes \( \mu_H^O(f) = \delta \mu_{H'}^{O'}(f'_{qs}) \) where \( O' \) is the regular unipotent class of \( M_{qs} \) and \( (T_{qs}, \kappa) \) defines the endoscopic group \( H' \). Thus \( \delta \) and
hence $f_{qs} = \delta f'_{qs}$ depends on $\gamma \in T(F)$, $\gamma_{qs} \in T_{qs}(F)$ only through $s \in S(F)$ in a sufficiently small neighborhood of the identity and sufficiently near $s$. The proposition follows.

As above let $G$ be a reductive group over a $p$-adic field $F$ of characteristic zero, and let $H$ be an endoscopic group of $G$. Suppose that up to a finite center $H$ is a product $H_1 \times H_2$ over $F$. Then we associate to $G$ and $H$ an intermediate group $M$ as follows.

Select a Cartan subgroup $T$ over $F$ in $H$. Corresponding to $H_1 \times H_2$ we write $T = T_1T_2$ with $T_1 \cap T_2$ finite. Identify $T = T_1T_2$ with a Cartan subgroup in $G$. It is determined up to stable conjugacy. Let $M = (C_G(T_1) \times C_G(T_2))/\{(x,x^{-1}) : x \in T\}$. For example, if $G = \text{Sp}(2n)$, $H = \text{Sp}(2i) \times \text{SO}(2(n-i))$, then $M = \text{Sp}(2i) \times \text{Sp}(2(n-i))$.

**Lemma 2.12.** $M_{qs}$ is independent of the choice of $T \subseteq H$.

**Proof:** It is enough to check that $T$ and $T' = T_1T_2$ give the same $M_{qs}$. Select $g \in Q(\overline{F})$ such that $T_2^g = T_2'$ where $Q = C_G(T_1)$. $C_{C_G(T_2)}(T_1T_2) = C_G(T_1T_2) = T_1T_2$ so $T_1T_2$ is a Cartan subgroup of $C_G(T_2)$. It follows that there is a Cartan subgroup $T'_{der}$ of $C_G(T_2)_{der}$ contained in $T_1$.

If $m \in C_G(T_2)_{der}$ $\sigma \in \text{Gal}(\overline{F}/F)$ then $\sigma(m)^g \sigma(g)^{-1} = \sigma(m^g)$, $\sigma(g)^{-1} = w_\sigma \in N_Q(T_2)$. By the definition of $Q$, $t^{w_\sigma} = t$ for $t \in T_{der}(\overline{F}) (\subseteq T_1(\overline{F}))$. Since $T_{der}$ is a Cartan subgroup, $w_\sigma \in T_{der}(\overline{F})$ so that $C_G(T_2)_{der}$ and $C_G(T_2')_{der}$ are inner forms. Using $TC_G(T_2)_{der} = C_G(T_2)$, $TC_G(T_2')_{der} = C_G(T_2')$ the result follows easily.

The following simple lemma is the key to what follows.

**Lemma 2.13.** Let $k$ be any field. Let $X,Y$ be sets endowed with decreasing filtrations $X = X_0 \supseteq X_1 \supseteq \ldots, Y = Y_0 \supseteq Y_1 \supseteq \ldots$. Let $\mathcal{F}_X, \mathcal{F}_Y$ be $k$-vector spaces of $k$-valued functions on $X$ and $Y$ respectively. Suppose a function $\varphi$ on $X \times Y$ has the form $\varphi = \sum_{i=1}^{p} a_i b_i, a_i \in \mathcal{F}_X, b_i \in \mathcal{F}_Y, i = 1,\ldots,p$. Suppose further that there exist functions $a'_i \in \mathcal{F}_X, b'_i \in \mathcal{F}_Y, i = 1,\ldots,q$ and an integer $n \geq 0$ such that for every $x \in X_n$ there is a $j_x$ such that

$$\varphi(x,\cdot) = \sum_{i=1}^{q} a'_i(x)b'_i \quad \text{as functions on } Y_{j_x}.$$

Then there exists a positive integer $N$ such that $\varphi = \sum_{i=1}^{q} a'_i b'_i$ on $X_N \times Y_N$. Moreover, if the $a'_1,\ldots,a'_p$ are linearly independent over $k$ on $X$, and if $b_1,\ldots,b_q$ are linearly independent over $k$ on $Y$, then there exists a matrix of constants $e_{ij} \in k, i = 1,\ldots,p, j = 1,\ldots,q$ and a positive integer $N'$ such that
\[ \varphi = \sum_{i,j} a'_{ij} e_{ij} b_j \text{ on } X_{N'} \times Y_{N'} \]

**Proof:** Choose \( r_1 > n \) so that the dimension of the span of the functions \( \{a_i\}_{i=1}^p \) or \( \{a'_i\}_{i=1}^q \) (resp. \( \{b_i\} \) or \( \{b'_i\} \)) is constant on \( X_i \) (resp. \( Y_i \)) for \( i \geq n_1 \). By combining terms in the sum we may assume that the \( \{b_i\} \) (resp. the \( \{a'_i\} \)) are 1-linearly independent for all \( Y_i, i \geq n_1 \) (resp. \( X_i, i \geq n_1 \)). Assuming now that \( a'_1, \ldots, a'_q \) are linearly independent on \( X_i, i \geq n_1 \), we may select \( x_1, \ldots, x_q \in X_{n_1} \) such that the matrix \( A' = (a'(x_i)) \) is invertible. Then if \( A = (a_j(x_i)), b = (b_1, \ldots, b_p)^t, b' = (b'_1, \ldots, b'_q)^t \) it is clear that \( Ab = A'b' \) on \( Y_j \) for \( j \geq n_2 \) \( \max_{x \in S}\). Set \( a = (a_1, \ldots, a_p), \ a' = (a'_1, \ldots, a'_q) \). Suppose there exists \( x \in X_{n_1} \) such that \( a(x) - a'(x)A' \neq 0 \). Then by the assumed linear independence of the \( b \)'s we have on \( Y_j, j \geq \max(j_x, n_2) \) \( 0 \neq (a(x) - a'(x)A' - 1 A)b = a(x)b - a'(x)b' = 0 \). This contradiction shows that \( a = a'A' - 1 A \) on \( X_{n_1} \). Consequently for \( N = \max(n_1, n_2) \) and on \( X_N \times X_N, \varphi = \sum a_i b_i = ab = a'A' - 1 A b = a'b' = \sum a'_{ij} b'_i \). For the second statement of the lemma we take \( e_{ij} = (A' - 1 A)_{ij} \).

Suppose that up to finite center \( H \) is equal to \( H_1 \times H_2 \). Let \( (T, \kappa) \) be a pair associated to \( H \) and let \( f \in C_c^\infty(G) \). Write \( T = T_1 T_2, T_1 \cap T_2 \) finite as above. Let \( A_1, \ldots, A_j \) be the germs of \( (T_{g \sigma}, \kappa) \) on \( C(T_2)_{g \sigma} \) normalized with transfer factors as in (2.9). These may be considered functions on \( T_1 \). Let \( B_1, \ldots, B_k \) be the normalized germs of \( (T_{g \sigma}, \kappa) \) on \( C(T_1)_{g \sigma} \), again considered as functions on \( T_2 \).

**Proposition 2.14.** Suppose \( G \) is quasi-split and that the \( \kappa \)-orbital integrals on \( G \) have quasi-split reduction. Suppose further that \( \kappa \)-orbital integrals have the form

\[ \varphi \overset{\text{def}}{=} \Delta_T^{T, \kappa} \Phi_G^{(T, \kappa)}(\gamma, f) = \sum a_i(\gamma_1) b_i(\gamma_2) \quad \gamma = \gamma_1 \gamma_2 \quad \gamma_i \in T_i \]

in a small neighborhood of the identity \( 1 \in T(F) \).

(1) There exist constants \( e_{ij} \) such that

\[ \varphi = \sum_{i,j} A_i(\gamma_1) e_{ij} B_j(\gamma_2) \]

in some possibly smaller neighborhood of the identity.

(2) \( \varphi = \Delta_{M_{q \sigma}}^{T, \kappa} \Phi_{M_{q \sigma}}^{T, \kappa}(f^M) \) for some function \( f^M \in C_c^\infty(M_{q \sigma}) \) on a sufficiently small neighborhood of the identity.
**PROOF:** (1) Proposition 2.11 shows that for every $\gamma_1 \in T_1(F)$ such that $C_{G}(\gamma_1) = C(T_1)$ there exists a function $F = F_{\gamma_1}$ such that

$$\varphi = \Delta_{C(T_1)}^{\gamma_1} \Phi_{C(T_1)}^{\gamma_1} (\gamma_{q_2}, F_{\gamma_1})$$

expanding the right side in a germ expansion

$$\varphi = \sum_i B_i \mu_i (F_{\gamma_1})$$

where $\mu_i$ are measures on unipotent orbits of $C(T_1)_{q_2}$. These expansions hold in some neighborhood $V \subseteq T_2(F)$ depending on $\gamma_1$. Thus Lemma 2.13 holds with $a'_i(\gamma_1) = \mu_i(F_{\gamma_1})$ $b'_i = B_i$, so that

$$\varphi = \sum A'_i B_i$$

for some $A'_i$ on some neighborhood of the identity. Interchanging the roles of $T_1, T_2$ we apply the lemma again to conclude

$$\varphi = \sum A_i B'_i$$

for some $B'_i$ on some neighborhood of the identity. By combining terms we may assume that the $A_i$ are linearly independent on every sufficiently small neighborhood of the identity. Likewise for $B_i$. We apply the last part of the lemma this time with

$$a'_i = A_i, \quad b'_i = B'_i, \quad b_i = B_i, \quad a_i = A'_i$$

to obtain $\varphi = \sum A_i \delta_{ij} B_j$ in some neighborhood of the identity.

(2) The germs on $M_{q_2}$ are $A_i B_j$. The result is immediate.

**§3. A VARIETY WHICH COMPUTES SHALIKA GERMS**

This section reviews a geometric approach to Shalika germs. If $X$ is a variety over $F$ we often write $x \in X$ instead of $x \in X(\overline{F})$.

Fix a curve $\Gamma$ in $T$ whose tangent direction at the identity does not lie in a singular hyperplane. Let $\lambda$ be a local parameter on $\Gamma$ such that $\lambda = 0$ gives the identity element of $T$. Suppose that $\Gamma(\lambda)$ is regular for $\lambda \neq 0$, and let $\Gamma^0 = \Gamma \setminus \{0\}$. There is a variety $Y_{\Gamma}$ over $\Gamma$ which fits into the diagram

$$\Gamma^0 \times T \setminus G \xrightarrow{\gamma} Y_{\Gamma} \xrightarrow{\varphi} G \xrightarrow{\pi} \Gamma.$$
The following properties of $Y_T$ are known [H], [L].

(3.1) If $T$ and $\Gamma$ are over $F$ then $Y_T, \varphi, \pi, i$ are also defined over $F$.

(3.2) $\pi : Y_T \to G$ is proper.

(3.3) $i$ embeds $\Gamma^0 \times T \setminus G$ as an open subvariety of $Y_T$.

(3.4) If $G$ is made to act on $\Gamma^0 \times T \setminus G$ by translations on the second factor, on $G$ by inner automorphisms and trivially on $\Gamma$ then there is a $G$ action on $Y_T$ which makes $i, \varphi, \pi$ into $G$-equivariant maps.

(3.5) $\pi \circ i(\gamma, g) = \gamma g; \quad \varphi \circ i(\gamma, g) = \gamma$.

(3.6) Let $E$ be an irreducible component of $\varphi^{-1}(0)$. Then $\pi(E)$ is the closure of a stable unipotent class $O$ in $G$. Call $E$ an $O$-divisor. If $O$ is regular (resp. subregular) we also call $E$ a regular (resp. subregular) divisor. There is exactly one irreducible component $E_0$ such that $\pi(E)$ coincides with the unipotent variety in $G$. $E_0$ is isomorphic to the Springer resolution $\{(u, B) : u \in B, u \text{ unipotent in } G, B \text{ a Borel subgroup}\}$ of the unipotent variety. Identifying $E_0$ and the Springer resolution, $\pi$ becomes $\pi(u, B) = u$.

(3.7) A Zariski open patch of $Y_T$ may be described in local coordinates as follows. It depends on a choice of opposite Borel subgroups $(B_\infty, B_0)$. Let $\Phi$ be the root system of $G$ with positive roots $\Phi^+$, let $p = |\Phi^+|$ be the number of positive roots and let $\Delta$ be the set of positive simple roots. Consider the affine space $A^{2p+1}$ the coordinates being labelled

\[
\begin{align*}
\varpi(\gamma) & \quad \gamma \in \Phi^+ \setminus \Delta \\
\varsigma(\alpha) & \quad \alpha \in \Delta \\
\varsigma(\gamma) & \quad \gamma \in \Phi^+ \\
\lambda & \quad (\text{identifying } \lambda \text{ with its pull-back to } Y_T).
\end{align*}
\]

There is an open set $Y_U = Y_U(B_\infty, B_0)$ of $Y_T$ which is isomorphic to an open set of the product $A^p \times Z$ where $Z = Z_\Phi$ is the Zariski closure in $A^{2p+1}$ of the variety defined for $\lambda \neq 0$ by the equations

\[
\lambda \varpi(\gamma) = \varsigma(\gamma) \Pi \varsigma(\alpha)^{m(\alpha)} \quad \gamma = \sum m(\alpha) \alpha \quad \alpha \in \Delta,
\]

where $\varpi(\alpha) = 1$ for $\alpha \in \Delta$.

We write $v(\gamma) \quad \gamma \in \Phi^+$ for coordinates on the factor $A^p$ of $A^p \times Z$.

(3.8) There is a $G$-invariant differential form of maximal degree $\omega_Y$ which is non-vanishing for $\lambda \neq 0$ which on $Y_U$ equals

\[
\omega_Y = d\lambda \bigwedge_{\gamma \in \Phi^+} d\varsigma(\gamma) \bigwedge_{\gamma \in \Phi^+} dv(\gamma).
\]
(3.9) Let $E$ be a splitting field of $T$ and suppose that $T$ is defined over $F$. Then the function $\sigma(g)g^{-1}$ for $\sigma \in \text{Gal}(E/F)$, $g \in (T \setminus G)(F)$ pushes forward to a function $t'_\sigma \in T(K_E)$, $K_E$ the rational functions on $Y_T \times \text{Spec}(F)$ $\text{Spec}(E)$. There is a cocycle $\delta^T_{\Gamma,\lambda}$ in $T(E(\lambda)) \subseteq T(K_E)$ such that $t_\sigma = \text{def} \delta^T_{\Gamma,\lambda} t'_\sigma$ satisfies condition (4) preceding Proposition 1.1. The factor $\delta^T_{\Gamma,\lambda}$ is determined for quasi-split groups by the condition that $\kappa(t_\sigma)$ extends generically to the regular divisor $E_0$ and

$$\int_{E_0} \pi^*(f) \kappa(t_\sigma) |\omega_{E_0}| = \mu^H_O(f) \quad \forall f \in C_c^\infty(G)$$

with $O$ the unique adjoint regular unipotent class and $\mu^H_O$ determined as in (2.2).

(3.10) $F(\lambda) = \int_{\phi^{-1}(\lambda)} \pi^*(f) \kappa(t_\sigma) \frac{|\omega_Y|}{|\phi^*(d\lambda)|} = \Delta^{T,\kappa} \Phi^{T,\kappa}(\Gamma(\lambda), f)$

for $\lambda$ sufficiently small and non-zero.

(3.11) The variety $Y_T$, form $\omega_Y$, functions $t_\sigma$, $\pi^*(f)$, etc., satisfy all of the conditions of Proposition 1.1 except that $Y_T$ is not in general smooth, and the irreducible components of $\phi^{-1}(0)$ are not in general divisors with normal crossings. By blowing up $Y_T$ one obtains a variety $\tilde{Y}_T$ proper over $Y_T$ for which all of the conditions of Proposition 1.1 are met.

It is often convenient to consider all possible tangent directions at the identity in $T$ simultaneously rather than fixing one direction. We introduce parameters $T_1, \ldots, T_\ell \quad \ell = \dim T$, and let the vector $(T_1, \ldots, T_\ell)$ denote the tangent direction (in Lie $T$). Let $R$ be the field of rational functions in $T_1, \ldots, T_\ell$ over $E$. There is then a natural action of the Weyl group on $R$ fixing $F$. We often consider the varieties $Y_T, G, \Gamma$, over $R$ instead of $F$. Set $K_R = KE \otimes_E R$.

(3.12) Another system of coordinate patches are $Y_1(B_\infty, B_0, \Sigma)$ parametrized by pairs $(B_\infty, B_0)$ of opposite Borel subgroups together with a map $\Sigma : \Delta \to \mathcal{W}$ where $\mathcal{W}$ is the set of Weyl chambers. As $B_\infty, B_0$ and $\Sigma$ vary, the patches $Y_1(B_\infty, B_0, \Sigma)$ cover $Y_T$. We also let $Y_1(B_\infty)$ denote the union of these patches for a fixed $B_\infty$. The coordinates on $Y_1(B_\infty, B_0, \Sigma)$ are

$$z_1(W, \alpha) \quad w \in \mathcal{W} \quad \alpha \in \Delta \quad \text{where} \quad z_1(\Sigma(\alpha), \alpha) = \frac{a(\gamma)-1}{\lambda}, \quad \gamma = \Gamma(\lambda).$$

$$z(\alpha) \quad \alpha \in \Delta$$

$$x(\gamma) \quad \gamma \in \Phi^+$$

$$v(\gamma) \quad \gamma \in \Phi^+$$

$$\lambda$$

The relation between $\Gamma^0 \times T \setminus G, Y_U$ and $Y_1(B_\infty, B_0, \Sigma)$ is the following. Let $T_0$ be the intersection of $B_0$ and $B_\infty$. Let $N_0$ and $N_\infty$ be the unipotent radicals of $B_0$ and $B_\infty$. Fix a Borel subgroup
$B$ containing $T$. The other Borel subgroups containing $T$ are then $B(W) \overset{def}{=} B^W$, where $W = \omega^{-1}W_+ \in \mathcal{W}$ and $W_+$ is the positive Weyl chamber with respect to $B$. Then for $g \in T \setminus G(\bar{F})$ such that $B(W)^g$ is opposite $B_\infty$ for all $W$ we may write for $t = \Gamma(\lambda)$ regular

$$(t^g, B(W)^g) = (t_0 \cdot n, B_0^n)^v,$$

with $n_W, v \in N_\infty$, $t_0 \in T_0$, $n \in N_0$.

Fixing an order on root vectors $X_\gamma$ we may write

$$n = \Pi \exp(z(\gamma)X_\gamma), v = \Pi \exp(v(\gamma)X_{-\gamma})$$

$$n_W = \exp(z(\alpha_k)z_1(W_k, \alpha_k)X_{-\alpha_k}) \cdots \exp(z(\alpha_1)z_1(W_1, \alpha_1)X_{-\alpha_1})$$

where $W = \omega^{-1}W_+$ with $\omega = \sigma_{\alpha_k} \cdots \sigma_{\alpha_1}$, a wall of type $\alpha_j$ separates the chambers $W_{i+1}$ and $W_i$ and $W_1 = W_+$. This gives the relation to $\Gamma^0 \times T \setminus G$.

To relate this to the patch $Y_U$, pick the function $\Sigma : \Delta \to \mathcal{W}$ such that $\Sigma(\alpha) = W_+$. Select $g_0$ so that $T^{g_0} = T_0$, $B(W_+)^{g_0} = B_0$, $B(W)^{g_0} = B_0(W)$, and set $t^{g_0} = t_0$. The set $g_0T_0N_\infty$ is open in $G$. If $g$ lies in this open set we have the decomposition $g = g_0t_0^n v$ which implies that $t^g = t_0^n v$. We obtain

$$(t_0 \cdot n, B_0^n)^v = (t_0^g, B_0(W)^n)^v.$$ 

Define $z(\gamma), z(\alpha)$ and $v(\gamma)$ as above and set $t_0^{-1}t_0^n = \Pi \exp((w(\gamma)\lambda)/(\Pi z(\alpha)^m(\alpha))X_\gamma)$ where $\gamma = \Sigma m(\alpha) \alpha$. By the relation $B_0^{nW} = B_0(W)^n$ we may consider the $w(\gamma)$ as rational functions of the variables $z_1(W, \alpha)$. The variables $w(\gamma)$ turn out to be independent of $z(\alpha)$. We identify the variables $z(\gamma), v(\gamma), z(\alpha)$ on $Y_U = Y_U(B_\infty, B_0)$ and $Y_1(B_\infty, B_0, \Sigma)$ and relate $w(\gamma)$ on $Y_U$ to $z_1(W, \alpha)$ on $Y_1(B_\infty, B_0, \Sigma)$ through these relations.

(3.13) When $G$ is split on an inner form of a split group, the action of $Gal(\bar{F}/F)$ on the variety $Y_T$ is given as follows. Fix a choice of $(B_\infty, B_0)$; we assume that $T_0 = B_0 \cap B_\infty$ is a Cartan subgroup defined over $F$. Let $G_*$ be a split inner form of $G$. Let $E$ be a Galois extension of $F$ which splits $T_0$ and $T$. Let $\varphi : G \to G_*$ be an isomorphism defined over $E$ carrying $T_0$ to $T_*$. We may assume that $T_*$ is split and that $T^{g_0} = T_0$ for some $g_0 \in G(E)$. We identify Weyl chambers of $T_0, T_1, T_* \in G_*$ by $g_0$ and $\varphi$.

Since $G_*$ is an inner form of $G$ we may assume $\varphi^{-1} \circ \sigma^{-1} \circ \varphi \circ \sigma$ is an automorphism of $G$ of the form $ad A^{-1}_\sigma A_\sigma \in N_{G_{adj}}(T_0)$ and that $A_\sigma = 1$ if $\sigma \in Gal(\bar{F}/E)$. In other words, we may take $A_{\sigma^{-1}}$ to be a cocycle of $Gal(E/F)$.
Let $B$ be a Borel subgroup containing $T$, for $\sigma \in \text{Gal}(E/F)$ define $\sigma_T$ in the Weyl group $W$ of $T$ by $\sigma(B^w) = B^{w\sigma_T^{-1}}$ for all $w \in W$. Notice that $\sigma_T$ exists because $G$ is an inner form of a split group. Also $\sigma \mapsto \sigma_T$ is a homomorphism. Let $W_T$ be the image of $\text{Gal}(E/F)$ in $W$. It is often convenient to identify $W_T$ (using $g_0$) with a subgroup of the Weyl group of $T_0$. A different choice of $g_0$ will lead to a conjugate subgroup of the Weyl group of $T_0$.

Define an action of $\sigma \in \text{Gal}(\overline{F}/F)$ on $Y_U$ by

$$\sigma \left( (g, (B_0^{nw}))^v \right) \overset{def}{=} \left( \sigma(g), (\sigma(B_0)^{\sigma(nw^{-1}w)})^\sigma(v) \right).$$

This extends to an action on $Y_T$. Notice that if $g^v = t^{g'}$, and $B_0^{nw} = B_0^{w_{g'}}$ with $t \in \Gamma^0(F) \cap (T \setminus G)(F)$ then $\sigma(t^{g'}, (B_0^{w_{g'}})) = (t^{g'}, (\sigma(B)^{\omega_T \sigma(g')})(t^{g'}, (B_0^{w_{g'}}))$ is an $F$-rational point. This action may be more conveniently described by twisting an action which is independent of $T$. To this end, define an action $\sigma^*$ of $\sigma \in \text{Gal}(\overline{F}/F)$ on $Y_U$ by

$$\sigma^* \left( (g, (B_0^{nw}))^v \right) \overset{def}{=} \left( \varphi^{-1} \circ \sigma \circ \varphi(g), B_0^{\varphi^{-1} \circ \sigma \circ \varphi(nw)} \right)^{nw^{-1} \circ \sigma \circ \varphi(v)}.$$

$T_*$ is split and $\varphi^{-1} \circ \sigma \circ \varphi(B_0) = B_0$. Then by the definition of $A_{\sigma}$

$$\sigma^{-1} \sigma \left( (g, (B_0^{nw}))^v \right) = \left( g, (B_0^{nw^{-1}w}) \right)^{vA_{\sigma}}.$$

The action of $A_{\sigma}$ by conjugation commutes with the action of $\sigma_T$ by permutation of Borel subgroups.

The action of $\sigma$ through $\sigma_T$ may be expressed as a $W_T$ action on $K_R$. If we wish to make the dependence of the coordinates $x(\gamma)$ on the elements $b \in B_0$ explicit we write $x_\gamma(b)$ for $x(\gamma)$. When $A_\sigma$ is trivial and $\sigma_T$ is a reflection $\sigma_{\alpha'}$ through a simple root $\alpha'$ then $\sigma_{\alpha'}$ acts on the coefficients $x_{\gamma'}(b)$ of $b$ by

$$\sigma_{\alpha'} : x_{\gamma'}(b) \mapsto x_{\gamma'}(b') \quad b' = b^{n_{W(\sigma_{\alpha'})}} \quad \gamma' \text{ a positive root}.$$

This together with $\sigma(\lambda) = \lambda$ and (3.7) determines the action of $\text{Gal}(\overline{F}/F)$ on $Y_U$ through $\sigma_T$.

To calculate the effect of $A_{\sigma}$ when $\sigma_T = 1$ and $A_{\sigma}$ maps to a simple reflection, write $vA_{\sigma} = b_0v'$ with $b_0 \in B_0$, $v' \in N_{\infty}$ by using the matrix identity

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 0 & -\zeta a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1/x & -\zeta \\ 0 & -\zeta x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\zeta x & 1 \end{pmatrix}.$$
(3.14) We consider cocycles on $W \times A$ instead of $\text{Gal}(\overline{F}/F)$ whenever convenient (meaning almost always). (We should warn the reader that sometimes we treat $W \times A$ as Galois groups and even apply results of class field theory to $W \times A$ with the understanding that the map from $\text{Gal}(\overline{F}/F)$ to $W \times A$ may be used to give precise meaning to our statements.) We have a function $t''_\sigma$ given on generators by

$$
\sigma_{\alpha'} \times A_{\alpha'} \mapsto \left( \frac{\xi_{\alpha'} + \alpha'(T)z(\alpha')}{z(\alpha')} \right)^{-\alpha'} \quad \sigma_{\alpha'} \mapsto (z(\alpha'))^{\alpha'} \quad \alpha' \text{ simple}
$$

where $A_{\alpha'}$ is an element of $A$ in the normalizer of $T_0$ which maps to the simple reflection $\sigma_{\alpha'}$ in the Weyl group. Here $\xi_{\alpha'}$ is the element such that $v = \exp(\xi_{\alpha'} X_{-\alpha'}) v^{\alpha'}$, $v^{\alpha'} \in N_{-\alpha'}$ the unipotent radical of the parabolic subgroup containing $B_\infty$ associated to the root $-\alpha'$. Also $\alpha'(T)$ is the root evaluated on the tangent direction. Using the cocycle relation $b_\sigma = b_\sigma b_\tau$ we have

$$
A_{\alpha'} \mapsto (\xi_{\alpha'})^{\alpha'}, \quad \sigma_{\alpha'} \mapsto (z(\alpha'))^{\alpha'}.
$$

We have $t_\sigma = t''_\sigma b_\sigma$ where $t_\sigma$ is the cocycle of (3.9) and $b_\sigma$ depends on points of $Y_\Gamma$ only through $\lambda$. To determine $b_\sigma$ for quasi-split groups we use the fact that the cocycle $t_\sigma$ restricted to the divisor $E_0$ becomes constant (3.9) on each regular unipotent class.

**LEMMA 3.15.** Define $b'_\sigma$ by $\sigma_{\alpha'} \mapsto (1/\lambda)^{\alpha'}$, $A_{\alpha'} \mapsto 1$. Then there is a factor $b''_\sigma$ which is independent of $\lambda$ with values in $T(E)$ such that $b'_\sigma b''_\sigma t''_\sigma = t_\sigma$.

**PROOF:** $t''_\sigma b'_\sigma$ is given by

$$
\sigma_{\alpha'} \mapsto (1/z(\alpha'))^{\alpha'}, \quad A_{\alpha'} \mapsto (\xi_{\alpha'})^{\alpha'} \quad \alpha' \text{ simple}.
$$

So it is enough to verify that the action of $W \times A$ on $z(\alpha')$, $\xi_{\alpha'}$ is independent of the tangent direction. For $\sigma \in W$ the action is given by (3.13). But $n_{W(\sigma_a)} = 1$ on $E_0$ so that by 3.13 $\sigma_a(z(\alpha')) = z(\alpha')$. By (3.13) the action of $\sigma \in W$ on $v$ is given by $\sigma : v \mapsto n_{\sigma_a}^{-1} W_+ v = v$ (since $n_{\sigma_a}^{-1} W_+ = 1$ on $E_0$). The action of $A$ is given by (3.13) as $(b, B_0) v \mapsto (b, B_0)^{A_\sigma}$ where $A_\sigma$ is an element of the normalizer of $T_0$ independent of the tangent direction. So on $E_0$ we have $(u, B_0) v \mapsto (u, B_0)^{A_\sigma}$ which is clearly independent of the tangent direction.

A factor $b''_\sigma \in T(E)$ may be needed to make $t_\sigma$ agree with the cocycles of the transfer factor [LS2] for various reasons. The boundary of $t''_\sigma \in T(K_R)$ lies in $T(E)$ but is not necessarily zero. Thus $b''_\sigma$ is needed to make $b'_\sigma b''_\sigma t''_\sigma$ a cocycle. Also the choice of measure $\mu_O^H$ is made by selecting an $F$-class $O_1$ in $O$ the regular class unipotent orbit for split groups. A factor $b''_\sigma$ will be needed.
to insure that $\kappa(t''_cb'_cb''_c) = 1$ on $O_1 \subseteq O \cap E_0$. Finally when $G$ is not quasi-split, our analysis has not been complete: the factor $b'_c$ will be chosen by the normalization of measures $\mu^H_O$ for an orbit $O$ which contains an $F$-rational element. At any rate, to prove the transfer of orbital integrals to an endoscopic group all that is needed is that $b''_c$ is constant which has been shown.

§4. SYMPLECTIC GROUPS

The argument in the remainder of the paper is divided into the following steps.

(1) List the endoscopic groups $H$ of $G$ and the pairs $(T, \kappa)$ associated to each $H$. Determine the stable and adjoint unipotent orbits of $H$ and $G$.

(2) Show that the patch $Y_U$ is regular in codimension 1 and determine its divisors.

(3) Obtain an explicit resolution of singularities of $Y_U$.

(4) Fix a stable unipotent class $O$ and look at all divisors $E$ meeting $Y_U$ which are $O$-divisors. By looking at the data defining the principal value integral

$$PV \int_E f m_{\theta,E} |\omega_E|$$

on $E$ either show that the principal value integral is zero or show that there is a decomposition on $Y_U \cap E$ of the type described in the hypotheses of (2.14). The following simple implication of (1.1) and (2.4.2) is often used. If $E$ is an $O$-divisor, $O$ is $r$-regular, and if $b(E)/a(E) - 1 \neq r$ then $E$ makes no contribution to the germ of $O$. Also by (2.4.2) no logarithmic terms appear in the expansion so that $F_r(\theta, \beta) = 0$ for $r > 1$.

(5) Show that the decomposition on $Y_U$ of step (4) extends to all points of the divisor $E$.

(6) Resolve the singularities on the rest of $Y_f$ and show that the principal value integral

$$\int_E f m_{\theta,E} |\omega_E|$$

is zero for any divisor not meeting $Y_U$.

The remainder of this section carries out steps (1), (2), (3) (step (2) for an arbitrary symplectic group). Sections 5 and 6 carry out step 4 for the subregular and 2-regular unipotent classes. Section 7 contains the arguments needed to extend the decomposition obtained in §5, §6 to points outside of the patch $Y_U$. The fact that the principal value integrals considered in this paper are not birational invariants means that it is not enough to restrict ourselves to the patch $Y_U$. However, the arguments become much more technical outside of $Y_U$ and these details, including steps 5 and 6 are relegated to the final section.

We ignore the regular unipotent class in all that follows. By (2.9) we see that the transfer factors are chosen so that the matching of regular germs on $G$ and $H$ is a triviality. We also
ignore the identity element. By [R2] the germ associated to the identity element is known explicitly. In particular, it is zero for a $\kappa$-orbital integral if $\kappa$ is nontrivial.

We begin with the simply connected semi-simple symplectic group $Sp(2n)$ of arbitrary rank (or its inner form) and specialize to the rank 2 case toward the end of the section. If all of the Cartan subgroups of $H$ may be identified with Cartan subgroups in a Levi factor $M$ of $G$ then (2.4.1, 2.4.3) shows that the $\kappa$-orbital integrals on $G$ may be realized inside $M$ (since $M_{der}$ is then also simply connected). We may then inductively assume that $H$ is cuspidal, that is, the $L$-group of $H$ is not contained in a parabolic subgroup of $LG$. The possibilities for $H$ have been computed [H]. They are:

\begin{align*}
F_{H_1} &= Sp(2i) \times SO(2(n-i)) \\
E_{H_i} &= Sp(2i) \times ESO(2(n-i)).
\end{align*}

$F_{H_n}$ is the quasi-split inner form of $G$, $F_{H_{n-1}}$ degenerates to a Levi factor and may be excluded. $E_{H_n}$ is also degenerate and may be excluded. Here $E$ is the quadratic field extension which splits the endoscopic group. Fixing $i$, the Cartan subgroups $T$ in $G_{qs}$ associated to $F_{H_1}$ are all conjugate to a Cartan subgroup of $M_i = Sp(2i) \times Sp(2(n-i)) \subset Sp(2n)$ and every Cartan subgroup $T$ in $M_i \subset Sp(2n)$ is associated to an endoscopic group $E_{H_i}$ for an appropriate choice of quadratic field extension $E$. Given $T$, $E$ is determined as follows.

Identify $W_{H_i}$ with the product of $W_{Sp(2i)}$ and the subgroup of $W_{Sp(2(n-i))}$ generated by short reflections. Then if $T \subset M_i$ and $T$ is a split group in $M_i$, pick $T^m = T$ and let $E$ be the quadratic extension associated to the homomorphism $Gal(\overline{F}/F) \to \sigma(m)m^{-1} \in W_{M_i}/W_{H_i} \sim \{\pm 1\}$. Then $T$ may be identified with a stable conjugacy class of Cartan subgroups in $E_{H_i}$. $H^1(Gal(\overline{F}/F), E_{H_i})$ has two elements. Fix the non-trivial character $\kappa$ on $H^1(Gal(\overline{F}/F), E_{H_i})$. Then pulling $\kappa$ back to $T$, $(T, \kappa)$ is associated to the endoscopic group $E_{H_i}$. (This is the only possible choice of $\kappa$ because for anisotropic $T$ one is guaranteed a non-trivial character on $H^1(Gal, T)$ which must be trivial in $H^1(Gal(\overline{F}/F), E_{H_i}$). When $n = 2$ we have $E_{H_0} = ESO(4)$, and $E_{H_1} = SL(2) \times U_E(1)$, where the letter $E$ indicates the quadratic extension splitting the group and $U_E(1)$ is a one dimensional torus.

Next we determine some prime divisors of $\varphi^{-1}(\lambda)$ in $Z = Z_{\Phi}$ for $\Phi = C_n$. See (3.7). Label the simple roots $\Delta$ of $\Phi$ by
Recall that the positive roots have the form \( \gamma(k, \ell) \triangleq \alpha_k + \alpha_{k+1} + \cdots + \alpha_\ell \quad \ell > k \) or \( \gamma^+(k, \ell) \triangleq \alpha_k + \alpha_{k+1} + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_\ell \quad \ell \geq k \). Let \( S \) be the set of possibly empty pairs \((S_1, S_2) \subseteq \Delta \times \Delta\), \( S_1 = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \quad (i_1 < i_2 < \cdots < i_r)\), \( S_2 = \{\alpha_{j_1}, \ldots, \alpha_{j_q}\} \quad (j_1 < \cdots < j_q)\) satisfying \( i_r < j_1 \) and \( j_q \neq n \). For each \((S_1, S_2) \in S\) we define an open patch \( Z(S_1, S_2) \) of \( Z \) by

(i) \( z(\alpha) \neq 0 \) if \( \alpha \notin S_1 \cup S_2 \) (write \( z(\alpha_{i_k}) = z_k \) \( z(\alpha_{j_k}) = z_k \)).

(ii) \( z(\alpha) \neq 0 \) if \( \alpha \in S_1 \)

(iii) \( x_k \triangleq z(\gamma(j_k, j_{k+1})) \neq 0 \quad k \neq q \)

(iv) \( w_k \triangleq w(\gamma(j_k, j_{k+1})) \neq 0 \quad k \neq q \)

Then on \( Z(S_1, S_2) \) if \( S_2 \neq \emptyset \) we have by \((3.7)\)

(i) \( w(\delta) = \frac{x(\delta)w(\gamma)\Pi z(\alpha)m(\alpha)}{x(\gamma)} \)

if \( x(\gamma) \neq 0 \) and \( \delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0 \)

(ii) \( x(\gamma) = \frac{\lambda w(\gamma)}{\Pi z(\alpha)m(\alpha)} \quad \text{if} \quad \gamma = \Sigma m(\alpha)\alpha \quad \alpha \notin S_1 \cup S_2 \)

(iii) \( \lambda = \frac{x_q}{w_q} z_q^2 (\Pi z(\alpha)m(\alpha)) \quad 2\alpha_{j_q} + \Sigma m(\alpha)\alpha = \gamma^+(j_q, j_q) \)
\( \text{T. C. HALE}
\)

(notice that \( x_q \neq 0, \Pi z(\alpha)^m(\alpha) \neq 0, \) so \( \lambda = 0 \) if and only if \( z_q = 0 \)).

(iv) \( z_k = z_{k+2} \left( \frac{x_{k+1} w_k}{w_{k+1}} \Pi z(\alpha)^m(\alpha) \right) \)
where \( \Sigma m(\alpha) \alpha + \alpha_{j_k+2} = \gamma(j_{k+1}, j_{k+2}) \) and \( \alpha_{j_k+2} + \Sigma m(\alpha) \alpha = \gamma(j_k, j_{k+1}) \)

\( z_q = z_q \left( \frac{w_{q-1} x_q \Pi z(\alpha)^m(\alpha)}{w_{q-1} w_q \Pi z(\alpha)^m(\alpha)} \right) \)
where \( 2 \alpha_{j_q} + \Sigma m(\alpha) \alpha = \gamma^+(j_q, j_q) \) and \( \alpha_{j_q-1} + \alpha_{j_q} + \Sigma m(\alpha) \alpha = \gamma(j_{q-1}, j_q) \).

(v) \( \gamma^' = \frac{\lambda}{x^{(a)}} = \frac{x_q z_q^2 (\Pi z(\alpha)^m(\alpha))}{w_q x^{(a)}} \) where \( 2 \alpha_{j_q} + \Sigma m(\alpha) \alpha = \gamma^+(j_q, j_q), \alpha_S \in S_1. \)

Notice that (iv), (v) give \( z(\alpha) = f_{\alpha} z_q^\alpha \) for some \( f_{\alpha} \) regular and non-vanishing and \( \epsilon \in \{0, 1, 2\} \).

(vi) Suppose \( \gamma = \Sigma m(\alpha) \alpha \) with \( \Pi z(\alpha)^m(\alpha) = f_{\gamma} z_q, f_{\gamma} \) regular and non-vanishing then

\( \begin{align*}
\frac{x(\gamma)}{f_{\gamma} w_q} = \frac{x_q z_q (\Pi z(\alpha)^m(\alpha)) w(\gamma)}{f_{\gamma} w_q}, \quad 2 \alpha_{j_q} + \Sigma m(\alpha) \alpha = \gamma^+(j_q, j_q).
\end{align*} \)

\( Z \) is defined by \( p \) equations and is consequently \( p + 1 \)-dimensional. A count of the equations in
(i) \ldots (vi) reveals \( p \) independent equations. In fact for \( \gamma = \Sigma m(\alpha) \alpha \in \Phi^+ \) let \( e(\gamma) \in Z \) be the exponent which makes \( \frac{\Pi z(\alpha)^m(\alpha)}{z_q} \) regular and invertible on \( Z(S_1, S_2) \) (by (iv) such an exponent exists). Let \( q' = \# \{ \gamma \in \Phi^+ \mid e(\gamma) = 0 \text{ or } 1 \} \). Then the above equations show that \( z(\alpha), \alpha \in \Delta \setminus (S_1 \cup S_2); z_q; w(\gamma), (e(\gamma) = 0, 1); x(\gamma) \) \( (e(\gamma) \geq 2) \) are \( p + 1 = (n - r - q) + (1 + (q' - n + r) + (p - q')) \) variables which generate the ring of \( Z(S_1, S_2) \). Hence \( Z(S_1, S_2) \) is regular. The divisor \( \lambda = 0 \) on \( Z(S_1, S_2) \) has the form \( \lambda = a(E(S_1, S_2)) E(S_1, S_2) \) with \( a(E(S_1, S_2)) = 2 \) where \( E(S_1, S_2) \) is the prime divisor given by \( z_q = 0 \).

If \( S_2 = \emptyset \) then on \( Z(S_1, \alpha) \) we have by (3.7)

(i) \( w(\delta) = x(\delta) w(\gamma) \Pi z(\alpha)^m(\alpha) \) if \( x(\gamma) \neq 0 \) and \( \delta = \gamma + \Sigma m(\alpha) \alpha \quad \alpha \geq 0. \)

(ii) \( x(\gamma) = \frac{\lambda w(\gamma)}{\Pi z(\alpha)^m(\alpha)} \) \( \gamma = \Sigma m(\alpha) \alpha \quad \alpha \notin S_1. \)

(iii) \( \lambda = z^r_q x(\alpha_{\ell r}). \)

(iv) \( z^r_q = z^r_q \left( \frac{x(\alpha_{\ell r})}{x(\alpha_{\ell r})} \right) \) \( \ell \neq r \quad \alpha_{\ell r} \in S_1. \)

For \( \gamma = \Sigma m(\alpha) \alpha \in \Phi^+ \) define \( e'(\gamma) \in Z \) to be the exponent which makes \( \frac{\Pi z(\alpha)^m(\alpha)}{z^r_q} \) regular and
invertible on $Z(S_1, \emptyset)$. Let $q' = \#\{\gamma \in \Phi^+ \mid e'(\gamma) = 0\}$. Then (i)-(iv) show that the coordinate ring of $Z(S_1, \emptyset)$ is generated by $z(\alpha), \alpha \in \Delta \setminus S_1; z'_\gamma; w(\gamma), (e'(\gamma) = 0); x(\gamma) (e'(\gamma) \geq 1)$ and hence by $p + 1 = (n - r) + (1) + (q' + r - n) + (p - q')$ variables. Again by a dimension count we see that $Z(S_1, \emptyset)$ is regular and that the divisor $\lambda = 0$ on $Z(S_1, \emptyset)$ has the form $(\lambda) = a(E(S_1, \emptyset))E(S_1, \emptyset)$ with $a(E(S_1, \emptyset)) = 1$ and $E(S_1, \emptyset)$ prime.

We extend the prime divisors $E(S_1, S_2)$ to all of $Z$ by taking their closures. Given one of the divisors $E$ above, $S_1$ is determined as the set of simple roots for which $x(\alpha) \neq 0$ generically on $E$, and $S_2$ is determined as the set of simple roots not in $S_1$ for which $z(\alpha) \equiv 0$ on $E$. Hence the map from pairs $(S_1, S_2)$ to divisors is injective.
LEMMA 4.2. (1) The codimension of the complement of $\bigcup_{(s_1,s_2) \in \delta} Z(s_1,s_2)$ in $Z$ is at least two.

(2) $Z$ is regular in codimension one.

(3) The divisor $(\lambda)$ on $Z$ is given by

$$
(\lambda) = \sum_{(s_1,s_2)} a(E(s_1,s_2))E(s_1,s_2)
$$

with $a(E(s_1,s_2)) = \begin{cases} 1 & S_2 = 0 \\ 2 & \text{otherwise.} \end{cases}$

PROOF: By the preceding remarks (2) and (3) follow immediately from (1). $Z$ is clearly regular for $\lambda \neq 0$ so we study $\lambda$ near points of $\lambda = 0$.

Consider a point $p$ in $Z$ where $\lambda = 0$. Let $S_1$ be the set of simple roots such that $x(\alpha) \neq 0$. Let $S_{\min}$ be the set of positive roots such that $x(\gamma) \neq 0$ and such that whenever $\gamma = \beta + \Sigma m(\alpha)\alpha$, $m(\alpha) \geq 0$ then $x(\beta) = 0$. Clearly $S_1 \subseteq S_{\min}$. We may write $\Phi^+ = S_{\min} \cup S^+ \cup S^-$, where $S^+ = \{\delta : \delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0\}$ for some $\gamma \in S_{\min}, \delta \notin S_{\min}\}$. $S^- = \{\beta : x(\beta) = 0\}$.

We have equations

(i) $x(\beta) = 0$ \quad $\beta \in S^-$

(ii) $w(\delta) = \frac{w(\gamma)x(\delta)\Pi x(\alpha)^{m(\alpha)}}{x(\gamma)} \quad \delta \in S^+, \gamma \in S_{\min}, \delta = \gamma + \Sigma m(\alpha)\alpha \quad m(\alpha) \geq 0.$

(iii) If $\gamma = \gamma(k,\ell) \text{ or } \gamma^+(k,\ell) \in S_{\min} \setminus S_1$. 

$$w(\gamma - \alpha_k)z(\alpha_k) = \frac{w(\gamma)x(\gamma - \alpha_k)}{x(\gamma)} = 0 \quad \text{by the definition of } S_{\min}$$

(iv) $z(\alpha) = \frac{\lambda}{x(\alpha)} = 0$ if $\alpha \in S_1$

(v) $\lambda = 0$.

Counting the number of equations we have at least $p + 1$ independent equations. For the proof of (1) we may exclude sets of codimension 2. Thus we may assume that (i)-(v) give exactly $p + 1$ independent solutions. In particular, we may assume that $S^-, S^+$ and $S_{\min}$ are disjoint and that in (iii) if $w(\gamma - \alpha_k) = 0$ then $z(\alpha_k) \neq 0$.

Suppose that $\delta = \gamma(k,\ell) \text{ or } \gamma^+(k,\ell) \in S_{\min} \setminus S_1, \ell \geq k$. Then as in (iii) we have $w(\delta - \alpha_\ell)z(\alpha_\ell) = 0$. Suppose that $z(\alpha_\ell) \neq 0$. Then $w(\delta - \alpha_\ell) = 0$. This gives $p + 2$ equations unless
\[ \delta' = \gamma(k - 1, \ell - 1) \quad \text{or} \quad \gamma^+(k - 1, \ell - 1) \in S_{\min} \setminus S_1 \] and the equation (iii) associated to \( \delta' \) is
\[ w(\delta' - \alpha_{k-1}) = w(\delta - \alpha_k) = 0 \] (and so \( z(\alpha_{k-1}) \neq 0 \)). But then
\[ w(\delta) = \frac{x(\delta) z(\alpha_k) w(\delta')}{x(\delta') z(\alpha_{k-1})} \]
gives \( p + 2 \) equations. This contradiction shows that \( z(\alpha_k) = 0 \).

Now if \( \gamma^+(r, \ell) \in S_{\min} \setminus S_1 \) then \( z(\alpha_k) = 0 \). This must be a consequence of (iii) so that \( \gamma(\ell, m) \) or \( \gamma^+(\ell, m) \in S_{\min} \). But this contradicts the definition of \( S_{\min} \) unless \( r = \ell = m \).

Also by the definition of minimality there is at most one \( \ell \) such that \( \gamma^+(\ell, \ell) \in S_{\min} \setminus S_1 \). Let \( \gamma(j_1, j_2) \in S_{\min} \setminus S_1 \), then \( z(\alpha_{j_2}) = 0 \) so there exists a \( \gamma(j_2, j_3) \) or \( \gamma^+(j_2, j_2) \in S_{\min} \setminus S_1 \).

Continuing this way we obtain a chain \( \gamma(j_1, j_2), \gamma(j_2, j_3) \ldots \gamma^+(j_q, j_q) \in S_{\min} \setminus S_1 \), which we may assume to be maximal in length. If also \( \gamma(k_1, k_2) \in S_{\min} \setminus S_1 \) we see that a new chain could be formed which must also end in \( \gamma^+(k_{q'}, k_{q'}) = j_{q'} \). But then \( \gamma(k_{q'-1}, k_{q'}) \), \( \gamma(j_{q-1}, j_q) \in S_{\min} \setminus S_1 \) and by the definition of \( S_{\min} \) \( \gamma(k_{q'-1}, k_{q'}) = \gamma(j_{q-1}, j_q) \). Continuing in this manner one finds that \( \gamma(k_1, k_2) = \gamma(j_{q'-q' + 1}, j_{q-q'+2}) \). Thus \( \{\gamma(j_1, j_2), \ldots \gamma^+(j_q, j_q)\} = S_{\min} \setminus S_1 \) \( \text{def} \) \( S_2 \). Since \( \gamma^+(j_q, j_q) \notin S_1, j_q \neq n \). Since (i)-(v) give all equations which hold identically at \( p \) (excluding sets of codimension 2) the inequalities defining \( Z(S_1, S_2) \) hold at the generic point of the variety defined by (i)-(v) together with \( x(\gamma) \neq 0, \quad \gamma \in S_{\min} \). This completes the proof.

Turn to the case \( G = C_2 \). We introduce new notation for these divisors. Set \( \alpha = \alpha_1, \beta = \alpha_2 \).

(4.3)

<table>
<thead>
<tr>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( E(S_1, S_2) )</th>
<th>( \pi(E(S_1, S_2)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {\alpha, \beta} )</td>
<td>( \phi )</td>
<td>( E_0 )</td>
<td>regular</td>
</tr>
<tr>
<td>( {\alpha} )</td>
<td>( \phi )</td>
<td>( E_\beta )</td>
<td>subregular</td>
</tr>
<tr>
<td>( {\beta} )</td>
<td>( \phi )</td>
<td>( E_\alpha )</td>
<td>Richardson class of ( P_\beta )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( {\alpha} )</td>
<td>( E_2 )</td>
<td>subregular</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( \phi )</td>
<td>( E_{id} )</td>
<td>identity</td>
</tr>
</tbody>
</table>

The projection of each divisor to \( G \) is the closure of a stable unipotent conjugacy class. The final column lists that conjugacy class. Write \( \gamma \overset{\text{def}}{=} \alpha + \beta, \quad \delta = 2\alpha + \beta \).

There are 4 stable unipotent classes in \( \text{Sp}(4) \) \[\text{Sp}\]: the regular, the subregular, the 2-regular and the 4-regular (the identity element). For a complete discussion see [Sp]. There is only one adjoint regular class, one adjoint 2-regular class, and one adjoint 4-regular class. The
adjoint subregular classes are in bijection with quadratic extensions (or elements of $F^\times/F^{\times 2}$).

To determine the quadratic extension associated to a subregular unipotent element $u \in G(F)$, conjugate $u$ by an element of $G(F)$ so that it has the form

$$
\begin{pmatrix}
I_2 & X \\
0 & I_2
\end{pmatrix} \quad \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X \in M_2(F),
$$

where $Sp(4)$ is defined using the skew form $J_0 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det X \in F^\times/F^{\times 2}$ depends only on the adjoint conjugacy class of $u$. Equivalently, $B_u$, the variety of Borel subgroups containing $u$, is a union of three projective lines. The lines of type $\beta$ are defined over a quadratic extension of $F$ depending only on the adjoint conjugacy class of $u$.

When $G$ is the inner form of $Sp(4)$ the regular, 2-regular, and 1-regular adjoint class associated to the trivial quadratic extension $F^{\times 2} \subseteq F^\times/F^{\times 2}$ do not exist. If $u \in G(F)$ is unipotent regular then there is a unique Borel subgroup over $F$ containing $u$. If $u \in G(F)$ is 2-regular unipotent, then the line of type $\beta$ in $B_u$ is defined over $F$ giving a parabolic of type $\beta$ over $F$; but the minimal parabolic of $G$ is of type $\alpha$. If $u \in G(F)$ is 1-regular corresponding to the trivial extension then the Borel subgroup corresponding to the intersection of a line of type $\alpha$ and a line of type $\beta$ in $B_u$ is defined over $F$. Thus such unipotent classes do not exist for the inner form $G$.

**Lemma 4.4.** (1) The variety $Z_\Phi \Phi = G_2$ is regular except possibly at points $p$ such that $\pi(p)$ is the identity element of $G$ or at points $p$ where

$$
\lambda = x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0.
$$

(2) The singularity at $\lambda = x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0$ may be resolved by blowing up once along the subvariety $x(\alpha) = x(\beta) = x(\gamma) = w(\gamma) = w(\delta) = z(\alpha) = z(\beta) = 0$. Let $E_B$ be the divisor introduced by blowing up.

(3) The divisors on the desingularized variety and their Igusa constants are given as follows (assume $\pi(p) \neq$ identity).

<table>
<thead>
<tr>
<th>$E$</th>
<th>$a(E)$</th>
<th>$b(E)$</th>
<th>$\pi(E)$</th>
<th>$E(B) - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>1</td>
<td>1</td>
<td>0-reg</td>
<td>0</td>
</tr>
<tr>
<td>$E_\alpha$</td>
<td>1</td>
<td>2</td>
<td>1-reg</td>
<td>1</td>
</tr>
<tr>
<td>$E_\beta$</td>
<td>1</td>
<td>2</td>
<td>1-reg</td>
<td>1</td>
</tr>
<tr>
<td>$E_2$</td>
<td>2</td>
<td>6</td>
<td>2-reg</td>
<td>2</td>
</tr>
<tr>
<td>$E_B$</td>
<td>2</td>
<td>5</td>
<td>2-reg</td>
<td>3/2</td>
</tr>
</tbody>
</table>

**Proof:** The divisors $E_0, E_\alpha, E_\beta, E_2, E_B$ are defined by
E₀ : \( z(\alpha) = z(\beta) = w(\gamma) = w(\delta) = 0 \)
E₁ : \( z(\alpha) = z(\beta) = 0, z(\alpha)x(\gamma) = w(\gamma)x(\beta), w(\delta)x(\gamma) = w(\gamma)x(\delta)x(\alpha) \)
E₁ : \( x(\alpha) = z(\beta) = x(\gamma) = 0 \)
E₂ : \( x(\alpha) = x(\beta) = x(\gamma) = 0 \)
E_{id} : \( x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0 \).

We consider several patches, showing that the coordinate ring on each is regular, and at the same time we identify the divisors on each patch. The defining relations (3.7) of \( Z \) are used repeatedly. The expression \( \lambda = z(\beta)x(\beta) \) (\( E₀E₇ \)) indicates that \( z(\beta) = 0 \) defines the divisor \( E₀ \), that \( x(\beta) = 0 \) defines the divisor \( E₇ \), and so forth.

(If \( x(\alpha) \neq 0 \))
\[
\lambda = z(\beta)x(\beta) \quad (E₀E₇), \quad z(\alpha) = z(\beta)x(\beta)/x(\alpha),
\]
\[
w(\gamma) = x(\gamma)z(\beta)/x(\alpha), w(\delta) = x(\delta)z(\alpha)z(\beta)/x(\alpha)
\]
generators of local ring: \( z(\beta), x(\beta), x(\gamma), x(\delta), z(\alpha) \).

(If \( x(\beta) \neq 0 \))
\[
\lambda = z(\alpha)x(\alpha) \quad (E₀E₆), \quad z(\beta) = z(\alpha)x(\alpha)/x(\beta),
\]
\[
w(\gamma) = x(\gamma)z(\alpha)/x(\beta), w(\delta) = x(\delta)z(\alpha)^2/x(\beta)
\]
generators of local ring: \( z(\alpha), x(\alpha), x(\beta), x(\gamma), x(\delta) \).

(If \( x(\gamma) \neq 0 \))
\[
\lambda = x(\alpha)z(\beta)w(\gamma)/x(\alpha) \quad (E₆E₀E₇), \quad z(\alpha) = w(\gamma), x(\beta)/x(\gamma),
\]
\[
z(\beta) = w(\gamma)x(\alpha)/x(\gamma), w(\delta) = w(\gamma)x(\delta)z(\alpha)/x(\gamma)
\]
generators: \( x(\beta), x(\alpha), w(\gamma), x(\gamma), x(\delta) \).

(If \( x(\alpha) \neq 0 \))
\[
\lambda = x(\beta)x(\beta) \quad (E₆E₁), \quad z(\alpha) = z(\beta)x(\beta)/z(\alpha),
\]
\[
x(\gamma) = w(\gamma)x(\alpha)/x(\beta), x(\delta) = x(\beta)w(\delta)/z(\alpha)^2,
\]
generators: \( x(\beta), x(\alpha), w(\gamma), z(\alpha), w(\delta) \).

(If \( w(\delta) \neq 0 \))
\[
\lambda = z(\alpha)^2z(\beta)x(\delta)/w(\delta) \quad (E₆²E₆E₁)
\]
\[
x(\alpha) = z(\alpha)z(\beta)x(\delta)/w(\delta), x(\beta) = z(\alpha)^2x(\delta)/w(\delta), x(\gamma) = z(\alpha)w(\gamma)x(\delta)/w(\delta)
\]
generators: \( z(\alpha), z(\beta), x(\delta), w(\gamma), w(\delta) \).

Since we are assuming \( \pi(p) \neq 1 \) and since the variety has now been shown to be nonsingular for \( x(\alpha), x(\beta) \) or \( x(\gamma) \neq 0 \), we may assume in the remaining cases that \( x(\delta) \neq 0 \).

(If \( w(\gamma) \neq 0 \))
\[
\lambda = x(\gamma)^2w(\delta)z(\beta)/(w(\gamma)^2x(\delta)) \quad (E₆²E₆E₀)
\]
\[
z(\alpha) = w(\delta)x(\gamma)/(w(\gamma)x(\delta)), x(\alpha) = z(\beta)x(\gamma)/w(\gamma), x(\beta) = z(\gamma)^2w(\delta)/(w(\gamma)^2x(\delta))
\]
generators: \( x(\gamma), x(\delta), z(\beta), w(\gamma), w(\delta) \).
(If $z(\beta) \neq 0$) $\lambda = w(\delta) x(\alpha)^2 / (z(\beta) x(\delta))$ ($E_\beta E_\delta^2$) $x(\beta) = w(\delta) x(\alpha)^2 / (z(\beta)^2 x(\delta))$,

$$z(\gamma) = w(\gamma) x(\alpha) / z(\beta), \quad z(\alpha) = w(\delta) x(\alpha) / (x(\delta) z(\beta))$$
generators: $x(\alpha), x(\delta), z(\beta), w(\gamma), w(\delta)$.

This completes the proof of (1). The validity of (2) is easily checked using the same seven patches. Details are omitted.

The only points that are not clear in (3) are the value of the constants $b(E)$ and the value of $a(E_B)$. $b(E) - 1$ is given as the order to which the form (3.8)

$$\omega_Y = d\lambda dx(\alpha)dz(\beta)dx(\gamma)dz(\delta)dv$$

$$dv = dv(\alpha)dv(\beta)dv(\gamma)dv(\delta)$$

vanishes along $E$. For example, the form when expressed in terms of the coordinates on the patch $(w(\gamma) \neq 0)$ becomes

$$\omega_Y = z(\gamma)^5 w(\delta) z(\beta) / (x(\delta)^2 w(\gamma)^4) dz(\beta) d(1/w(\gamma)) dw(\delta) dx(\gamma) dz(\delta) dv \quad (E_\beta^2 - 1 E_\delta^2 - 1 E_\alpha^{-1})$$

Thus $b(E_2) = 6, b(E_\beta) = 2, b(E_\alpha) = 2$. The Igusa constants $b(E)$ for $E = E_B, E_0$ are similarly calculated using a blown up region. Introduce projective coordinates $X_\alpha, X_\beta, X_\gamma, Z_\alpha, W_\delta, Z_\beta, W_\gamma$. On $X_\alpha = 1$ we have $X_\beta x(\alpha) = x(\beta), X_\gamma x(\alpha) = x(\gamma)$. The patch $(x(\alpha) \neq 0)$ becomes (dropping the assumption that $x(\alpha) \neq 0$).

$$\lambda = x(\alpha)^2 Z_\beta X_\beta (E_\beta^2 E_0 E_\beta), \quad Z_\alpha = Z_\beta X_\beta, \quad W_\gamma = X_\gamma Z_\beta, \quad W_\delta = x(\delta) Z_\alpha Z_\beta$$

$$\omega_Y = z(\beta) x(\alpha)^5 - 1 Z_\beta^1 - 1 X_\beta^2 - 1 dZ_\beta dx(\alpha) dX_\beta dX_\gamma dx(\delta) dv$$

so that $b(E_B) = 5, b(E_0) = 1, b(E_\beta) = 2$.

This completes the proof.

§5. THE SUBREGULAR GERMS ON $G = \text{Sp}(4)$

We begin with a summary of the results of this section. Let $G$ be Sp(4) or its inner form. By (2.5) we will have determined all of the germs of $GSp(4)$ once we have calculated those of $Sp(4)$. Details on normalizations of measures are found below. Let $\alpha'$ be a root of Sp(4), let $E$ be a quadratic extension (possibly trivial of $F$), and let $H$ be an endoscopic group. Let $T(\alpha')$ be a
Cartan subgroup corresponding to the homomorphism \( \text{Gal}(\overline{F}/F) \rightarrow W \) given by \( \sigma \in \text{Gal}(E/F) \) \((\sigma \neq 1) \sigma \mapsto \sigma_{\alpha'} \in W \) (or let \( T(\alpha') \) be split if \( E = F \)). Let \( U_E(1) \) be a one dimensional unitary group split by \( E \).

Let \( O_{E'} \) be the subregular adjoint unipotent class associated to the quadratic extension \( E' \) of \( F \) (see discussion following 4.3). Let \( \Gamma(E', E, \alpha') \) be the germ of \( O_{E'} \) for the pair \((T(\alpha'), st)\) where \( st \) is the trivial character on \( H^1(\text{Gal}(\overline{F}/F), T(\alpha')) \). Set \( \Gamma(E', E, \alpha') = 0 \) if \( T(\alpha') \) does not exist. The function \( \Gamma_1(E', E'', \beta) \) is defined in (5.18). Set \( \delta_G = 1 \) if \( G \) is quasi-split, 0 otherwise; set \( \epsilon_G = 2\delta_G - 1 \); and set \( \delta(E, E') = 1 \) if \( E = E' \), 0 otherwise. Then we have for various \((T, \kappa)\) in \( G \) associated to the nondegenerate cuspidal endoscopic groups \( H \) (4.1)

\[
\begin{array}{c|c|c|c}
H & T & \text{Germ of } O_{E'} \text{ if } E' \neq F & \text{Germ of } O_E \\
\hline
E_{H_1} E \neq F & U_{E''}(1) \times U_E(1) & \delta(E, E')\delta_G \Gamma_1(E', E'', \beta) & 0 \\
E_{H_0} E \neq F & \text{arbitrary} & 0 & 0 \\
F_{H_2} T/(\pm 1) = & U_{E''}(1) \times U_E(1) & \Gamma(E', E, \alpha) & \Gamma(F, E, \alpha) \\
F_{H_0} & \text{arbitrary} & \epsilon_G \Gamma_{O_{E'}}^{T_{\alpha', st}} & \delta_G \Gamma_{O_{E'}}^{T_{\alpha', st}}.
\end{array}
\]

Quasi-split reduction will be useful in identifying the germs of the subregular unipotent classes. For a group of semi-simple rank 2, quasi-split reduction is a statement about the matching of inner forms of rank 1 groups. This situation is well understood. See for example [LS]. In particular it follows immediately from the well-known matching of orbital integrals on a rank 1 group with orbital integrals on its quasi-split inner form that \( G \) has quasi-split reduction.

The variety of Borel subgroups containing a given subregular unipotent element in a group of type \( C_2 \) is a union of three projective lines and is called the Dynkin curve. There is one line corresponding to the short root and two lines corresponding to the long root. If \( u \) is contained in \( G(F) \) then \( \text{Gal}(\overline{F}/F) \) permutes the lines \( \ell_\beta, \ell'_\beta \) corresponding to the long root. We say that a subregular class is distinguished if the quadratic extension associated to it is the trivial extension, that is, all three lines are defined over \( F \). Most of the analysis of this section will be devoted to the classes which are not distinguished.

**Lemma 5.2.** Suppose that \( H \) is an endoscopic group of \( G \) which is split by a nontrivial quadratic extension \( E \) of \( F \). Let \((T, \kappa)\) be a pair associated to \( H \). Then the subregular germ of the unipotent class \( O_{E'} \) associated to the quadratic extension \( E' \) for the pair \((T, \kappa)\) is zero unless \( E' = E \).
PROOF: Let $Z$ be the center of $G$. Let $u \in O(F)$ and let $C_G(u)_{\text{red}}$ be the reductive centralizer of $u$. If $\kappa$ is nontrivial on $\ker(H^1(\text{Gal}(\overline{F}/F), Z) \to H^1(\text{Gal}(\overline{F}/F), C_G(u)_{\text{red}}))$ pick $h$ so that $O^h = O$, $\kappa(\sigma(h)h^{-1}) \neq 1$ so that by (2.1) $\Gamma^{\kappa}_{\text{red}} = 0$. It was mentioned in (2.1) that $\kappa$ restricted to $H^1(\text{Gal}(\overline{F}/F, Z)$ depends only on the endoscopic group associated to to $(T, \kappa)$. We see this directly as follows in the case that $G$ is the inner form of a split group and $Z = \{\pm 1\}$ and $H$ split by a quadratic extension $E$ of $F$. Directly from the definition of endoscopic groups we see easily that $E = F$ if and only if $H$ (up to a central factor) is an endoscopic group of $G$ which in turn is true if and only if $\kappa$ restricted to $H^1(\text{Gal}(\overline{F}/F, Z)$ is trivial. If $E$ is a nontrivial extension then $\kappa$ is nontrivial but trivial over the quadratic extension $E$ of $F$. In other words, $\kappa$ is trivial on the image of the corestriction map from $H^1(\text{Gal}(\overline{F}/E), Z)$ to $H^1(\text{Gal}(\overline{F}/F, Z)$. Identifying $H^1(\text{Gal}(\overline{F}/F, Z)$ with $F^\times/F^\times 2$ it follows that $\kappa$ may be identified with the nontrivial character on $F^\times/NE$ where $NE$ is by definition the group of norms of nonzero elements of $E$. To complete the proof it is sufficient to show that $\ker(H^1(\text{Gal}(\overline{F}/F), Z) \to H^1(\text{Gal}(\overline{F}/F, C_G(u)_{\text{red}}))$ is $NE'/F^\times 2 \subseteq F^\times/F^\times 2$. Note that $Z$ fixes the lines $\ell_\beta$, $\ell'_\beta$ so that the image of $H^1(\text{Gal}(\overline{F}/F, Z)$ lies in $H^1(\text{Gal}(\overline{F}/F, C_G(u)^{\text{red}}_{\text{red}})$ where $C_G(u)^{\text{red}}_{\text{red}}$ is the subgroup of $C_G(u)_{\text{red}}$ which fixes the lines $\ell_\beta$, $\ell'_\beta$. In the situation at hand it is known [Sp] that $C_G(u)^{\text{red}}_{\text{red}} = C_G(u)^0_{\text{red}}$ the identity component of $C_G(u)_{\text{red}}$, so that it is sufficient to work with $C_G(u)^0_{\text{red}}$.

The connected reductive centralizer is computed when $G$ is split as follows. Let $B$ be a Borel subgroup over $F$ in the line of type $\alpha$ in the Dynkin curve of $u$. Conjugating $B$ to the upper triangular matrices we see that

$$u = \begin{pmatrix} I_2 & XJ \\ 0 & I_2 \end{pmatrix}$$

where $X = ^tX \in \text{GL}_2(F)$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows immediately that

$$C_G(u) = \begin{pmatrix} ^tA & B \\ 0 & JA^{-1}J \end{pmatrix} \quad \text{and} \quad C_G(u)_{\text{red}} = \begin{pmatrix} ^tA & 0 \\ 0 & JA^{-1}J \end{pmatrix}$$

where $^tAXA = X$. Thus $C_G(u)^0_{\text{red}} \xrightarrow{\sim} U_{E'}(1)$ a one dimensional torus split by $E'$. $\ker(H^1(Z) \to H^1(C_G(u)^0_{\text{red}}))$ is then identified with $NE'/F^\times 2 \subseteq F^\times/F^\times 2$ and the lemma follows for $G$ split.

When $G$ is not split we make the following modifications. $G$ contains a parabolic subgroup $P_\alpha$ over $F$ which is identified modulo a Borel subgroup $B$ with the line of type $\alpha$ in the Dynkin curve of $u$. Thus we may assume that $u$ has the same form. Over the quadratic extension $E'$ of $F$ $G$ splits and $u$ becomes distinguished so that over $E'$ it follows that $C_G(u)^0_{\text{red}} \xrightarrow{\sim} \mathbb{G}_m$. But if $C_G(u)^0_{\text{red}}$ were isomorphic to $\mathbb{G}_m$ over $F$ we would have a split torus in $\begin{pmatrix} ^tM & 0 \\ 0 & JM^{-1}J \end{pmatrix}$, where
det \( M = 1 \), which would contradict the hypothesis that \( G \) is not split. Thus \( C_G(u)_\text{red} \cong U_{E'}(1) \) as before, and \( E' \neq F \). This completes the proof of (5.2).

We have seen there are two divisors which contribute to the subregular germ. On the patch \( Y_U \) given by (3.7)

\[
\begin{align*}
\lambda &= z(\alpha)x(\alpha) \\
\lambda &= z(\beta)x(\beta) \\
\lambda w(\gamma) &= z(\alpha)z(\beta)x(\gamma) \\
\lambda w(\delta) &= z(\alpha)^2z(\beta)x(\delta).
\end{align*}
\]

\( E_\alpha \) is described by the equations \( x(\alpha) = z(\beta)w(\gamma) - z(\alpha)x(\gamma) = z(\beta)w(\delta) - z(\alpha)^2x(\delta) = 0 \) and \( E_\beta \) is described by the equations \( x(\beta) = z(\alpha) = w(\delta) = z(\alpha)w(\gamma) - z(\beta)x(\gamma) = 0 \).

**Lemma 5.3.** If \( u_0 \) is not distinguished then \( E_\beta \) contains no \( F \)-rational points over \( u_0 \) and consequently makes no contribution to the germ associated to \( u_0 \).

**Proof:** The lemma is an immediate consequence of two facts. First, if \( (u_0, (B(W))) \in E_\beta \) then every Borel subgroup \( B(W) \) lies in the same line \( \ell_\beta \) of type \( \beta \) in the Dynkin curve. For select local coordinates as in (3.12) such that \( B_0 \) lies in \( \ell_\beta \) and \( \ell_\beta \) contains \( B(W_+) \). Then \( B(W) = B_0^{n_wv} \) where \( n_w \) has the form

\[
(5.4) \quad n_w = \exp(z(\alpha)z_kX_{-\alpha})\exp(z(\beta)z_{k-1}X_{-\beta})\cdots \exp(z(\alpha)z_1X_{-\alpha})
\]

for appropriate values of \( z_1, \ldots, z_k \). So \( z(\alpha) = 0 \) implies that the product (5.4) collapses to \( n_w = \exp(z(\beta)z'X_{-\beta}) \) for some \( z' \). Also \( B_0^v = B(W_+) \in \ell_\beta \) and \( B_0, B(W_+) \subset P_\beta \) where \( B\setminus P_\beta = \ell_\beta \) so \( P_\beta^v = P_\beta \) which implies that \( v \in N_\infty \cap P_\beta \) or \( v = \exp(\xi X_{-\beta}) \) for some \( \xi \), where \( N_\infty \) is the unipotent radical of \( B_\infty \). Thus \( n_wv \in P_\beta \) and \( B(w) = B_0^{n_wv} \in P_\beta \). Thus if \( \ell_\beta \) and \( \ell'_\beta \) are the lines of the Dynkin curve of \( u_0 \) we may separate points \( (u_0, (B(W))) \) of \( E_\beta \) into \( \ell_\beta \)-points and \( \ell'_\beta \)-points according to where the Borel subgroups lie. Second, the action of \( \text{Gal}(\overline{F}/F) \) on the points of \( E_\beta \) interchanges \( \ell_\beta \)-points with \( \ell'_\beta \)-points. For the action (3.13) is given by \( (u_0, (B(W))) \mapsto (u_0, (\sigma(B(\sigma^{-1}_T W)))) \) for \( \sigma \in \text{Gal}(\overline{F}/F), u_0 \in G(F) \). If \( B(W) \) lies in \( \ell_\beta \) then \( B(\sigma^{-1}_T W) \) lies in \( \ell_\beta \) and \( \sigma(B(\sigma^{-1}_T W)) \) lies in \( \sigma(\ell_\beta) = \ell'_\beta \) provided \( \sigma \) has nontrivial image in \( \text{Gal}(E/F) \) where \( E \) is the quadratic extension trivializing the action on the lines \( \ell_\beta, \ell'_\beta \).

We now devote our attention to the divisor \( E_\alpha \). We fix \( u_0 \) a subregular element which is not distinguished and let \( B_0 \) be a Borel subgroup at the intersection of the line of type \( \alpha \) and
a line of type β in the Dynkin curve of u₀. As in the discussion of the divisor Eβ we find that
if \((u, B_0^{nw})^v\) lies in \(E_α\) over the subregular element u₀ then \(u^v = u₀\) and \(v \in P_α \cap N∞\) so that
\(v = exp(ξX_−α)\) for some ξ.

Select a Cartan subgroup \(T₀ \subseteq B₀\) over \(F\) such that if \(σ_α\) is the simple reflection associated
to the simple root \(α\), \(B_0^{σ_α}\) is equal to the Borel subgroup at the intersection of \(ℓ_α\) and \(ℓ'_β\) (the
second line of type β in the Dynkin curve). Such a Cartan subgroup may be found by selecting
any Cartan over \(F\) in \(B₀\) and in a Levi factor over \(F\) of \(P_α\) where \(B \setminus P_α = ℓ_α\). Let \(B∞\) be
the Borel subgroup opposite \(B₀\) through \(T₀\). We consider the patch \(Y_U = Y_U(\B_∞, B₀)\) for this
choice of \((\B_∞, B₀)\).

The choice of \(B₀\) forces \(u₀\) to have the form \(u₀ = \begin{pmatrix} I₂ & X \\ 0 & I₂ \end{pmatrix}\), \(X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\). The choice
of \(B∞\) forces \(σ_αu₀σ_α⁻¹\) to have the same form which implies that \(y = 0\) so that

\[
(5.5) \quad u₀ = \begin{pmatrix} I₂ & xI₂ \\ 0 & I₂ \end{pmatrix}
\]

We choose a Haar measure on \(O_{u₀}\) as follows. We select coordinates \(v' = \begin{pmatrix} I₂ & 0 \\ N & I₂ \end{pmatrix}\),
\(y = \begin{pmatrix} I₂ & Y \\ 0 & I₂ \end{pmatrix}\), \(N = \begin{pmatrix} n_γ & n_β \\ n_δ & n_γ \end{pmatrix}\), \(Y = \begin{pmatrix} y_γ & y_δ \\ y_β & y_γ \end{pmatrix}\) so that \(y^{v'}\) describes points on an open
set of \(O_{u₀}\). Then

\[
(5.6) \quad \frac{1}{2} dy dv' \overset{def}{=} \frac{1}{2} dy_β dy_γ dy_δ dn_β dn_γ dn_δ
\]

is an invariant measure on \(O_{u₀}\).

We consider the patch \(x(γ) \neq 0\) on \(Y_U\) so that the equations (3.7) take the form

\[
(5.7) \quad λ = \frac{x(α)x(β)w(γ)}{x(γ)}, \quad z(α) = \frac{x(β)w(γ)}{x(γ)}, \quad z(β) = \frac{x(α)w(γ)}{x(γ)}, \quad w(δ) = \frac{x(β)w(γ)^2x(δ)}{x(γ)^2}
\]

224
By (1.1), (3.8), and a change of variables

\[ \omega_{E_\alpha} = \frac{x(\gamma)}{x(\beta)} \frac{dw(\gamma)}{w(\gamma)^2} dx(\beta) dx(\gamma) dx(\delta) dv = \frac{\omega_Y}{\lambda_2 \frac{dx(\alpha)}{x(\alpha)}}. \]

We divide by the Haar measure \( \omega_{\Omega_{\omega_0}} = (1/2) dx \ dv' = (1/2) dy \ dv' \) to obtain a differential form \( \omega_{E_\alpha(u_0)} \) on the fibre \( E_\alpha(u_0) \) over \( u_0 \). Making use of the fact that \( v = \exp(\xi X) \) on \( E_\alpha(u_0) \) this gives

\[ \omega_{E_\alpha(u_0)} = \frac{2x(\gamma)}{x(\beta)} \frac{dw}{w^2} d\xi \quad w = w(\gamma). \]

Next we note that if \( (u, B(W))^\ast \) is a point of \( E_\alpha(u_0) \) with coefficients \( x_\beta(u) = x(\beta) \quad x_\gamma(u) = x(\gamma) \), \( x_\delta(u) = x(\delta) \) we must have \( u^v = u_0 \) with \( v = \exp(\xi X) \). By (5.5) this implies that \( x(\delta) = 0 \) and \( x(\gamma)/x(\beta) = 1/2 \xi \) so that (5.8) becomes

\[ \omega_{E_\alpha(u_0)} = \frac{dw}{w^2} \frac{d\xi}{\xi}. \]

Note that it is always possible to recover the action of Gal(\( \overline{F}/F \)) from the action of \( W \times A \) and the action \( \sigma_\ast \) for a split Cartan subgroup and \( B_0 \) over \( F \). The equation \( \sigma_{\alpha'}(x_\gamma(b)) = x_\gamma(b') \) of (3.13) gives for an arbitrary divisor.

\[ (5.10) \]

\[ \sigma_\alpha(x(\alpha))/x(\alpha) = 1 \]
\[ \sigma_\alpha(x(\beta))/x(\beta) = 1 + 2(T_1 - T_2)w(\gamma) + (T_1 - T_2)^2 w(\delta) \]
\[ \sigma_\alpha(x(\gamma))/x(\gamma) = 1 + (T_1 - T_2)x(\alpha)x(\delta)/x(\gamma) = 1 + (T_1 - T_2)w(\delta)/w(\gamma) \]
\[ \sigma_\alpha(x(\delta))/x(\delta) = 1 \]
\[ \sigma_\beta(x(\alpha))/x(\alpha) = 1 - 2T_2 w(\gamma) \]
\[ \sigma_\beta(x(\beta))/x(\beta) = 1 \]
\[ \sigma_\beta(x(\gamma))/x(\gamma) = 1 \]
\[ \sigma_\beta(x(\delta))/x(\delta) = 1. \]

Here \( T_1, T_2 \) represents the tangent direction of the curve \( \Gamma \) in \( T \) at the identity and \( \sigma_\alpha \) and \( \sigma_\beta \) are simple reflections associated to the simple roots \( \alpha \) and \( \beta \). From the choice of \( B_0 \) made following lemma 5.3 the image of the group \( A \) in the Weyl group is the subgroup of order two generated by the reflection \( \sigma_\alpha \). Let \( \sigma_0 \) be the representative of \( \sigma_\alpha \) in \( A \). Then (3.13) and (5.10)
specializing to the divisor $E_\alpha$ give

(5.11) \begin{align*}
\sigma_\beta(\xi) &= \xi \\
\sigma_\alpha(\xi) &= w_A \xi \\
\sigma_0(\xi) &= 1/\xi \\
\sigma_\beta(w) &= w/w_B \\
\sigma_\alpha(w) &= w/w_A \\
\sigma_0(w) &= -w/w_A
\end{align*}

where we introduce abbreviations

(5.12) \begin{align*}
w_A &= 2(T_1 - T_2)w + 1 \\
w_B &= -2T_2w + 1 \\
w_D &= 2T_1w + 1.
\end{align*}

and $\zeta$ is an element of $F^\times$. Since $\sigma_0, \sigma_\alpha$ and $\sigma_\beta$ generate $W \times A$ we easily deduce the following charts:
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\sigma(w)$</th>
<th>$\sigma(w_A)$</th>
<th>$\sigma(w_B)$</th>
<th>$\sigma(w_D)$</th>
<th>$\sigma(\xi)$</th>
<th>$t_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$w$</td>
<td>$w_A$</td>
<td>$w_B$</td>
<td>$w_D$</td>
<td>$\xi$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>$\sigma_\alpha$</td>
<td>$w$</td>
<td>$\frac{1}{w_A}$</td>
<td>$w_B$</td>
<td>$w_D$</td>
<td>$\frac{1}{w_A} \xi$</td>
<td>$(\xi w, \frac{1}{\xi w})$</td>
</tr>
<tr>
<td>$\sigma_\beta \sigma_\alpha$</td>
<td>$w$</td>
<td>$w_B$</td>
<td>$\frac{1}{w_A}$</td>
<td>$w_A$</td>
<td>$\frac{1}{w_D} \xi$</td>
<td>$(\xi w, \frac{1}{\xi w})$</td>
</tr>
<tr>
<td>$\sigma_\alpha \sigma_\beta \sigma_\alpha$</td>
<td>$\frac{w}{w_D}$</td>
<td>$\frac{w_B}{w_D}$</td>
<td>$\frac{w_A}{w_D}$</td>
<td>$\frac{1}{w_B} \xi$</td>
<td>$(\xi w^2, \frac{1}{w_A})$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta$</td>
<td>$w$</td>
<td>$\frac{1}{w_D}$</td>
<td>$w_B$</td>
<td>$w_A$</td>
<td>$\frac{1}{w_D} \xi$</td>
<td>$(\xi w^2, \frac{1}{w_A})$</td>
</tr>
<tr>
<td>$\sigma_\beta \sigma_\alpha \sigma_\beta$</td>
<td>$w$</td>
<td>$w_A$</td>
<td>$w_D$</td>
<td>$w_B$</td>
<td>$\frac{1}{w_B} \xi$</td>
<td>$(\xi w^2, \frac{1}{w_A})$</td>
</tr>
<tr>
<td>$\sigma_\alpha \sigma_\beta$</td>
<td>$w$</td>
<td>$w_A$</td>
<td>$w_B$</td>
<td>$\frac{1}{w_B} \xi$</td>
<td>$(\xi w^2, \frac{1}{w_A})$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>$w$</td>
<td>$w_B$</td>
<td>$\frac{1}{w_B} \xi$</td>
<td>$(\xi w^2, \frac{1}{w_A})$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
$\zeta$ is a constant in $F^\times$. It is a norm of an element in the quadratic extension $E$ (associated to the unipotent element $u_0$) if and only if $G$ is split. To see this we argue as follows. $w = 0$ defines the intersection of $E_\alpha$ with $E_0$. By (3.6) the fibre $E_\alpha(u_0) \cap E_0$ may be identified with a subvariety of the Springer variety. This fibre is 1-dimensional, closed, and consists of $(u_0, (B(W))) \in E_\alpha(u_0) \cap E_0$ with $B(W) = B(W_+) \in \ell_\alpha$ for all $W$, and hence it is isomorphic to $\ell_\alpha$ with coordinate $\xi$. Pull the action of $W \times A$ back to $Gal(\overline{F}/F)$ (3.13) and note that since when $w = 0$, then $w_A = w_B = w_D = 1$ it follows that $\xi$ is a coordinate over the quadratic extension $E$ of $F$ whose absolute Galois group maps to the identity of $A$. The condition $\sigma^2 = 1$, for $1 \neq \sigma \in Gal(E/F)$ together with $\sigma_0(\xi) = \frac{1}{\xi^2}$ forces $\xi \in F^\times$ and the elements of $E_\alpha(u_0) \cap E_0 \cong \ell_\alpha$ are in bijection with elements of $E$ with norm $\zeta$. Also $\ell_\alpha$ has $F$-rational points if and only if $B \setminus P_\alpha$ contains a Borel subgroup over $F$ which is true if and only if $G$ is quasi-split. Thus $\zeta$ is a norm if and only if $G$ is quasi-split. This fact will account for the signs $\epsilon_G$ (5.1) that enter into the formulas for the germs.

It is shown in [H] that $\kappa$ is trivial on the cocycle

\begin{equation}
\sigma_\alpha \mapsto \lambda^{\alpha^*} \sigma \beta \mapsto 1 \sigma_0 \mapsto 1.
\end{equation}

and $\kappa$ is trivial on the cocycle $\sigma_\alpha \mapsto 1 \sigma_\beta \mapsto \lambda^{\beta^*} \sigma_0 \mapsto 1$ if and only if $H$ is split.

Let $E$ be the quadratic extension associated to the fixed unipotent element $u_0$. We define a cocycle taking values in the 1-dimensional torus $U_E(1)$ split by $E$ as follows. For a fixed $w = w_0$, the fibre over $w$, which is isomorphic over $F$ to $\mathbf{P}^1$, does not necessarily have any $F$-rational points. It does if and only if the cocycle depending on $w$

\begin{equation}
\sigma_\alpha : \sigma \mapsto \sigma(\xi)(\xi)^{-1} \in U_E(1)
\end{equation}

is a coboundary when evaluated at $w = w_0$. The germ, being a principal value integral over $E_\alpha(u_0)$ may be expressed as a double integral over $\xi$ and then $\xi$ if we integrate only over those points for which (5.15) is a coboundary. Let $\eta_E$ be the non-trivial character on $H^1(\text{Gal}(\overline{F}/F), U_E(1))$. Then the subregular germ is equal to

\begin{equation}
|\lambda| \int_{E_\alpha(u_0)} \frac{dw}{w^2} \frac{d\xi}{\xi} \kappa(t_\sigma) = |\lambda| \int_{N_{m1}} \frac{d\xi}{|\xi|} \int_{\mathbf{P}^1(F)} \left( \frac{1 + \eta_E(\alpha^*)}{2} \right) \frac{dw}{|w|^2} \kappa(t_\sigma)
\end{equation}

\begin{equation}
= \frac{|\lambda|}{2} \int_{N_{m1}} \frac{d\xi}{|\xi|} \int_{\mathbf{P}^1(F)} \kappa(t_\sigma) \frac{dw}{|w|^2} + \frac{|\lambda|}{2} \int_{N_{m1}} \frac{d\xi}{|\xi|} \int_{\mathbf{P}^1(F)} \kappa(t_\sigma) \eta_E(\alpha^*) \frac{dw}{|w|^2}.
\end{equation}
Here $\bar{\xi}$ ranges over the set of norm one elements in $E$: $Nm1 = \{\xi \in E \mid \sigma(\xi)\xi = 1 \text{ for } \sigma \in Gal(E/F) \sigma \neq 1\}$. (In particular $\int_{Nm1}$ is independent of the tangent direction.) Notice that in (5.16) we integrate over $\mathbb{P}^1 \times \mathbb{P}^1$. To justify integration over $\mathbb{P}^1 \times \mathbb{P}^1$ rather than over the rational surface $E\alpha(u_0)$ we must consider coordinate patches other than $Y_U$ and check that the principal value integral is unaffected. This is done in [H]. It is also a simple consequence of calculations done in section 7 on other coordinate patches.

Identify roots in $ESO(2n)$ and $Sp(2n)$ as in the beginning of §4.

**Proposition 5.17.** Let $(T, \kappa)$ be a pair associated to the non-split group $ESO(4)$. Then the subregular germs on $G$ for the pair $(T, \kappa)$ are zero.

**Remark:** The group $ESO(4)$ has no $F$-rational subregular elements. Proposition 5.17 gives the transfer of the subregular germ in this case.

**Proof:** By (5.2) we may assume that $ESO(4)$ is split by the quadratic extension $E'$ associated to the unipotent class $O_{E'}$ (that is $E'=E$). Thus the group $W_T \times A$ is equal to a subgroup of

$$W_T \times A \subseteq \left\{ 1, \sigma_\alpha, \sigma_\beta, \sigma_\alpha^\sigma_\beta, \sigma_\beta^\sigma_\alpha, \sigma_\alpha, \sigma_\beta, \sigma_0, \sigma_\alpha^\sigma_\beta \sigma_0, \sigma_\alpha^\sigma_\beta \sigma_\alpha \sigma_\beta \sigma_0, \sigma_\alpha^\sigma_\beta \sigma_0 \right\} = \Omega_1$$

where $W_T = \{\sigma_T \mid \sigma \in Gal(\overline{F}/F)\}$.

We introduce the new variable $\xi' = w_D \xi$ and remark that $\sigma(\xi') = \xi'$ or $1/\xi' \xi'$ if $\sigma \in \Omega_1$ (by the chart (5.13)) so that the cocycle $a_\sigma$ is trivial if $G$ is split and non-trivial if $G$ is not split. The factor $\frac{1 + \eta_E(a_\sigma)}{2}$ of (5.16) is then identically 1 if $G$ is split, 0 otherwise. This proves the proposition when $G$ is not split.

Let $t = (\frac{w}{w_D}, 1)$, $b_\sigma = \sigma(t)t^{-1}$. It is given on generators of $\Omega_1$ by $\sigma_\alpha \mapsto (\frac{w_D}{w}, \frac{w}{w_D})$, $\sigma_\beta \sigma_0 \mapsto (-w_D, 1)$. $t_\sigma$ and $b_\sigma t_\sigma$ have the same class and $b_\sigma t_\sigma$ is given by

$$\sigma_\alpha \mapsto (\xi', \xi'^{-1}) \quad \sigma_\beta \sigma_0 \mapsto (-\xi', 1/\sigma(\gamma)).$$

It follows that $b_\sigma t_\sigma$ for $\sigma \in \Omega_1$ is independent of $w$. We may integrate in (5.16) first over $w$. But $\int_{\mathbb{P}^1(F)} \frac{dw}{|w|^2} = 0$ [LS1] so that the germ is equal to zero. This completes the proof. We remark that the proof does not use the fact that $\kappa$ is a nontrivial character.

Next we consider the endoscopic group $H = SL(2) \times U_E(1)$. By (5.2) the subregular germ for pairs $(T, \kappa)$ associated to $H$ are zero except when $H$ is split by the quadratic extension associated to the unipotent class. We assume that $E$ splits $H$. We identify the nontrivial element of the Weyl group of $H$ with the simple reflection $\sigma_\beta$ and the automorphism which
twists the torus $U_E(1)$ with the element $\sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha$. The factor $\sigma_0$ reflects the fact that by (5.2) we may assume $E' = E$. Then we have $W_T \times A \subseteq \{ 1, \sigma_\beta, \sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha, \sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta \}^{\text{def}} = \Omega_1$.

**Proposition 5.18.** (1) Let $(T, \kappa)$ be a pair associated to the non-split endoscopic group $H = SL(2) \times U_E(1)$. Then the nonzero subregular germ on $G$ for the pair $(T, \kappa)$ considered as a function of the tangent direction $(T_1, T_2)$ of the curve $\Gamma$ at the identity element depends only on $T_2$ not $T_1$.

(2) The subregular germ of the unipotent class $O_E$ on $G$ for pairs $(T, \kappa)$ associated to $SL(2) \times U_E(1)$ are given by the first row of (5.1) where

$$
\Gamma_1(E, E'', \beta) = |\lambda| \int_{\text{Nm}1} \frac{d\xi}{|\xi|} \int_{F^*} \eta_E(w^2/w_B) \frac{dw}{|w|^2}.
$$

Here the action of $\text{Gal}(\overline{F}/F)$ on $w$ is determined by the rule $\sigma_\beta(w) = w/w_B$, $\sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta (w) = -w$. $\text{Nm}1$ denotes the group of norm 1 elements of $E$.

(3) The transfer (2.8) of subregular germs near the identity is compatible for various choices of $(T, \kappa)$ associated to $SL(2) \times U_E(1)$. (In other words the function $f^H$ may be chosen independent of $(T, \kappa)$.)

**Proof:** The character $\kappa$ of $H^1(\text{Gal}(\overline{F}/F), T)$ may be identified with the nontrivial character on $H^1(\text{Gal}(\overline{F}/F), U_E(1))$ under the projection $T \subseteq H = SL(2) \times U_E(1) \to U_E(1)$. We then see using (5.13) that $\kappa(\tau_{\alpha})$ equals $\eta_E$ applied to $w^2/w_A w_D \in F^\times$. (We use $\eta_E$ indiscriminately for the nontrivial character on $U_E(1)$ and the nontrivial character on $F^\times$ trivial on the norms of $E$.) Again by (5.13) $\eta_E(\tau_{\alpha})$ of (5.15) is equal to $\eta_E$ applied to $w_A w_D \zeta \in F^\times$. By (5.16) the germ is equal to

$$
\frac{|\lambda|}{2} \int \frac{d\xi}{|\xi|} \int \eta_E \left( \frac{w^2}{w_A w_D} \right) \frac{dw}{|w|^2} + \frac{|\lambda|}{2} \int \frac{d\xi}{|\xi|} \int \eta_E \left( \frac{w^2}{w_B} \right) \frac{dw}{|w|^2}.
$$

$w_B = -2T_2 w + 1$ depends only on $T_2$ not $T_1$ and by (5.13) $\sigma(w), \sigma(w_B)$ for $\sigma \in \Omega_1$ depend only on $T_2$ not $T_1$. Thus the second term of (5.19) depends only on $T_2$.

To show that the first term of (5.19) is independent of $T_1$ we introduce the variable $w' = w/2T_1 w + 1 = w/w_D$. Simple calculations give

$$
\frac{dw}{w^2} = \frac{dw'}{w'^2}, \quad \frac{w^2}{w_A w_D} = \frac{w'^2}{(-2T_2 w' + 1)},
$$

and writing $w'_B = -2T_2 w' + 1$,

$$
\begin{array}{cccc}
\sigma(w) & \sigma(w') & \sigma(w_B) & \sigma(w'_B) \\
\sigma_\beta & \frac{w}{w_B} & \frac{w'}{w'_B} & \frac{1}{w_B} \\
\sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta & -w & -w' & w_B
\end{array}
$$

230
We see that the first term is independent of $T_1$ (because $w' w_B, \sigma(w')$ and $\sigma(w_B')$ are). In fact, the second term of (5.19) is equal to $\eta_E(\xi)$ times the first. In particular, if $G$ is not quasi-split the germ is zero; if $G$ is quasi-split then the germ is

$$|\lambda| \int \frac{d\xi}{|\xi|} \int \eta_E\left(\frac{w^2}{w_B}\right) \frac{dw}{|w|^2}.$$  

(3) To prove compatibility it would be easy enough at this point to identify the principal value integrals with those of [LS]. But instead, we use the fact that the subregular germ is invariant in the direction of $T_1$. Thus a function supported on regular and subregular elements satisfies $\Delta^T_G \Phi^T_G(\gamma, f) = \Delta^T_G \Phi^T_G(z_0 \gamma, f)$ (assuming that $z_0 = \exp(z T_1)$ and $\gamma$ are sufficiently small), so that the germ expansion at the identity coincides with the germ expansion at $z_0$. By (2.11) $\Delta^T_G \Phi^T_G(z_0 \gamma, f) = \Delta^T_{M_\gamma} \Phi^T_{M_\gamma}(\gamma, f)$. $\kappa$ is trivial on $(T \setminus M)(F)$ and $M_{der} \sim H_{der}$ so the result follows.

Next we consider the case that $H = Sp(4)$, $G$ is not quasi-split and the integrals are all stable.

**Proposition 5.22.** If $(T, 1)$ is a Cartan subgroup of $G$ and $G$ is not quasi-split, $\Gamma^{(T, 1)}_{q^r}$ is the germ on the quasi-split inner form and $\Gamma^{(T, 1)}$ is the germ on $G$. Then $\Gamma^{(T, 1)} = -\Gamma^{(T, 1)}_{q^r}$.  

**Proof:** The factor $\kappa(t_\sigma)$ of (5.16) is constant so that by the result $\int_{P^1(F)} \frac{dw}{|w|^2} = 0$ it follows that the first term of (5.16) is zero. The cocycle $a_\sigma$ and $a_\sigma^{q^r}$ for the quasi-split inner form differ only by the factor $\eta$. Thus we have

$$\Gamma^{(T, 1)} = \frac{|\lambda|}{2} \int \frac{d\xi}{|\xi|} \int_{P^1(F)} \eta_E(a_\sigma) \frac{dw}{|w|^2} = \frac{|\lambda|}{2} \eta_E(\xi) \int \frac{d\xi}{|\xi|} \int \eta_E(a_\sigma^{q^r}) \frac{dw}{|w|^2}$$

$$= \eta_E(\xi) \Gamma^{(T, 1)}_{q^r} = -\Gamma^{(T, 1)}_{q^r}$$

since $\eta_E(\xi) = -1$ because $\xi$ is not a norm if $G$ is not quasi-split.

Finally we consider the endoscopic group $H = SO(4)$. Since $H$ is split we must consider all subregular unipotent classes including the distinguished one. The Weyl group of $H$ may be identified with $\{1, \sigma, \sigma_0, \sigma_0 \sigma, \sigma_0 \sigma_0 \sigma_0 \sigma \}$. The group $W_T \times A$ is a subgroup of

$$\Omega_1 \stackrel{def}{=} \left\{ 1, \sigma, \sigma_0, \sigma_0 \sigma, \sigma_0 \sigma_0 \sigma, \sigma_0 \sigma_0 \sigma_0, \sigma_0 \sigma_0 \sigma_0 \sigma_0 \sigma_0 \sigma_0 \sigma_0 \right\}.$$  

**Proposition 5.23.** (1) Let $(T, \kappa)$ be a pair associated to the endoscopic group $H = SO(4)$. Then the germ on $G$ for the pair $(T, \kappa)$ and a subregular unipotent class associated to a nontrivial
quadratic extension $E$ of $F$ breaks into a sum of two terms. The first term depends on $T_1 + T_2$ but not $T_1 - T_2$. The second term depends on $T_1 - T_2$ but not $T_1 + T_2$.

(2) These germs are given as in (5.1) and the transfer (2.8) to the endoscopic group $SO(4)$ is valid for those germs.

**Proof:** As in the discussion of the endoscopic group $H = ESO(4)$ we use coordinates $\xi' = wD\xi$ and $w$ instead of $\xi$ and $w$. We have by (5.13) $\sigma_\alpha(\xi') = \xi'$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(\xi') = \xi'$, $\sigma_\alpha(\xi') = \left(\frac{w_Bw_D}{w_A}\right)(\frac{1}{\xi'})$. Thus $\eta_E$ evaluated on the cocycle $a_\sigma$ of (5.15) equals $\eta_E\left(\frac{w_Bw_D}{w_A}\right)$, $\frac{w_Bw_D}{w_A} \in F^\times$.

$H = SO(4)$ is an endoscopic group (up to a central factor) even if $G$ is of adjoint type, so we may calculate the cocycle in the adjoint group. We form a cocycle in $U_{E'}(1)$ (where $E'$ is the invariant field of $\sigma_\alpha$ and $\sigma_0$) by replacing the cocycle $t_\sigma : \sigma \mapsto (t_1, t_2)$ by $e_\sigma : \sigma \mapsto t_1 t_2 \in U_{E'}(1)$. The fact that $W_T \times A$ is a subgroup of $\Omega_1$ insures that this is well-defined. By (5.13) we find that $e_\sigma$ is given by

\[
\sigma_\alpha \mapsto 1, \quad \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \mapsto w^2/\gamma^2 w_B w_D \in F^\times, \quad \sigma_\alpha \mapsto 1.
\]

Thus $\kappa(t_\sigma) = \eta_{E'}(e_\sigma)$. Since $x(\gamma) \in F^\times$, $x(\gamma)^2 \in F^{\times 2}$ and it may be removed without affecting $\eta_{E'}(e_\sigma)$. Write $e'_\sigma$ for the cocycle so obtained.

We make the change of coordinates $w = w'/(2T_2 w + 1)$. Then $e'_\sigma$ becomes $\sigma_\alpha \mapsto 1$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \mapsto w'^2/(2(T_1 + T_2)w + 1) \in F^\times$, $\sigma_\alpha \mapsto 1$. The action of $\Omega_1$ on the variable $w'$ is easily seen to be $\sigma_\alpha(w') = w'$, $\sigma_0(w') = -w'/2(T_1 + T_2)w + 1$, $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta(w') = w'/2(T_1 + T_2)w + 1$. Also $dw/w = dw'/w'^2$. It follows that the first term of (5.16) is $\frac{1}{2} \int \frac{d\xi}{|\xi|} \int \eta_{E'}(e'_\sigma) \frac{dw'}{w'^2}$ which is independent of $T_1 - T_2$ because the action of $\Omega_1$ on $w'$ and the cocycle $e'_\sigma$ are independent of $T_1 - T_2$. Moreover the action of $\sigma_\alpha$ on $w'$, $(T_1 + T_2)$ and $2(T_1 + T_2)w + 1$ is trivial, so the data defining the principal value integral is the same for $W_T = \langle \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta, \sigma_\alpha \rangle$ and $\langle \sigma_\beta\sigma_\alpha\sigma_\beta \rangle$ and $\langle \sigma_\alpha \rangle$ as for $W_T = \langle \sigma_\alpha \rangle$ as for $W_T = \langle \sigma_\alpha \rangle$.

To prove that the second term of (5.16) is independent of $T_1 + T_2$ we return to the coordinates $(w, \xi')$. Note that $kw_B/w$ lies in the quadratic extension $E''$ which is invariant by $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$, $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0$ where $k$ is a constant chosen to satisfy $\sigma_\alpha(k) = k, \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_0(k) = -k$. Also if $\sigma$ is a nontrivial element of $Gal(E''/F)$ $\sigma(kw_B/w)(kw_B/w) = |\sigma(k)|kw_B w_D/w^2$. Thus up to a constant $w_B w_D/w^2$ modulo the norms of $E'$ is trivial or nontrivial according as $w_B w_D/w^2$ modulo the norms of $E$. The result is that $\kappa(t_\sigma) = \eta_{E'}(e'_\sigma) = \eta_E(e'_\sigma)$, and $\kappa(t_\sigma)\eta_E(a_\sigma) = \eta_E(e'_\sigma a_\sigma) = \eta_E(e'_\sigma^{-1} a_\sigma)$ $(\eta_E$ has order 2$)$ $= \eta_E\left(\frac{w^2}{w_B w_D} \cdot \frac{w_B w_D}{w_A}\right) = \eta_E\left(\frac{w^2}{w_A}\right)$. Special argument is required if $E'$ or $E'' = F$ but it is easy to check that the conclusion $\kappa(t_\sigma)\eta_E(a_\sigma) = \eta_E(w^2/w_A)$
still holds. Thus $\kappa(t_\sigma)\eta_\beta(a_\sigma)$ depends only on $T_1 - T_2$ (through $w_A = 2(T_1 - T_2)w + 1$) and not on $T_1 + T_2$. Furthermore an examination of (5.13) shows that the action of $\Omega_1$ on $w$ and $w_A$ depends only on $T_1 - T_2$ and not $T_1 + T_2$. Moreover the action of $\sigma_\beta\sigma_\alpha\sigma_\beta$ on $w, T_1 - T_2$ and $w_A$ is trivial, so the data defining the principal value integral is the same for $W_T = \langle \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta \rangle$ as for $W_T = \langle \sigma_\beta\sigma_\alpha\sigma_\beta, \sigma_\alpha\sigma_\alpha\sigma_\alpha\sigma_\beta \rangle$ and $W_T = \langle \sigma_\alpha \rangle$, and the same for $W_T = \langle \sigma_\beta\sigma_\alpha\sigma_\beta \rangle$ as for $W_T = \{1\}$. Consequently, the second term of (5.13) depends only on $T_1 - T_2$ and not $T_1 + T_2$.

We conclude that the germs are given by the entries of (5.1).

Compatibility of (2.8) for various pairs $(T, \kappa)$ is known in the special case that $T$ lies in a parabolic subgroup of $G$. Since we have expressed the subregular germ in terms of the germs associated to $T$ lying in a parabolic subgroup compatibility for all pairs $(T, \kappa)$ associated to $SO(4)$ follows.

The following proposition will complete our study of the subregular germs.

**Proposition 5.25.** (1) Let $(T, \kappa)$ be a pair associated to the endoscopic group $SO(4)$. Then the germ on $G$ for the pair $(T, \kappa)$ and the distinguished subregular class breaks into a sum of two terms. The first depends on $T_1 + T_2$ but not $T_1 - T_2$. The second term depends on $T_1 - T_2$ but not $T_1 + T_2$.

(2) These germs are given as in (5.1) and the transfer (2.8) to the endoscopic group $SO(4)$ is valid for these germs.

**Proof:** In the analysis of the distinguished subregular class (5.2) does not apply so that there are two divisors to consider. We select $B_0, B_\infty$ as before, noting that $B_0, B_\infty$ are now over $F$ so that the group $A$ is trivial ($\sigma_0$ does not occur). Recall that $B_0$ was selected to lie in a line $\ell_\beta$ of type $\beta$ in the Dynkin curve of $u_0$ and that the points $(u_0, (B(W)))$ of $E_\beta(u)$ are such that $B(W) \in \ell_\beta$ for all $W$ or $B(W) \in \ell_\beta'$ for all $W$. Let $E_\beta(\ell_\beta)$ or $E_\beta(\ell_\beta, u_0)$ denote the points of $E_\beta(u_0)$ for which the first condition holds.

The function $\kappa(t_\sigma)$ is seen to be identically 1 on $E_\alpha(u_0)$ and $E_\beta(u_0)$ as follows. By definition $t_\sigma$ is given by (3.15)

$$\sigma_\alpha \mapsto (1/x(\alpha))^{a_\sigma}$$

$$\sigma_\beta \mapsto (1/x(\beta))^{b_\sigma}$$

Since $H$ is split, $\kappa$ evaluated on the cocycle $\sigma_\alpha \mapsto 1, \sigma_\beta \mapsto x^{b_\sigma}, (x \in F^\times)$ is trivial (5.14). Thus
we may take $t_{\sigma}$ to be given by

$$
\sigma_{\alpha} \mapsto \left( \frac{1}{x(\alpha)} \right)^{\alpha^*} \\
\sigma_{\beta} \mapsto \left( \frac{\lambda}{x(\beta)} \right)^{\beta^*} = \left( \frac{x(\alpha)w(\gamma)}{x(\gamma)} \right)^{\beta^*}.
$$

We adjust this by the coboundary $c_{\sigma} = \sigma(t)t^{-1} = (1, x(\alpha))$ to obtain $t_{\sigma}c_{\sigma}$ given by $\sigma_{\alpha} \mapsto 1$, $\sigma_{\beta} \mapsto w/(w_Bx(\alpha)x(\gamma))$. By (5.10) together with the fact that $w(\delta) = 0$ on $E_{\alpha}(u_0) \cap Y_U$ and $E_{\beta}(u_0) \cap Y_U$ we see that $w/(w_Bx(\alpha)x(\gamma)) \in F^\times$. Thus by (5.14) $\kappa(t_{\sigma})$ is identically 1 on the open patches $E_{\alpha}(u_0) \cap Y_U$ and $E_{\beta}(u_0) \cap Y_U$ and so identically 1 everywhere: $\kappa(t_{\sigma}) \equiv 1$.

We turn to the form on $E_{\beta}$. Recall that $E_{\beta}$ is given by the equations $x(\beta) = x(\alpha) = w(\delta) = x(\alpha)w(\gamma) - z(\beta)x(\gamma) = 0$. For $x(\gamma) \neq 0$ we use coordinates given on the patch $x(\gamma) \neq 0$ in the proof of (4.4). The form $\omega_{E_{\beta}}$ on $E_{\beta}$ is given by

$$
\omega_{E_{\beta}} = \text{Res}_{E_{\beta}}(\omega_Y/\lambda^2) = \text{Res} \left( \frac{d\lambda}{\lambda^2} dx(\alpha) dx(\beta) dx(\gamma) dx(\delta) dv \right)
$$

$$
= x(\gamma) \frac{dw(\gamma)}{w(\gamma)^2} \frac{dx(\alpha)}{x(\alpha)} dx(\gamma) dx(\delta) dv.
$$

We take an invariant measure on the subregular class $O$ to be

$$
\omega_0 = dx(\alpha) dx(\gamma) dx(\delta) dv(\alpha) dv(\gamma) dv(\delta)
$$

where we use coordinates $x(\alpha), x(\gamma), x(\delta), v(\alpha), v(\gamma)$, on an open set of the double cover of $O$. Set

$$
m(w, x, y, z) = \begin{pmatrix} 1 & x & y & z \\
1 & w & * & \\
1 & * & 1 \\
1 & & & 1
\end{pmatrix} \in Sp(4), \ J_0 = \begin{pmatrix} 1 & 0 & & \\
& & & \\
& -1 & 1 & \\
& & & -1
\end{pmatrix}.
$$

Then $(x(\alpha), x(\gamma), v(\alpha), v(\beta), v(\delta))$ are coordinates for the element $m(0, x(\alpha), x(\gamma), x(\delta))^{v_{\beta}}$, where $v_{\beta} = J_0^{-1}m(0, v(\alpha), v(\gamma), v(\delta))J_0$.

We take the quotient $\omega_{E_{\beta}}/\omega_0$ to obtain the form on $E_{\beta}(\ell_{\beta})$

$$
\frac{x(\gamma)}{x(\alpha)} \frac{dw(\gamma)}{w(\gamma)^2} d\xi_{\beta} \quad \text{where } \xi_{\beta} \text{ is given as follows.}
$$

If $(u, B_0^{nw})^v \in E_{\beta}(\ell_{\beta})$ then $B_0^v \in \ell_{\beta}$, $B_0 \in \ell_{\beta}$ which implies that $v \in P_{\beta} \cap N_{\infty}$ so that $v = \exp(-\xi_{\beta} X_{-\beta})$. Also recall that $u_0$ has the form (5.5) $u_0 = m(0, 0, x, 0)$ provided $B_0$ is taken to be upper triangular and $B_{\infty}$ lower triangular. Thus $u_0^{v^{-1}} = u = m(0, \xi, x, 0)$ so that on $E_{\beta}(\ell_{\beta})$

$$
\frac{x(\gamma)}{x(\alpha)} = \frac{1}{\xi_{\beta}}.
$$
Finally we must normalize the form of \((5.27)\) so that it is compatible with form on \(E_\alpha(u_0)\). The compatibility condition is

\[
\text{Res}_{E_\alpha} \omega_{E_\alpha} = \text{Res}_{E_\beta} \omega_{E_\beta}
\]

and provided the forms on \(E_\alpha(u_0)\) and \(E_\beta(\ell_\beta, u_0)\) are defined with respect to the same invariant form \(\omega_0\) on \(O\)

\[
\text{Res}_{E_\beta}(\ell_\beta) \omega_{E_\alpha}(u_0) = \text{Res}_{E_\alpha}(u_0) \omega_{E_\beta}(\ell_\beta).
\]

This condition forces the normalization

\[
(5.28) \quad \omega_{E_\beta}(\ell_\beta) = \frac{dw(\gamma)}{w(\gamma)^2} \frac{d\xi_\beta}{\xi_\beta}
\]

for then \(\text{Res}_{E_\alpha}(u_0) \omega_{E_\beta}(\ell_\beta) = \frac{dw(\gamma)}{w(\gamma)^2}\).

Write \(w_\beta\) for the restriction of \(w(\gamma)\) to \(E(\ell_\beta)\). Then by \((5.10)\)

\[
(5.29) \quad 1 \quad \sigma_\alpha \quad \sigma_\beta \sigma_\alpha \sigma_\beta \quad \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta
\]

\[
\sigma(\xi) \quad \xi \quad w_A \xi \quad w_D \xi / w_B \quad w_A w_D \xi / w_B
\]

\[
\sigma(w) \quad w \quad w / w_A \quad w \quad w / w_A
\]

\[
\sigma(w_A) \quad w_A \quad 1 / w_A \quad w_A \quad 1 / w_A
\]

\[
\sigma(w_B) \quad w_B \quad w_B / w_A \quad w_D \quad w_D / w_A
\]

\[
\sigma(w_D) \quad w_D \quad w_D / w_A \quad w_B \quad w_B / w_A
\]

\[
\sigma(\xi_\beta) \quad \xi_\beta \quad \xi_\beta \quad w_\beta,B \xi_\beta / w_\beta,D \quad w_\beta,B \xi_\beta / w_\beta,D
\]

\[
\sigma(w_\beta) \quad w_\beta \quad w_\beta / w_\beta,A \quad w_\beta \quad w_\beta / w_\beta,A
\]

\[
\sigma(w_\beta,A) \quad w_\beta,A \quad 1 / w_\beta,A \quad w_\beta,A \quad 1 / w_\beta,A
\]

\[
\sigma(w_\beta,B) \quad w_\beta,B \quad w_\beta,B / w_\beta,A \quad w_\beta,D \quad w_\beta,D / w_\beta,A
\]

\[
\sigma(w_\beta,D) \quad w_\beta,D \quad w_\beta,D / w_\beta,A \quad w_\beta,B \quad w_\beta,B / w_\beta,A
\]

where \(w_{\beta,A} = 2(T_1 - T_2) w_\beta + 1\), \(w_{\beta,B} = -2T_2 w_\beta + 1\), and \(w_{\beta,D} = 2T_1 w_\beta + 1\).

We must also consider coordinates patches with \(B_0, B_\infty / F\) such that \(B_0^{\sigma_\beta}\) is the point of intersection of \(\ell_\beta\) and \(\ell_\alpha\) (resp. \(\ell_\beta'\) and \(\ell_\alpha\)). On this patch \(z_c \overset{\text{def}}{=} z(\beta)\) and \(\xi_c \overset{\text{def}}{=} v(\beta)\) serve as coordinates on \(E(\ell_\beta, u_0)\) (resp. \(z_c', \xi_c'\) on \(E(\ell_\beta', u_0)\)).

To distinguish coordinates on various patches we add subscript \(\alpha\)'s to coordinates on \(E(\ell_\alpha, u_0)\) and subscript \(\beta\)'s to coordinates on \(E(\ell_\beta, u_0)\) (resp. \(E(\ell_\beta', u_0)\)). On \(E(\ell_\beta, u_0)\) we have
(5.30)
\[ w_\beta = z_c/(\xi_c + 2T_2 z_c), \quad \xi_\beta = -1/\xi_c \]
\[ z_c = -w_\beta/\xi_\beta(1 - 2T_2 w_\beta), \quad \xi_c = -1/\xi_\beta. \]

On \( E(t_\alpha, u_0) \) we have

(5.31)
\[ \xi_\alpha = -1/\xi'_\alpha, \quad w_\alpha = -w'_\alpha/(2(T_1 - T_2)w'_\alpha + 1) \]
\[ \xi'_\alpha = -1/\xi_\alpha, \quad w'_\alpha = -w_\alpha/(2(T_1 - T_2)w_\alpha + 1). \]

On \( E(t'_\beta, u_0) \) we have

(5.32)
\[ w'_\beta = z'_c/(\xi'_c + 2T_2 z'_c), \quad \xi'_\beta = -1/\xi'_c \]
\[ z'_c = -w'_\beta/\xi'_\beta(1 - 2T_2 w'_\beta), \quad \xi'_c = -1/\xi'_\beta. \]

The form on \( E(t_\beta, u_0) \) is \(-dz_c d\xi_c/z^2_c = dw_\beta d\xi_\beta/w_\beta^2 \xi_\beta \) and the form on \( E(t'_\beta, u_0) \) is \(-dz'_c d\xi'_c/z^2'_c = dw'_\beta d\xi'_\beta/w'_\beta^2 \xi'_\beta \).

Truncate near \( E(t_\beta, u_0) \cap E(t_\alpha, u_0) \) in \( E(t_\beta, u_0) \) (The \( t_\alpha \)-pole) by

\[ \left| -2T_2 w_\beta + 1/(T_1 - T_2) w_\beta + 1 \right| \xi_\beta \geq q^{-m}. \]

By (5.29) \( -2T_2 w_\beta + 1/(T_1 - T_2) w_\beta + 1 \xi_\beta \) is a variable over \( F \). Since by (5.30)

\[ \frac{1}{\xi_c + (T_1 + T_2)z_c} = \frac{(-2T_2 w_\beta + 1)\xi_\beta}{(T_1 - T_2) w_\beta + 1}. \]
This diagram should help to clarify coordinates patches. It also identifies how the principal value integral will be broken into five pieces by truncating the integrals near the poles.

This region is the same as $|\xi_c + (T_1 + T_2)z_c| \leq q^m$. Let $\overline{\xi}_c = \xi_c + (T_1 + T_2)z_c$. Then $z_c$ and $\overline{\xi}_c$ are variables over $F$ and

$$PV_1 = \int_{|\overline{\xi}_c| \leq q^m} \frac{d\overline{\xi}_c}{F} \int_{|z_c| \leq q^m} \frac{dz_c}{|z_c^2|} = 0.$$
Truncate in \( E(\ell'_\beta, u_0) \) near the \( \ell'_\alpha \)-pole by 
\[
-\frac{2 T_2 w'_\beta + 1}{(T_1 - T_2) w'_\beta + 1} \xi'_\beta
\geq q^{-m}.
\] Then similarly \( PV_5 = 0 \).

Truncate near the \( \ell_\delta \)-pole in \( E(\ell_\alpha, u_0) \) by 
\[
-\frac{2(2(T_1 - T_2) w_\alpha + 1)}{(-2T_2 w'_\alpha + 1)} \xi'_\alpha
\geq q^{-m}.
\] (2\(T_1 w_\alpha + 1\)) \( \xi'_\alpha \) and \( (2(T_1 - T_2) w'_\alpha + 1) \xi'_\alpha \) are variables over \( F \). Then since 
\[
\frac{1}{2(T_1 w_\alpha + 1)} \xi'_\alpha = \frac{-2(T_1 - T_2) w'_\alpha + 1}{(-2T_2 w'\alpha + 1)} \xi'_\alpha
\] let \( \bar{\xi}_\alpha = (2T_1 w_\alpha + 1) \xi_\alpha \).

\[
PV_3 = \int_{q^{-m} \leq |\bar{\xi}_\alpha| \leq q^m} \frac{d \xi_\alpha}{|\bar{\xi}_\alpha|} \int_{F_1} \frac{d w_\alpha}{|w_\alpha|^2} = 0.
\]

The contribution \( PV_4 \) to the germ at the pole \( \ell'_0 \cap \ell_\alpha \) is given by (1.3). The function \( M \) is given by \( m(\alpha_0) + 2m \) where \( \alpha_0 \) is defined by:

\[
\alpha_0 = \lim_{\lambda \to 0} \frac{\lambda}{\xi'_\beta \xi_\alpha} \left\{ \frac{x(\alpha) x(\beta) w(\gamma)}{x(\gamma)} \right\}
\]

Here \( \bar{\xi}'_\alpha \) and \( \bar{\xi}'_\beta \) are variables over \( F \). Their definitions – if not clear – may be read off from the denominator of this limit. On \( E_\alpha(u_0) \) \( x(\gamma)/x(\beta) = 1/2 \xi'_\alpha \) (5.8), on \( E(\ell'_\beta, u_0) \) \( x(\gamma)/x(\alpha) = 1/\xi'_\beta \) (5.27) and \( w'_\alpha = w'_\beta = w(\gamma) \) on their intersection so that

\[
\alpha_0 = \lim_{\lambda \to 0} \frac{2x(\gamma) w(\gamma)((T_1 - T_2) w(\gamma) + 1)}{(2(T_1 - T_2) w_\gamma + 1)} = \frac{2x(\gamma) w_3}{(1 - (T_1 - T_2)^2 w^2)}
\]

where \( w_3 = w'_\alpha/((T_1 - T_2) w_\gamma + 1) \) is a coordinate over \( F \). Thus \( M \) and hence \( PV_4 \) depend only on \( T_1 - T_2 \).

Similarly the factor \( \alpha \) at the \( (\ell_\beta \cap \ell_\alpha)_\)-pole is up to inessential factors is

\[
\frac{w}{(T_1 - T_2) w + 1} = \frac{w_4}{(1 - (T_1 + T_2)^2 w_4^2)}
\]

where \( w_4 = \frac{w}{(T_1 - T_2) w + 1} \) is a variable over \( F \), so that \( PV_2 \) depends only on \( T_1 + T_2 \). As in the proof of (5.23) we see that \( PV_2 \) is expressed in terms of the corresponding terms \( PV_2 \) associated to a Cartan subgroup \( T' \) contained in a parabolic subgroup \( (W_T = \{ \sigma_\alpha \} \text{ or } \{ 1 \}) \) for which compatibility is known. The term \( PV_4 \) is analyzed similarly. Hence the proposition follows.
There are no 2-regular elements in $G(F)$ if $G$ is not split. Consequently, we assume in this section that $G$ is split. We also take $G = \text{Sp}(4)$ but since we only prove the transfer of germs (2.8) for $G\text{Sp}(4)$ we assume that the endoscopic groups of $G$ are cuspidal and split. This leaves the group $H = SO(4)$. We may make this assumption by the remark of (2.5) and the fact that all the endoscopic groups of $G\text{Sp}(4)$ are split. (They are split because the derived group of the dual of $G\text{Sp}(4)$ is simply connected.)

Recall that all unipotent 2-regular elements are conjugate by $PSp(4,F)$ or equivalently by $G\text{Sp}(4,F)$. We have seen that there is one divisor $E_2$ of the Igusa data which contributes to the 2-regular germ. On the patch $Y_{\nu}$, $E_2$ is described by the equation $z(\alpha) = x(\alpha) = x(\beta) = x(\gamma) = 0$. On the smaller patch where $x(\delta), w(\gamma) \neq 0$ the equations become

\begin{equation}
\lambda = x(\gamma)^2 w(\delta) z(\beta)/(w(\gamma)^2 x(\delta))
\end{equation}

\begin{align*}
x(\alpha) &= z(\beta) x(\gamma)/w(\gamma) \\
x(\beta) &= x(\gamma)^2 w(\delta)/(w(\gamma)^2 x(\delta)) \\
z(\alpha) &= \frac{x(\gamma)^2 w(\delta)}{w(\gamma) x(\delta)}
\end{align*}

and $E_2$ is described simply by $x(\gamma) = 0$. Coordinates on $E_2$ on this patch are $x(\delta), w(\gamma), w(\delta), z(\beta), v(\alpha), v(\beta), v(\gamma), v(\delta)$.

By [Sp, p. 148] $B_{u_0}$ the 2-dimensional variety of Borel subgroups containing a given 2-regular element $u_0$, contains a unique projective line $\ell_\beta$ of type $\beta$ and $B_{u_0}$ is a union of lines of type $\alpha$ which intersect the line of type $\beta$. We consider the variety $E_2(u_0)$ of all points $p$ of $E_2$ above $u_0 = \pi(p) \in G$. We fix a Borel subgroup $B_0$ in the line $\ell_\beta$ (whose points are Borel subgroups) and select any $B_\infty$ opposite $B_0$. Then by the choice of $B_0 x_\alpha(u_0) = x_\beta(u_0) = x_\gamma(u_0) = 0$. Also if in the notation of (3.12) we have $(u, B_0^w)^v \in E_2(u_0)$, then $x_\alpha(u) = x_\beta(u) = x_\gamma(u) = 0$ and $u^v = u_0$. This forces $v = exp(\xi x_{-\beta})$ for some $\xi$. It follows that $x(\delta), v(\alpha), v(\gamma), v(\delta)$ serve as coordinates on an open set of the conjugacy class $O_{u_0}$ of $u_0$ while $z = \text{def} z(\beta), \xi = \text{def} v(\beta), w = \text{def} w(\gamma)$ and $\bar{w} = \text{def} w(\delta)$ serve as coordinates on an open set of the fibre $E_2(u_0)$. We also set

\begin{align*}
w_A &= 2(T_1 - T_2)w + 1, \quad w_B = -2T_2 w + 1, \quad w_D = 2T_1 w + 1 \\
\ell &= \bar{w}/w_B
\end{align*}
The differential form on $E_2$ is $\omega_{E_2} = \text{Residue}_{E_2}(\omega_Y/\lambda^\beta) = b(E_2)/a(E_2) = 3$. Using the coordinates of (6.1) we obtain

$$\omega_Y/\lambda^3 = -x(\delta) dz/d\bar{w} dw/(\bar{w}) d\xi d\alpha d\gamma d\delta$$

and

$$(6.3) \quad \omega_{E_2} = -x(\delta) dz/d\bar{w} dw/(\bar{w}) d\xi d\alpha d\gamma d\delta.$$  

We separate this form into two parts: an invariant form on the conjugacy class $O_{u_0}$ and a form on the fibre $E_2(u_0)$. $\omega_0 = -x(\delta) dz/\bar{w} dw/(\bar{w}) d\alpha d\gamma d\delta$ is easily seen to be an invariant form on $O_{u_0}$ and $\omega_{E_2}(u_0) = dz/d\bar{w} dw/(\bar{w}) d\xi$ is then the form on the fibre $E_2(u_0)$.

Next we consider the action of the group $W \times A$ on the variables. By our choice of Borel subgroup $B_0$, as with the subregular germs, $A$ has two elements $\{1, \sigma_0\}$.

Equations (5.10) give the first two rows of (6.4)

$$(6.4) \quad \begin{array}{ccccccc}
\sigma_\alpha & w & \ell & z & \xi & T_1 & T_2 \\
\sigma_\beta & w/w_B & \ell & z/\bar{w}_A & \xi & T_2 & -T_2 \\
\sigma_0 & (\xi + z)/(\xi + 2T_2 z) & \ell & z/(\xi(2T_2 z + \xi)) & -1/\xi & T_1 & T_2
\end{array}$$

We have used the abbreviations $\ell = \bar{w}/w_B$, $w = w + (T_1 - T_2) \bar{w}$, $\bar{w}_A = w_A + (T_1 - T_2)^2 \bar{w}$. The last row of (6.4) as well as the relation $\sigma_0(x(\alpha))/x(\alpha) = c\xi^{-1}$ (for some immaterial constant $c$ depending on the representative $A_\sigma$ in the normalizer), used below in the calculation of $c_\sigma$, is found by the method described at the end of (3.13). The cocycle $t_\sigma$ is given by $\sigma_\alpha \mapsto (1/x(\alpha))^{\alpha^*}$, $\sigma_\beta \mapsto (1/x(\beta))^{\beta^*}$, $\sigma_0 \mapsto (\xi \beta)^{\beta^*}$. We adjust $t_\sigma$ by $c_\sigma = \sigma(t) t^{-1}$, $t = (1, x(\alpha))$

$$c_\sigma: \quad \sigma_\alpha \mapsto (z(\alpha))^{\alpha^*}, \sigma_\beta \mapsto (w_B x(\alpha)^2)^{-\beta^*}, \sigma_0 \mapsto \xi^{-\beta^*}$$

$$c_{\sigma} t_\sigma: \quad \sigma_\alpha \mapsto 1, \sigma_\beta \mapsto (w_B x(\alpha)^2 x(\beta))^{-\beta^*}, \sigma_0 \mapsto 1.$$  

$\kappa$ is trivial on $d_\sigma$: $\sigma_\alpha \mapsto 1, \sigma_\beta \mapsto (\lambda^2)^{\beta^*}, \sigma_0 \mapsto 1$. So we may consider

$$(6.5) \quad c_{\sigma} t_\sigma d_\sigma = \sigma_\alpha \mapsto 1, \sigma_0 \mapsto 1, \sigma_\beta \mapsto \left(\frac{z(\alpha)^2}{w_B x(\beta)}\right)^{\beta^*} = \xi^{\beta^*}.$$  

$\kappa(t_\sigma) = \kappa(t_{\sigma}')$ depends only on $\ell$.

We recall the fact that $a(E_2) = 2$ implies that there is a term of the asymptotic expansion $F_1(\theta, 3)$ for every character $\theta$ in $\mathcal{F} \times$ of order 2.
PROPOSITION 6.6. $F_1(\theta, 3) = 0$ if $\theta$ is non-trivial.

PROOF: Let $T_0$ be a split Cartan subgroup. Then the action of $G$ on $Y_T$ gives an action of $T_0 \subseteq G$ on $Y_T$. To make this explicit select two Borel subgroups $B_0, B_\infty$ over $F$ with $B_0 \cap B_\infty = T_0$. Then the action of $t_0 \in T_0(\overline{F})$ is given on $Y_1(B_\infty, B_0, \Sigma)$ by

$$(b, B_0^{nw})^\nu \rightarrow (b, B_0^{nw})^{\nu t_0}.$$  

In particular it acts on the fibres $\varphi^{-1}(\lambda)$ in $Y_T$. Supposing $t_0 \in C_G(u_0)(\overline{F})$, it is not difficult to see that the action on points $E_2(u_0)$ above $u_0 \in G(F)$ in $E_2$ is given by

$$z \mapsto sz$$
$$\xi \mapsto s\xi$$
$$w \mapsto w$$
$$\ell \mapsto \ell$$

where $s = \beta(t_0)$. By (6.4) this morphism from $E_2(u_0)$ to $E_2(u_0)$ is defined over $F$ provided $s \in F^\times$. Call this morphism $\varphi_s$.

The morphism $\varphi_s$ is easily seen to carry the form

$$\omega_{E_2(u_0)} = \frac{dz}{z^2} \frac{d\tilde{w}}{\tilde{w}^2} dw \, d\xi$$

to itself.

The action of $\varphi_s$ on $m_{\theta, E}$ is also easily calculated. By (6.4) $\sigma(x(\gamma) x(\gamma)^{-1}$ for $\sigma \in Gal(\overline{F}/F)$ evaluated on $E_2$ is a function of $w, \ell$ and the tangent direction of $\Gamma$ but not of $\xi$ or $z$. By Hilbert’s 90th we write $x(\gamma) = x(\gamma)a$ with $a$ a function of $\xi, \ell$ and the tangent direction, and $\tilde{x}(\gamma)$ a variable over $F$. Then

$$\lambda = \frac{x(\gamma)^2 \tilde{w}z}{w^2 x(\delta)} = \frac{\tilde{x}(\gamma)^2 \tilde{w}z}{a^2 w^2 z(\delta)}$$

and $\theta(\lambda) = \theta(\frac{\tilde{w}z}{a^2 w^2 z(\delta)})$. Thus

$$m_{\theta, E} \overset{def}{=} \lim_{\lambda \rightarrow 0} \frac{\kappa(t_\sigma)}{\theta(\lambda)} = \frac{\kappa(t_\sigma)}{\theta(\frac{\tilde{w}z}{a^2 w^2 z(\delta)})}.$$ 

$t_\sigma$ depends only on $\ell$ so that $\varphi_s$ carries $m_{\theta, E}$ to

$$\frac{\kappa(t_\sigma)}{\theta(\frac{\tilde{w}z}{a^2 w^2 z(\delta)})} = \frac{m_{\theta, E}}{\theta(s)}.$$ 

241
A change of coordinates does not change a principal value integral. Changing coordinates on $E_2(u_0)$ by the automorphism $\varphi_s$ gives

$$F_1(\theta, 3) = \left[ \int_{E_2(u_0)} m_{\delta, E} |\omega_{E_2(u_0)}| \right] \mu^H(f) = \frac{1}{\theta(s)} \left[ \int m_{\delta, E} |\omega_{E_2(u_0)}| \right] \mu^H(f).$$

So that if $\theta(s) \neq 1$ for some $s \in F^\times$ we must have $F_1(\theta, 3) = 0$.

We turn to the Cartan subgroups associated to the endoscopic group $SO(4)$. We define a birational map from $E_2(u_0)$ to $(\mathbb{P}^1)^4$ by

$$w_2 = \frac{-T_1w_B + \ell_1}{T_2w_B} \quad \ell_2 = \frac{f_T \ell}{f_T \ell + 2} \quad w_B = (-2T_2w + 1) \quad \ell_1 \overset{def}{=} (1 + T_2(T_1 + T_2))$$

$$(6.7)$$

$$\xi_1 = \xi \quad f_T \overset{def}{=} (T_2^2 - T_1^2) \quad \xi_2 = \xi + \rho z \quad \rho \overset{def}{=} \frac{2T_2\ell_1}{-T_1w_B + \ell_1}$$

where $w_2, \ell_2, \xi_1, \xi_2$ are coordinates on $(\mathbb{A}^1)^4 \subseteq (\mathbb{P}^1)^4$. The map is birational, for it may be inverted by inverting the relations for $\ell_2, \xi_1, w_2$ and $\xi_2$ in that order. The form $\omega_{E_2(u_0)}$ in these coordinates is found to be

$$(6.8) \quad \frac{(T_2^2 - T_1^2)}{2} \frac{d\ell_2}{\ell_2^2} \frac{dw_2}{w_2} \frac{d\xi_1 d\xi_2}{(\xi_1 - \xi_2)^2} = \omega_{E_2(u_0)}$$

Transferring the action of $W \times A$ via (6.7) to the coordinates $w_2, \ell_2, \xi_1, \xi_2$ gives:

$$(6.9)$$

$$\begin{array}{cccccc}
1 & \sigma_\alpha & \sigma_\beta \sigma_\alpha \sigma_\beta & \sigma_0 \\
w_2 & w_2 & w_2/f^2 & 1/w_2 & (\xi_2/\xi_1)w_2 \\
\ell_2 & \ell_2 & -\ell_2 & -\ell_2 & \ell_2 \\
\xi_1 & \xi_1 & \xi_1 & \xi_2 & -1/\xi_1 \\
\xi_2 & \xi_2 & \xi_2 & \xi_1 & -1/\xi_2 \\
\end{array}$$

where $f = \left( \frac{1 + \ell_2}{1 - \ell_2} \right)$. $t'_\sigma$ is given by $\sigma_0 \to 1, \sigma_\alpha \to 1, \sigma_\beta \to \ell^{\sigma}$, $\ell^{\sigma} = \left( \frac{2\ell_2}{f_T} \frac{1}{1 - \ell_2} \right)^{\sigma}$. The factor $2\ell_2/f_T$ lies in $F^\times$ so that $\kappa(t_\sigma) = \kappa(t'_\sigma)$ where $t'_\sigma$ is defined by $\sigma_0 \to 1, \sigma_\alpha \to 1, \sigma_\beta \to (1 - \ell_2)^{-\sigma}$. This
implies that \( \sigma_\beta \sigma_\alpha \sigma_\beta \to ((1 + \ell_2)^{-1}, (1 - \ell_2)^{-1}, 1 - \ell_2, 1 + \ell_2) \) so that as in the proof of (5.23) \( \kappa(t_\sigma) = \eta_{E'}(1 - \ell_2^2) \) where \( E' = \text{Inv}(\sigma_0, \sigma_\alpha) \) the field invariant by \( \sigma_\alpha \) and \( \sigma_0 \).

We may always choose the homomorphism \( \text{Gal}(\overline{F}/F) \to A \) so that \( \xi_1, \xi_2 = 0, \infty \) are not rational points. Thus \( \xi_2/\xi_1 \neq 0, \infty \) and is well defined. Assume that such a choice is made. Also if \( \text{Im}(\text{Gal}(\overline{F}/F) \subseteq W) = \langle \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta \rangle \) then \( \ell_2 = \pm 1, f = 0, \infty \) are \( F \)-rational points so that the action of \( \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta \) on \( w_2 \) for \( f = 0, \infty \) is not defined. But this has an effect because the integrals are principal value integrals.

The following lemma is proved in section 7.

**Lemma 6.10.** The principal value integral \( \int_{E_2(u_0)} m_{K,E} |\omega_{E_2}(u_0)| \) is preserved under the birational map (6.7).

**Corollary 1.** The 2-regular germ is zero if \( \kappa \) is trivial and \( W_T \subseteq \langle 1, \sigma_\alpha, \sigma_\beta \sigma_\alpha \sigma_\beta \rangle \).

**Proof:** For \( s \in F^x \) the automorphism of \( E_2(u_0) \) sending \( w_2 \) to \( w_2 \), \( \ell_2 \) to \( s\ell_2 \), \( \xi_1 \) to \( \xi_1 \) and \( \xi_2 \) to \( \xi_2 \) is defined over \( F \). It takes the form \( \omega_{E_2}(u_0) \) to \( \omega_{E_2}(u_0)/s \) so that changing coordinates using this automorphism

\[
\int_{E_2(u_0)} |\omega_{E_2}(u_0)| = \frac{1}{|s|} \int_{E_2(u_0)} |\omega_{E_2}(u_0)| = 0 \quad \text{if } |s| \neq 1.
\]

**Corollary 2.** The 2-regular germs have the form \( |T_1^2 - T_2^2| c(T, \kappa) \) for some constants \( c(T, \kappa) \) depending on \( (T, \kappa) \).

**Proof:** The data (6.9) is independent of \( T_1, T_2 \) and \( |\omega_{E_2}(u_0)| = |T_1^2 - T_2^2| \) \( |\omega'| \) where \( \omega' \) is a form independent of \( T_1, T_2 \).

**Corollary 3.** The transfer of orbital integrals in (2.8) holds for a function \( f^H \) independent of \( (T, \kappa) \) associated to \( H = \text{SO}(4) \).

**Proof:** To prove the compatibility of functions \( f^H \) for various choices of \( (T, \kappa) \) associated to \( \text{SO}(4) \) we reason as follows. We find it convenient to pass to the adjoint group \( G/Z, Z = Z(G) \) in which \( T/Z \) becomes a product of two tori. Germs are not affected (2.5) but we must replace \( \text{SO}(4) \) by the product \( H = \text{PSL}(2) \times \text{PSL}(2) \). We write \( T/Z = S_1 \times S_2 \) where the tangent direction in \( S_1 \) is \( T_1 - T_2 \) and \( T_1 + T_2 \) in \( S_2 \). Fix a function \( f \in C_c^\infty(G) \) such that \( \mu_0^H(f) = 0 \) if \( O \) is not 2-regular. Using transfer factors \( \Delta_{H}^{T_1, \kappa} \) we fix normalized subregular germs \( A_{S_1}, B_{S_2} \) on \( H \). \( A_{S_1} \) are functions on \( S_1 \) of the form \( \lambda |T_1 - T_2| a_{S_1}, a_{S_1} \) a constant independent of \( S_2 \) (see [LS]). Similarly \( B_{S_2} = \lambda |T_1 + T_2| b_{S_2} \). By (2.9)

\[
\Delta_{G}^{T_1, \kappa} \Phi_{G}^{T_1, \kappa} = \lambda^2 |T_1^2 - T_2^2| c(T, \kappa) = \lambda |T_1 - T_2| a_{S_1} \mu_0^{H'}(f_{\kappa}).
\]

243
It was argued in (2.9) that \( f_{q_5} \) is independent of \( T_1 - T_2 \). The choice of \( f_{q_5} \) may also be made independently of \( S_1 \). To see this we use the explicit formula for \( f^M \) in \( [R] \) and note that the choice of \( f_{q_5} \) in the definition of QSR (2.10) may be chosen independently of \( S_1 \). Thus we conclude that \( \mu_N^f(f_{q_5}) = \lambda|T_1 + T_2|b'_S a'_{S_2} \) or \( c(T, \kappa) = a_{S_1} b_{S_2} \) for some constants \( a'_{S_1} \). Reversing the roles of \( S_1 \) and \( S_2 \) we have \( c(T, \kappa) = a'_{S_2} b_{S_1} \) for constants \( a'_{S_2} \). We conclude that \( c(T, \kappa) = \alpha_{1} a_{S_1} b_{S_2} \) for some constant \( \alpha_1 \). Thus the 2-regular germ is \( \alpha_{1} \lambda^2 |T_1^2 - T_2^2| a_{S_1} b_{S_2} = \alpha_{1} A_{S_1} B_{S_2} \) which up to scalar \( \alpha_1 \) is the 2-regular germ on \( H \).

§7 SPURIOUS DIVISORS AND SOME TECHNICAL DETAILS

This final section contains the remaining details of the proof of the transfer of orbital integrals on \( \text{GSp}(4) \). It is shown that the construction in the preceding sections satisfies the conditions of (1.1), none of the divisors outside of \( Y_U \) contributes to the germ expansion, any amount of blowing up along subvarieties of divisors outside of \( Y_U \) has no effect on the subregular germs, and the birational map \( E_2(u_0) \rightarrow (\mathbb{P}^1)^4 \) of (6.7) preserves principal value integrals.

To obtain coordinates outside \( Y_U \) we construct the variety \( Y_T \) from scratch. The variety of stars is defined on an open patch \( S(B_{\infty}, B_0) \) by

\[
\epsilon_{-\alpha}(s_4)\epsilon_{-\beta}(r_4)\epsilon_{-\alpha}(s_3)\epsilon_{-\beta}(r_3)\epsilon_{-\alpha}(s_2)\epsilon_{-\beta}(r_2)\epsilon_{-\alpha}(s_1)\epsilon_{-\beta}(r_1) = 1.
\]

where \( \epsilon_{-\alpha}(s) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ s & 1 & -s & 1 \end{pmatrix} \) and \( \epsilon_{-\beta}(r) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \end{pmatrix} \).

Multiplying out the relation (7.1) we obtain

\[
\begin{align*}
s_1 + s_2 + s_3 + s_4 &= 0 \\
r_1 + r_2 + r_3 + r_4 &= 0 \\
r_2s_1 + r_3(s_1 + s_2) + r_4(s_1 + s_2 + s_3) &= 0 \\
r_2s_1^2 + r_3(s_1 + s_2)^2 + r_4(s_1 + s_2 + s_3)^2 &= 0
\end{align*}
\]

There is a \( G_m \times G_m \) action on \( S(B_{\infty}, B_0) \) given by \((s_i, r_i) \mapsto (ss_i, rr_i)\). There is an action of the cyclic group of order four given by \((s_i, r_i) \mapsto (s_{i+1}, r_{i+1})\) with indices read modulo 4 and an action of an involution given by \((s_i, r_i) \mapsto (-s_{1-i}, -r_{2-i})\). They combine to give an action of the Weyl group on \( S(B_{\infty}, B_0) \) which commutes with the \( G_m \times G_m \) action.

We note for future reference that the equation (7.2) imply

\[
\begin{align*}
s_1 + s_2 + s_3 + s_4 &= 0 \\
r_1 + r_2 + r_3 + r_4 &= 0 \\
r_2s_1 + r_3(s_1 + s_2) + r_4(s_1 + s_2 + s_3) &= 0 \\
r_2s_1^2 + r_3(s_1 + s_2)^2 + r_4(s_1 + s_2 + s_3)^2 &= 0
\end{align*}
\]
There is a proper morphism $S_1 \xrightarrow{\varphi} S$ where $S_1(B_\infty, B_0) = \varphi^{-1}S(B_\infty, B_0)$ is covered by coordinate patches $U(m, n) \quad m, n \in \{1, 2, 3, 4\}$. The coordinates and relations on $U(m, n)$ are

\begin{align*}
r, s, R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4 & \quad \text{satisfying the relations} \\
R_m = S_n = 1 \\
S_1 + S_2 + S_3 + S_4 &= 0 \\
R_1 + R_2 + R_3 + R_4 &= 0 \\
R_2S_1 + R_3(S_1 + S_2) + R_4(S_1 + S_2 + S_3) &= 0 \\
R_2S_1^2 + R_3(S_1 + S_2)^2 + R_4(S_1 + S_2 + S_3)^2 &= 0
\end{align*}

The morphism $\varphi$ from $U(m, n)$ to $S(B_\infty, B_0)$ is given by $(S_i, R_j, s, r) \mapsto (sS_i, rR_j)$. It is proved in [H] that $\varphi$ is a proper map from $S_1$ to $S$.

**Lemma 7.5.** (a) $S_1$ is covered by patches isomorphic to $U(1, 1)$

(b) $S_1$ is nonsingular.

**Proof:** (a) The patches $S(B_\infty, B_0)$ as $(B_\infty, B_0)$ vary are all isomorphic. So we may fix $(B_\infty, B_0)$.

Let $Z = \{R_1, R_2, R_3, R_4, S_1, S_2, S_3, S_4\}$ and for $p \in U(m, n)$ set $Z_p = \{z \in Z \mid z(p) = 0\}$. The possibilities for $Z_p$ are calculated in [H]. Up to a symmetry of the Weyl group acting on the indices they are

\begin{equation}
a + b + c = 0, y \neq 0. \quad \text{If } a = 0 \text{ then } bc \neq 0.
\end{equation}
(7.7) \[ a + b + c = 0 \quad x \neq 0. \text{ If } b = 0 \text{ then } ac \neq 0. \]

(7.8) \[ x \neq 0. \text{ If } y = 0 \text{ then } z \neq 0. \]

(7.9) \[ y \neq 0. \text{ If } x = 0 \text{ then } z \neq 0. \]

where the variables are arranged around the squares as follows:

\[ \begin{array}{ccc}
S_2 & R_2 & S_1 \\
S_3 & R_3 & S_4 \\
R_1 & R_4 & R_1 \\
\end{array} \]

Notice that (7.3) now states that the product of variables along an edge is the negative of the product of variables along the opposite edge.
By examining (7.6)-(7.9) it is evident that there are always two adjacent variables on the square that are non-zero. By the action of the Weyl group, we may take the adjacent variables to be \( R_1, S_1 \). Such a point is contained in a patch isomorphic to \( U(1,1) \). This proves (a).

(b) By (a) it is enough to prove that \( U(1,1) \) is nonsingular. We consider two patches \( A \) and \( B \).

(A) Let \( x = S_4, y = S_3R_4 \). Then inverting the relations (7.2) we find

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= xy + y - 1 \\
S_3 &= -x - y - xy \\
S_4 &= x
\end{align*}
\]

(7.10)

We see that \( x \) and \( y \) are local coordinates, unless \((x,y) = (0,1) \) or \((0,0) \). (Unless the numerator and the denominator of \( R_2, R_3, \) or \( R_4 \) simultaneously vanish, \( x \) and \( y \) describe a point in \( S_1(B_\infty, B_0) \) although not necessarily a point in \( U(1,1) \).) By (7.10) it follows that

\[
t \overset{\text{def}}{=} \frac{y}{x} = \frac{S_3R_4}{S_4} = \frac{-R_4}{S_2R_3} = \frac{R_2}{R_3S_3S_4}.
\]

(7.11)

Near \((x,y) = (0,0) \) we may select coordinates \((x,t) \) or \((y,1/t) \) unless \( R_2 = R_4 = S_4 = 0, \ R_1 = S_1 = 1, \ S_2R_3 = 0 \). But the list of possible patterns \( Z_p \) in (7.6)-(7.9) shows that this never occurs. We conclude that patch (A) (together with the variants \((x,t) \) or \((y,1/t) \)) covers points such that \((x,y) \neq (0,1) \), i.e., \( S_4 = 0, S_3R_4 = 1 \).

(B) Let \( S_1 = R_1 = 1, \ a = S_2, \ b = R_3 \). Then inverting the relations (7.4) we find

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= a \\
S_3 &= (1 + a)/(ab - 1) \\
S_4 &= -ab(1 + a)/(ab - 1)
\end{align*}
\]

(7.12)

We see \( a \) and \( b \) are local coordinates unless \( ab = 1 \) and \( 1 + a = 0 \), i.e., \( a = -1, \ b = -1 \) (\( S_2 = -1, S_1 = 1, R_3 = -1 \)).

Points in the complement of both patches must satisfy \( S_4 = 0; R_1, S_1, S_2, R_3, S_3, R_4 \neq 0 \). This is impossible by the list of possible patterns (7.6)-(7.9).

Next we turn to a calculation of \( t^{-1}n^{-1}t \ n \in N \) for \( n \in N, t \in T \). Using the notation following (5.26) we let

\[
\begin{align*}
n &= m(n_\beta, n_\alpha, n_\gamma, n_\delta) \\
y &= m(y_\beta, y_\alpha, y_\gamma, y_\delta) \\
t &= \text{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \in T.
\end{align*}
\]
LEMMA 7.13.

\[ y_\alpha = (1 - t_2t_1^{-1})n_\alpha \]
\[ y_\beta = (1 - t_2^{-2})n_\beta \]
\[ y_\gamma = (1 - t_2^{-1}t_1^{-1})n_\gamma + (t_2^{-1}t_1^{-1} - t_2t_1^{-1})n_\alpha n_\beta \]
\[ y'_\gamma = (1 - t_2^{-1}t_1^{-1})n_\gamma + (t_2^{-2} - 1)n_\alpha n_\beta \]
\[ y_\delta = (1 - t_1^{-2})n_\delta + (t_2^{-1}t_1^{-1} - t_2t_1^{-1})n_\alpha n'_\gamma \]

provided \( y = t^{-1}n^{-1}t\ n \).

PROOF: Elementary matrix computation.

Identify \( B_0 \) with the group of upper triangular matrices and fix an ordering on the positive roots a choice of root vectors by the condition

\[ \prod_{\gamma'} \exp(x(\gamma')X_{\gamma'}) = m(x(\beta), x(\alpha), x(\gamma), x(\delta)). \]

By definition (3.12) \( w(\alpha) = w(\beta) = 1, w(\gamma) = \lambda y_\gamma/y_\alpha y_\beta \) and \( w(\delta) = \lambda^2 y_\delta/y_\alpha^2 y_\beta \) so that with \( \alpha = t_1t_2^{-1}, \beta = t_2^2, \gamma = t_1t_2, \delta = t_1^2 \) we have:

(7.14)

\[ w(\gamma) = \frac{(1 - \gamma^{-1})\lambda}{(1 - \alpha^{-1})(1 - \beta^{-1})} \left( \frac{n_\gamma}{n_\alpha n_\beta} \right) + \frac{(\gamma^{-1} - \alpha^{-1})\lambda}{(1 - \alpha^{-1})(1 - \beta^{-1})} \left( \frac{n'_\gamma}{n_\alpha n_\beta} \right) \]
\[ w(\delta) = \frac{\lambda^2(1 - \delta^{-1})}{(1 - \alpha^{-1})^2(1 - \beta^{-1})} \left( \frac{n_\delta}{n_\alpha^2 n_\beta} \right) + \frac{\lambda^2(\gamma^{-1} - \alpha^{-1})}{(1 - \alpha^{-1})^2(1 - \beta^{-1})} \left( \frac{n'_\gamma}{n_\alpha n_\beta} \right). \]

To complete the description of \( w(\gamma) \) and \( w(\delta) \) it is necessary to compute \( n_\gamma/(n_\alpha n_\beta) \) and \( n_\delta/(n_\alpha^2 n_\beta) \) in terms of coordinates on the star variety. The functions \( n_\alpha, n_\beta, n_\gamma \) and \( n_\delta \) are determined by the relation \( Bn_Wn^{-1} \in B\omega \) for all \( \omega \in W \) (the Weyl group) with \( W = W(\omega) \) (a Weyl chamber).

LEMMA 7.15. \( n'_\alpha/n_\alpha n_\beta = S_4/S_1, n_\gamma/(n_\alpha n_\beta) = (S_4/S_1) + 1, n_\delta/(n_\alpha^2 n_\beta) = (R_1S_4)/(R_4S_3), \)
\[ z(\alpha) = -s_4\lambda/(1 - \alpha^{-1}), z(\beta) = r_1\lambda/(1 - \beta^{-1}). \]

PROOF: The final two relations are found in both [L] and [H]. By \( n_Wn^{-1} \in B\omega \) we have (by writing out these matrices)

\[ n_\alpha = -1/s_4, n_\beta = 1/r_1, n'_\gamma = -n_\beta/s_1 = -1/r_1s_1, n_\gamma = n'_\gamma + n_\alpha n_\beta = -1/r_1s_1 - 1/r_1s_4, \]
\[ r_4s_4n_\delta = r_4s_4n_\gamma = -1/s_4 - r_4/r_1s_1 - r_4/r_1s_4 = -(r_1s_1 + s_4r_4 + r_4s_1)/r_1s_1s_4 = \]

248
\[-r_3s_2/(r_1s_4) = -r_3s_2/(-s_2r_3s_3) = 1/s_3, n_5 = 1/(s_3r_4s_4) = -1/(r_2s_1s_2).\] We have made use of (7.2) and (7.3). This completes the proof.

Now combining (7.14) and (7.15) with \( A = (1 - \alpha^{-1})/\lambda, \ B = (1 - \beta^{-1})/\lambda, \ C = (1 - \gamma^{-1})/\lambda, \ D = (1 - \delta^{-1})/\lambda \) we find

\[
\begin{align*}
\lambda w(\gamma) &= \frac{C}{AB} \left( \frac{s_4}{s_1} + 1 \right) + \frac{(A - C)}{AB} \\
&= \frac{C}{AB} \frac{s_4}{s_1} + \frac{1}{B} \\
w(\delta) &= \frac{D}{A^2B} \left( \frac{r_1s_4}{r_4s_3} \right) + \frac{(A - C)s_4}{(A^2B)s_1} \\
&= \frac{D}{A^2B} \frac{r_1s_4}{r_4s_3} + \frac{(A - C)s_4}{A^2B} \text{ for } \lambda = 1.
\end{align*}
\]

We are now in a position to identify all divisors and calculate their associated constants \( a(E), b(E), \beta(E) \). By (7.5) we may assume that \( R_1 = S_1 = 1 \).

Begin with the assumption that \( g = \text{def} \left( \frac{D}{A^2B} + \frac{A - C}{A^2B} R_4s_5 \right) \neq 0 \) at \( p \), a point on the divisor \( E \). We consider several patches. We begin by using (7.15), (7.16) to rewrite the equations

\[
\begin{align*}
\lambda w(\delta) &= z(\alpha)^2 z(\beta) z(\delta) \\
x(\alpha) &= \frac{\lambda}{z(\alpha)} \\
x(\beta) &= \frac{\lambda}{z(\beta)} \\
x(\gamma) &= \frac{\lambda w(\gamma)}{z(\alpha)z(\beta)}
\end{align*}
\]

in the form.

\[
\begin{align*}
\lambda &= \frac{R_4S_3S_4s_2r x(\delta)}{A^2Bg} \\
x(\alpha) &= -\frac{S_3r_4x(\delta)s}{ABg} \\
x(\beta) &= \frac{S_3R_4S_4s_2x(\delta)}{A^2g} \\
x(\gamma) &= -\frac{S_3r_4x(\delta)}{A^2g} \left( \frac{C}{AB} S_4 + \frac{1}{B} \right)
\end{align*}
\]
\[ \lambda = \frac{yzs^2rx(\delta)}{(A^2Bg)} \]

The form \( \omega = d\lambda \, dx(\alpha) \, dx(\beta) \, dx(\gamma) \, dx(\delta) \, dv \) in these coordinates becomes (we may treat \( A, B, C, D \) as constants as \( \lambda \to 0 \))

\[ \omega = d\lambda \, d\left( \frac{\lambda A}{-S_4^4} \right) \, d\left( \frac{\lambda B}{r} \right) \, d\left( \frac{\lambda \omega(\gamma)AB}{-S_4s} \right) \, dx(\delta) \, dv \]

\[ \omega = \frac{\lambda^3x(\delta)}{g^2} \frac{CD}{A^3B} \, ds \, dr \, dx(\delta) \, dv \]

We obtain the divisors on this patch

\( E_a \) defined by \( x = 0; \quad a(E_a) = 1, \quad \beta(E_a) = 3, \quad S_4 = R_2 = R_3 = 0, \)

\[ x(\beta) = 0, \quad x(\alpha) = \frac{-yrsx(\delta)}{ABg}, \]

\[ x(\gamma) = \frac{-yssx(\delta)}{A^2g} \left( \frac{C}{AB} S_4 + \frac{1}{B} \right) \]

\( E_b \) defined by \( y = 0; \quad a(E_b) = 1, \quad \beta(E_b) = 4, \quad R_2 = R_4 = 0, \quad x(\alpha) = x(\beta) = x(\gamma) = 0. \)

\( E_2 \) defined by \( s = 0; \quad a(E_2) = 2, \quad \beta(E_2) = 3, \quad x(\alpha) = x(\beta) = x(\gamma) = 0. \)

\( E_\alpha \) defined by \( r = 0; \quad a(E_\alpha) = 1, \quad \beta(E_\alpha) = 2, \quad x(\alpha) = 0, \quad \text{etc.} \)

\( E_{id} \) defined by \( x(\delta) = 0; \quad x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0. \)

Now we drop the assumption that \( (x, y) \neq (0, 0) \) and introduce the assumptions \( g \neq 0, \)

Patch A, coordinates \( t = y/x, \quad x = S_4, S_3R_4 = y \)

\[ (7.18) \quad \lambda = \frac{tx^2s^2rx(\delta)}{(A^2Bg)} \]

\[ (7.19) \quad \omega = \frac{\lambda^3x(\delta)CD}{A^3B} \, dt \, ds \, dr \, dx(\delta) \, dv \]

250
We have previously considered the divisors given by \( t = 0, \quad s = 0, \quad r = 0, \quad x(\delta) = 0 \). We obtain the new divisor given by \( x = 0 \).

\[ E_c \text{ given by } x = 0; \quad a(E_c) = 2, \quad \beta(E_c) = 7/2 \]

Now we drop our previous assumptions and introduce the assumptions \( g \neq 0 \), Patch A, coordinates \( u = x/y, y \quad (x = uy) \).

Equation (7.18) becomes

\[ \lambda = \frac{uy^2 s^2 rx(\delta)}{A^2 Bg}. \]

It is easy to see that all of these divisors meet a patch previously considered.

Again we drop our previous assumptions and assume that we are on patch B with coordinates \( a, b \) and that \( g \neq 0 \). We have by (7.10) and (7.12)

\[ x = \frac{-ab(1 + a)}{ab - 1}, \quad y = \frac{(1 + a)(1 - ab)}{1 + ba^2}. \]

Since \( a = S_2 \) and \( b = R_3 \). If \( ab = 1 \) then \( S_2R_3 = 1 \). But \( 1 = S_2R_3 = \frac{x}{xy + y + x} \) so that \((x, y) \neq (0, 1)\). Likewise if \( 1 + ba^2 = 0 \) then \( S_2^2R_3 = -1 \). But \(-1 = S_2^2R_3 = \frac{x(xy + y - 1)}{xy + y + x} \) so that again \((x, y) \neq (0, 1)\). Since we have already investigated the divisors for \( g \neq 0 \) \((x, y) \neq (0, 1)\) we may assume \( ab - 1 \neq 0 \) and \( 1 + ba^2 \neq 0 \). (7.18) and (7.19) become

\[ \lambda = \frac{ab(1 + a)^2 s^2 rx(\delta)}{(1 + ba^2)(A^2 Bg)} \]

\[ \omega = \lambda^3 x(\delta) \left( \frac{1 - ab}{ab(1 + a)} \right) \left( \frac{1 + a)(1 - ab)}{1 + ba^2} \right) \frac{ds}{s} \frac{dr}{r^2} \frac{d(x(\delta))}{dx(\delta)}. \]

The three divisors \( s = 0, \quad r = 0, \) and \( x(\delta) = 0 \) have already been considered. If \( 1 + a = 0 \) then \( R_2 = S_3 = S_4 = 0 \). Thus \((x, y) = (0, 0)\) and we find that \( 1 + a = 0 \) defines the divisor \( E_c \) considered above. If \( b = 0 \) then \( S_4 = R_3 = R_2 = 0 \). \((x, y) = (0, -1 - a)\) and we find that
\( b = 0 \) defines the divisor \( E_a \). Finally if \( a = 0 \) then \( S_2 = S_4 = 0 \) and \((x, y) = (0, 1)\). Call this divisor \( E_d \). We have \( a(E_d) = 1 \). Moreover, writing \( (1 + a)(1 - ab)/(1 + ba^2) = am_1 + 1, \)
\( b(1 + a)/(1 - ab) = m_2 \) we have

\[
d \left[ \frac{(1 + a)(1 - ab)}{1 + ba^2} \right] d \left[ \frac{ab(1 + a)}{1 - ab} \right] = d(am_1) d(am_2) = \frac{\partial(am_1, am_2)}{\partial(a, b)} da db.
\]

Clearly \( a^{-1} \frac{\partial(am_1, am_2)}{\partial(a, b)} \) is regular at the generic point of \( E_d \). Consequently \( \beta(E_d) > 3 \). At this point we drop the assumption that we are on patch B with \( g \neq 0 \) with coordinates \( a, b \).

If \( g = 0 \) at \( p \in E \) then by the definition of \( g \) we have \( S_3 R_4 \neq 0 \). Also the assumption that the tangent direction is regular implies that \( S_3 R_4 \neq 1 \), or that \((x, y) \neq (0, 1), (0, 0)\). Thus we are on Patch A. Also the assumption that \( E \) does not meet \( Y \) of (3.7) together with \( S_3 R_4 \neq 0, S_3 R_4 \neq 1, S_1 = R_1 = 1 \) implies that \( S_4 = 0 \) at \( p \). In fact, using (7.3) twice we have \( 0 = S_1 S_2 S_3 S_4 R_1 R_2 R_3 R_4 = -S_1 S_2 S_4 R_1 R_2 R_3 R_4 = (S_1 R_1 S_3 R_3) S_3 S_4 \) hence \( S_4 = 0 \) \((S_1, R_1, S_3, R_4 \neq 0)\).

Since \( w(\gamma) = \frac{C}{AB} S_4 + \frac{1}{B} \) we have that \( w(\gamma) \neq 0 \) at \( p \). The relations (3.7)

\[
\lambda = \frac{z(\alpha)z(\beta)z(\gamma)}{w(\gamma)}, \quad x(\alpha) = \frac{z(\beta)z(\gamma)}{w(\gamma)}, \quad x(\beta) = \frac{z(\alpha)z(\gamma)}{w(\gamma)}, \quad x(\gamma)w(\delta) = w(\gamma)z(\alpha)x(\delta)
\]

become using (7.15) and (7.16)

\[
\lambda = -\frac{xsx(\gamma)}{wAB}, \quad x(\alpha) = \frac{rsx(\gamma)}{wB}, \quad x(\beta) = -\frac{xsx(\gamma)}{wA}, \quad x(\gamma)g = \frac{-ywsx(\delta)}{A}.
\]

If \( x(\gamma) \neq 0 \), then this last equation, together with the definition of \( g \), yields

\[
\frac{1}{y} = \frac{BA^2}{D} \left[ -\frac{wsx(\delta)}{Ax(\gamma)} - \left( \frac{(A - C)}{A^2 B} \right) \right].
\]

This is non-zero on patch A so that \( y \) is given by the reciprocal of the right hand side of this equation. If \( x(\delta) \neq 0 \), then we obtain

\[
s = \frac{-Ax(\gamma)g}{yw(\delta)}.
\]

If \( x(\gamma) = x(\delta) = 0 \) then \( x(\alpha) = x(\beta) = x(\gamma) = x(\delta) = 0 \) and we obtain the divisor \( E_{id} \) which is irrelevant for our study of subregular and 2-regular germs. The only possible new divisor on this patch is that given by \( x = 0 \). But \( x = 0 \) implies \( x(\beta) = 0 \), and by (7.6)-(7.9), \( R_2 = R_3 = S_4 = 0 \).

We recognize this as the divisor \( E_a \).

We summarize the results in the first part of the following proposition.
PROPOSITION 7.20. Let \( V \) be the open subvariety of elements in \( Y_1(B_{\infty}) \) which do not lie over the identity element. Then

(a) \( V \) is non-singular,
(b) divisors have normal crossings in \( V \),
(c) divisors which do not meet \( Y_U \cap V \) make no contribution to the germ expansion of \( \kappa \)-orbital integrals for \( \kappa \) nontrivial,
(d) If \( E \) is a divisor which does not meet \( Y_U \), \( \beta(E) \leq 3 \), and \( T \) is an elliptic Cartan subgroup then \( E \) has no \( F \)-rational points.

PROOF: (a) and (b) have been verified on each patch.
(d) Up to conjugacy by the Weyl group the only divisors not meeting \( Y_U \) are:

\[
\begin{align*}
E_a & \quad \beta(E_a) - 1 = 2 \\
E_b & \quad \beta(E_b) - 1 = 3 \\
E_c & \quad \beta(E_c) - 1 = 5/2 \\
E_d & \quad \beta(E_d) - 1 > 2.
\end{align*}
\]

The condition that \( \beta(E) \leq 3 \) forces \( E = E_a \) on which \( S_4 = R_2 = R_3 = 0 \). The corresponding pattern is (7.7). Let \( p \) be an \( F \)-rational point of a divisor under the action defined in (3.12). It is clear by (3.12) that if \( z_1(W, \alpha_1) = 0 \) at \( p \) for some \( W \) and simple root \( \alpha_1 \) then \( z_1(\sigma T^{-1}W, \alpha_1) = 0 \) at \( p \). In other words the set \( Z_p \) of (7.5) is invariant under the Weyl group elements \( \sigma T \in W_T \). The symmetries of pattern (7.7) fix a chamber wall, hence the elements \( \sigma T \) of \( W_T \) do as well. This chamber wall corresponds to a proper parabolic subgroup over \( F \) and \( W_T \) may be identified with a subgroup of the Weyl group of \( P \). It follows that \( T \) is stably conjugate to a Cartan subgroup of \( P \) and hence that \( T \) is not elliptic.

(c) The germ expansion, excluding the germ of the identity element, has terms of homogeneity 0, 1, and 2. A divisor \( E \) contributes a term of homogeneity \( \beta(E) - 1 \) (1.1). The only possibility then is \( E_a \) and homogeneity \( \beta(E_a) - 1 = 2 \). But we have seen that if \( E_a \) has \( F \)-rational points \( T \) is not elliptic. If \( T \) is not elliptic, the germ expansion reduces to an expansion on a Levi factor for which the terms have homogeneity 0 and 1 but not 2. The proposition follows.

PROPOSITION 7.21. The morphism defined by (6.7) preserves principal value integrals.

PROOF: First we remark that the morphism extends to \( E_2(u_0) \to \mathbb{P}^1 \times \mathbb{P}^1 (\xi_1, \xi_2) \) with \( \xi_1, \xi_2 \) defined by (6.7). In fact, this morphism has a simple geometric interpretation. Let \( (b, (B(W))) = (b_0, B_0)^w \). By the discussion following (6.1) \( n_W v = \exp(z_W X_\beta) \) for appropriate choices of \( z_W \) for points of \( E_2(u_0) \). \( \xi_1 \) and \( \xi_2 \) have been chosen so that \( \xi_1 = z_{W_+} \) and \( \xi_2 = z_{W_-} \) where
$W_-$ is the Weyl chamber opposite $W_+$. Thus $E_2(u_0) \to \mathbb{P}^1 \times \mathbb{P}^1 (\xi_1, \xi_2)$ may be identified with the morphism $E_2(u_0) \to \mathbb{P}^1 \times \mathbb{P}^1 = B_0 \setminus P_\beta \times B_0 \setminus P_\beta, (b,(B(W))) \to (B(W_+),B(W_-))$. From the definition of $\xi(= \xi_1)$ following (6.1) it is clear that $\xi_1 = z_{W_+}$. But $\xi_2 = \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta (\xi_1) = \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta (z_{W_+}) = z_{W_-}$ by (6.9) and the fact that $\sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta (B(W_+)) = B(W_-)$.

Next we prove that the birational map $E_2(u_0) \to \mathbb{P}^1 (\ell_2)$ extends to $Y_1(B_\infty, B_0) \cap E_2(u_0)$. Up to a linear fractional transformation (6.7), (7.16) $\ell_2$ is equal to $(S_1 R_1)/(R_4 S_3)$. By (7.3)

$$\frac{S_1 R_1}{R_4 S_3} = \frac{R_3 S_3}{R_2 S_1} = \frac{-R_1 S_4}{R_2 S_2} = \frac{-R_3 S_2}{R_4 S_4}.$$ 

The morphism does not extend only if all the numerators and denominators in (7.22) simultaneously vanish. But if $S_i \neq 0$ then $R_i, R_{i+1} = 0$ which implies that $R_{i+2}, R_{i+3} \neq 0$ so that $S_{i+1} = S_{i+2} = S_{i+3} = 0$ contradicting $S_1 + S_2 + S_3 + S_4 = 0$. Thus the map extends.

Next we check that the birational map $E_2(u_0) \to \mathbb{P}^1 (w_B)$ extends to $Y_1(B_\infty) \cap E_2(u_0)$. Up to a scalar $w_B$ equals (7.16), (7.3) $S_4/S_1 = -R_2 S_2/(R_4 S_3)$. By the patterns of (7.6)-(7.9) it is not possible for $S_4 = S_1 = R_2 S_2 = R_4 S_3 = 0$ so that the map extends.

By considering the zero patterns of (7.6)-(7.9) and assuming that $T$ is elliptic ($T$ is elliptic when the 2-regular germ is non-zero) we may assume that we find ourselves at an $F$-rational point so that one of the following holds:

(i) $S_i \neq 0 \quad R_i \neq 0 \quad \forall i$
(ii) $S_1 = S_3 = 0 \quad S_i, R_j \neq 0$ otherwise
(iii) $S_2 = S_4 = 0 \quad S_i, R_j \neq 0$ otherwise
(iv) $R_1 = R_3 = 0 \quad S_i, R_j \neq 0$ otherwise
(v) $R_2 = R_4 = 0 \quad S_i, R_j \neq 0$ otherwise

Next we check that the birational map $E_2(u_0) \to \mathbb{P}^1 (w_2)$ given by (6.7) extends to $E_2(u_0) \cap Y_1(B_\infty)$. Up to a linear fractional transformation it is enough to consider $\ell_1/w_B$ or even

$$q \overset{\text{def}}{=} \left(1 - \frac{s_1 r_1}{r_4 s_3}\right)/\left(\frac{s_4}{s_1}\right).$$

On the patch (B) of (7.12) $q$ is calculated to be $q = \frac{1 - b - 2ab}{b(1+a)^2}$. On patch $B \quad (a,b) \neq (-1,-1)$ so that this gives a well-defined point in $\mathbb{P}^1$. If $w_B = 0$, then $S_4 = 0$ so that we fall into case (iii). But in case (iii) $R_1 = S_1 = 1, S_2 \neq -1$ so that points of case (iii) lie in patch $B$. This case has already been treated. If $w_B = \infty$ then $S_1 = 0$ so that we fall into case (ii). But in case (ii)

$$\left(1 - \frac{s_1 r_1}{r_4 s_3}\right) = \left(1 + \frac{R_1 S_4}{R_2 S_2}\right)$$
is finite, so that the map to \( \mathbf{P}^1 \) extends to such points. Finally, if \( w_B \neq 0, \infty \) it is easy to see that the map extends.

We consider coordinate transformations from one chart to another. We compare coordinates for the pair \( (B_\infty, B_0) \) with those for the pair \( (B'_\infty, B_0) = (B'^n_\infty, B_0) \) with \( n \in N_0 \). We have

\[
(b, B'^n_\infty) = (b''n, B^m_0) v''n
\]

where \( n_w, v \in N_\infty, n'^n_w, v''n \in B^n_\infty, b'^n_w, v''n \in B_0 \). Write \( v^{-1} = b''v'' \). Then \( b^n = b'' \). Write \( b_n = s_y, s = diag(s_1, s_2, s_2^{-1}, s_1^{-1}) \ y = m(y_\beta, y_\alpha, y_\gamma, y_\delta) \). Then

\[
\frac{x(\alpha)''}{x(\alpha)} = s_2 s_1^{-1}, \quad \frac{x(\beta)''}{x(\beta)} = \frac{2T_2z y_\beta s_2^2 + 1}{s_2^2}, \quad \frac{x(\gamma)''}{x(\gamma)} = \frac{s_2 y_\beta z + w}{s_1 s_2 w}, \quad \frac{x(\delta)''}{x(\delta)} = s_2^{-2}
\]

\[
w''_B = \frac{w_B}{(2T_2 z s_2^2 y_1 + 1)}, \quad \tilde{w}'' = \frac{\tilde{w}}{(2T_2 z s_2^2 y_1 + 1)}, \quad \ell'' = \ell
\]

where doubly primed objects are the corresponding variables on the new coordinate patch. Also

\[
z'' = \frac{s_2^2 z}{2T_2 z s_2^2 y_1 + 1}.
\]

We remark that this is the most general coordinate transformation that need be considered. For let \( B_\infty, B'_\infty \) be any two Borel subgroups. If there are any points of \( E_2(u_0) \) on \( Y_1(B_\infty), Y_1(B'_\infty) \), say \( (p, (B(W))) \) and \( (p', (B(W))) \), then \( B(W_+) \in \ell_\beta, B(W_+) \in \ell_\beta \) the line of type \( \beta \) in the variety of Borel subgroups contain \( u_0 \). Since the set of Borels opposite \( B_\infty \) is open and \( B(W_+) \in \ell_\beta \) and open sets of the Borels in \( \ell_\beta \) are opposite \( B_\infty \). Likewise for \( B'_\infty \). Thus there is a \( B_0 \in \ell_\beta \) opposite both \( B_\infty \) and \( B'_\infty \), so that \( B'^n_\infty = B^n_\infty \) for \( n \in N_0 \subseteq B_0 \).

We have

\[
b_n v'' = v'' = \exp(\xi X_{-\beta}) \exp(-u X_\beta) \mod \ N_\beta
\]

\[
= \left( \begin{array}{cc} 1/(1 - \xi u) & -u \\ 0 & 1 - \xi u \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \xi/(1 - \xi u) & 1 \end{array} \right) \mod \ N_\beta
\]

So that \( s_1 = 1, s_2 = 1/(1 - \xi u), v'' = \exp(\xi'' X_{-\beta}), \xi'' = \xi/(1 - \xi u), s_2 y_\beta = -u, \)

\[
\frac{x(\alpha)''}{x(\alpha)} = \frac{1}{(1 - \xi u)}, \quad \frac{x(\beta)''}{x(\beta)} = (1 - \xi u)(1 - \xi u - 2T_2 z u), \quad \frac{x(\gamma)''}{x(\gamma)} = -uz + (1 - \xi u)w, \quad \frac{x(\delta)''}{x(\delta)} = 1.
\]

\[
w''_B = \frac{(1 - \xi u) w_B}{(-2T_2 z u + 1 - \xi u)}, \quad \tilde{w}'' = \frac{(1 - \xi u) \tilde{w}}{(1 - \xi u - 2T_2 z u)} \ell'' = \ell, \quad \ell''_1 = \ell_1, \quad \ell''_2 = \ell_2.
\]
\[ z'' = \frac{z}{(1 - \xi u - 2T^2u)(1 - \xi u)^3}, \quad \frac{\ell''_1}{w_B''} = \frac{(1 - (\xi + 2T_2)u)}{(1 - \xi u)} \cdot \frac{\ell_1}{w_B}. \]

Since \( \ell'_2 = \ell_2 \) the birational map \( E_2(u_0) \to \mathbb{P}^1(\ell_2) \) extends. Selecting \( u \) so that \( (1 - \xi u) \neq 0 \) \( (1 - (\xi + 2T_2u) \neq 0) \) the birational map \( E_2(u_0) \to \mathbb{P}^1(\ell_1/w_B) \) extends so that \( E_2(u_0) \to \mathbb{P}^1(w_2) \) does as well.

**BIBLIOGRAPHY**


