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INTERTWINING OPERATORS AND UNITARY REPRESENTATIONS II

M. W. Baldoni-Silva* and A. W. Knapp**

For a linear semisimple Lie group G with a compact Cartan subgroup, the paper [5] derived explicit formulas for the action of intertwining operators on standard induced representations. These formulas had been announced earlier ([1], [2], [4]) and had been used in combination with some known results to classify irreducible unitary representations in certain situations (arbitrary such representations for $SU(N,2)$, as well as most Langlands quotients obtained from maximal parabolic subgroups for general G). The present paper gives the derivation of the remaining previously-announced formulas, handling the case of standard induced representations attached to a maximal cuspidal parabolic subgroup when G has no compact Cartan subgroup. These formulas were announced in [3] and [4] and were used in classifying irreducible unitary representations in further situations (groups of real rank two with restricted roots of type A_2, as well as other Langlands quotients obtained from maximal parabolic subgroups for general G).

The background for the formulas is as follows: The Langlands classification describes the irreducible "admissible" representations as the unique irreducible quotients ("Langlands quotients") of standard induced representations. The irreducible unitary representations in turn are those Langlands quotients that admit invariant Hermitian inner products. It is known when there exists an invariant Hermitian form, and the question is one of deciding positivity of the form. The form is unique up to scalars, if it exists, and it lifts to the standard induced representation.

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Relative to the (non-invariant) L^2 inner product on the standard induced representation, the form is given by an explicit integral (or singular-integral) intertwining operator. The question is whether the intertwining operator is semidefinite.

Rather than try to evaluate the integral operator, we follow a strategy that was used extensively by Klimyk, often in collaboration with Gavrilik, for particular groups (see, e.g., [9]). The strategy is to take advantage of the intertwining property to relate the operator on one subspace to the operator on another subspace. Indeed it turns out in principle to be possible to compute the form globally just by computing the L^2 inner product of an arbitrary iterated representation-image of one particular function with itself.

In the present paper we consider the case in which the standard induced representation comes from a maximal cuspidal parabolic subgroup of G. If this subgroup is all of G, then the standard representation is a discrete series or limit of discrete series and hence is unitary. Thus we may assume G has no compact Cartan subgroup. We shall derive two formulas, one for the effect of a single step within the Lie algebra and one for the single step followed by a step back to the start. In two applications we give special cases that correspond to two previously announced results (Lemma 14.3 of [4] and Proposition 4.1 of [3]).

Contents. 1. Occurrence of K types in a tensor product. 2. Representations to be studied. 3. Necessary conditions for unitarity. 4. General formula. 5. Application to $\mathfrak{so}$(odd,odd). 6. Application to certain groups of real rank two.

1. Occurrence of K types in a tensor product

Let G be a linear connected reductive Lie group, and let K be a maximal compact subgroup. We denote Lie algebras by corresponding lower case German letters, and we write $\mathbb{C}$ as a superscript to indicate complexifications. Let θ be a Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as the corresponding Cartan decomposition. We fix on $\mathfrak{g}$ a nondegenerate symmetric bilinear form B_0 invariant under θ such that ad $\mathfrak{g}$ acts by skew transformations, B_0 is negative on $\mathfrak{k} \times \mathfrak{k}$, B_0 is positive on $\mathfrak{p} \times \mathfrak{p}$, and $B_0(\mathfrak{k}, \mathfrak{p}) = 0$. We extend B_0 to $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ so as to be complex bilinear.

Throughout this paper we assume that $\mathfrak{g}$ has no simple factor of type G_2 (real or complex). We fix a maximal torus B in K and let $\mathfrak{t}$ be the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$. Then $\mathfrak{t}$ is a maximally compact θ-stable Cartan subalgebra of $\mathfrak{g}$ (see p. 129 of [12]), and we can write $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$ with $\mathfrak{a} \subseteq \mathfrak{p}$. Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. The form B_0 induces an inner product $\langle \cdot, \cdot \rangle$ on the set of linear functionals on $\mathfrak{t}^{\mathbb{C}}$ that are real-valued on $i\mathfrak{b} \oplus \mathfrak{a}$, and we write $\mu' \perp \mu''$ if $\langle \mu', \mu'' \rangle = 0$.

In [5] we worked with the special case $\mathfrak{t} = \mathfrak{b}$. Thus for now our results will generalize those in [5]. Starting in §2, we shall introduce further assumptions that make the situation in this paper disjoint from the one in [5].

If β is a root, we often write $\beta = \beta_R + \beta_I$ with $\beta_R = \beta|_{\mathfrak{a}}$ and $\beta_I = \beta|_{\mathfrak{b}}$. No root has $\beta_I = 0$ since $\mathfrak{t}$ is maximally compact. (See Proposition 11.16a of [12].) Let

$$\Delta_B = \{\beta \in \Delta \mid \beta|_{\mathfrak{a}} = 0\} .$$

The root vectors for the members of Δ_B lie either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$, and we call the corresponding roots <u>compact</u> or <u>noncompact</u>, respectively. Let

$$\Delta_{B,c} = \{\text{compact roots in } \Delta_B\}$$

$$\Delta_{B,n} = \{\text{noncompact roots in } \Delta_B\} .$$

We use a bar to denote the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. If $\beta = \beta_R + \beta_I$ is a root, then we are led naturally to roots $\overline{\beta}$ and $\theta\beta$, and these are given by $\overline{\beta} = \beta_R - \beta_I$ and $\theta\beta = -\beta_R + \beta_I$. Hence $\theta\overline{\beta} = -\beta$.

Using [7, pp. 155-156], we can select root vectors X_β for β in Δ in such a way that

$$B_0(X_\beta, X_{-\beta}) = 2/|\beta|^2 \tag{1.1a}$$

and

$$\theta \overline{X}_\beta = -X_{-\beta} . \tag{1.1b}$$

For a real-valued linear functional μ on $i\mathfrak{b} \oplus \mathfrak{a}$, we let H_μ be the member of $i\mathfrak{b} \oplus \mathfrak{a}$ such that $\mu(H) = B_0(H_\mu, H)$ for all H. (<u>Warning</u>: This normalization is different from the one in [5].) Then it follows from (1.1) that

$$[X_\beta, X_{-\beta}] = 2|\beta|^{-2}H_\beta \tag{1.2}$$

for $\beta \in \Delta$.

Proposition 1.1. The selection of root vectors as in (1.1) can be done in such a way that $\theta X_\beta = X_{\theta\beta}$ for all $\beta \in \Delta - \Delta_B$.

Remark. We assume henceforth that the selection is made in this way.

Proof. Under the assumption $\beta \notin \Delta_B$, β is neither real nor imaginary. Hence $\{\beta, -\beta, \theta\beta, -\theta\beta\}$ is a set of four distinct roots. First let us make a selection of $X_\beta, X_{-\beta}, X_{\theta\beta}, X_{-\theta\beta}$ so that (1.1) holds. Now we shall normalize this choice. Let us write

$$\theta X_\beta = aX_{\theta\beta} \quad \text{and} \quad \theta X_{-\beta} = bX_{-\theta\beta}.$$

Then (1.1a) implies

$$\begin{aligned} 2ab|\beta|^{-2} &= abB_0(X_{\theta\beta}, X_{-\theta\beta}) = B_0(\theta X_\beta, \theta X_{-\beta}) \\ &= B_0(X_\beta, X_{-\beta}) = 2/|\beta|^2, \end{aligned}$$

and $ab = 1$. Thus we have

$$\theta X_\beta = aX_{\theta\beta} \quad \text{and} \quad \theta X_{-\beta} = a^{-1}X_{-\theta\beta},$$

and application of θ gives

$$\theta X_{\theta\beta} = a^{-1}X_\beta \quad \text{and} \quad \theta X_{-\theta\beta} = aX_{-\beta}.$$

Using (1.1b), we obtain

$$-X_{-\beta} = \theta\overline{X}_\beta = \overline{aX}_{\theta\beta} = -\bar{a}\theta X_{-\theta\beta} = -\bar{a}aX_{-\beta},$$

and $a\bar{a} = 1$. Let us leave X_β and $X_{-\beta}$ unchanged, and let us redefine

$$X_{\theta\beta} = \bar{a}\theta X_\beta \quad \text{and} \quad X_{-\theta\beta} = a\theta X_{-\beta}.$$

Then (1.1) is unaffected for β, and $a\bar{a} = 1$ implies (1.1a) holds for the new $X_{\theta\beta}$ and $X_{-\theta\beta}$. Moreover (1.1b) holds for $X_{\theta\beta}$ and $X_{-\theta\beta}$ since

$$\theta\overline{X}_{\theta\beta} = \theta(a\theta\overline{X}_\beta) = a\overline{X}_\beta = -a\theta X_{-\beta} = -X_{-\theta\beta}$$

and

$$\theta \overline{X}_{-\theta\beta} = \theta(\overline{a}\theta \overline{X}_{-\beta}) = \overline{aX}_{-\beta} = -\overline{a}\theta X_\beta = -X_{\theta\beta} .$$

This completes the proof.

Let us fix some lexicographic ordering for $\mathfrak{a}$. If β is in $\Delta - \Delta_B$, we can form the four roots $\beta, -\beta, \theta\beta, -\theta\beta$ of the preceding proof. Two of these will have positive equal restrictions to $\mathfrak{a}$, and the other two will have negative equal restrictions. The relevant pairs are $(\beta, -\theta\beta)$ and $(-\beta, \theta\beta)$. From the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{b}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}} \oplus \sum_{\beta\in\Delta-\Delta_B} \mathbb{C}X_\beta \oplus \sum_{\beta\in\Delta_B} \mathbb{C}X_\beta ,$$

we thus obtain

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{b}^{\mathbb{C}} \oplus \sum_{\substack{\beta\in\Delta-\Delta_B \\ \beta|_{\mathfrak{a}} > 0}} \mathbb{C}(X_\beta + X_{\theta\beta}) \oplus \sum_{\beta\in\Delta_{B,c}} \mathbb{C}X_\beta \tag{1.3a}$$

$$\mathfrak{p}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \sum_{\substack{\beta\in\Delta-\Delta_B \\ \beta|_{\mathfrak{a}} > 0}} \mathbb{C}(X_\beta - X_{\theta\beta}) \oplus \sum_{\beta\in\Delta_{B,n}} \mathbb{C}X_\beta . \tag{1.3b}$$

Let

$$\Delta_K = \{\beta_I \mid \beta \in \Delta - \Delta_{B,n}\}$$

$$\Delta_n = \{\beta_I \mid \beta \in \Delta - \Delta_{B,c}\} .$$

From (1.3a) it follows that we can identify Δ_K with the root system $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{b}^{\mathbb{C}}$. From (1.3b) we can identify Δ_n as the set of nonzero weights for the action of $Ad(K)$ on $\mathfrak{p}^{\mathbb{C}}$; moreover each of these weights has multiplicity one. Note that $\Delta_K \cap \Delta_n \neq \emptyset$ as soon as Δ contains complex roots.

Lemma 1.2. *If β is in Δ, then $|\beta_I|^2 = c|\beta|^2$ with $c = \frac{1}{4}, \frac{1}{2}$, or 1.*

Proof. Calculation gives

$$\frac{2\langle\beta, \overline{\beta}\rangle}{|\beta|^2} = 2 - 4c . \tag{1.4}$$

The left side of (1.4) is an integer from -2 to $+2$, since

$|\beta| = |\bar{\beta}|$. The fact that there are no real roots implies (1.4) is neither $+2$ nor -1: If it were $+2$, β would be a real root, while if it were -1, $\beta + \bar{\beta}$ would be a real root. Hence $2 - 4c$ is $+1$, 0, or -2, and the result follows.

Lemma 1.3. *If γ_I is in Δ_K and β_I is in Δ_n, then either $\beta_I = \pm 2\gamma_I$ with $\beta_I \in \Delta_{B,n}$, or else*

$$2\langle \beta_I, \gamma_I \rangle / |\gamma_I|^2 \tag{1.5}$$

is an integer from -2 to $+2$.

Proof. It is an integer since β_I is a weight of $\mathfrak{p}^{\mathbb{C}}$ under the action of $\mathrm{Ad}(K)$. Choose β and γ in Δ with $\beta = \beta_R + \beta_I$ and $\gamma = \gamma_R + \gamma_I$. By Lemma 1.2, $|\gamma_I|^2 = c|\gamma|^2$ with $c = \frac{1}{4}$, $\frac{1}{2}$, or 1.

If $c = 1$, then $\gamma = \gamma_I$, and (1.5) equals $2\langle \beta, \gamma \rangle / |\gamma|^2$, which is between -2 and $+2$. (Recall we are excluding G_2 from our considerations.)

If $c = \frac{1}{2}$, then (1.5) is

$$= \frac{2\langle \beta, \gamma \rangle}{|\gamma|^2} - \frac{2\langle \beta, \bar{\gamma} \rangle}{|\gamma|^2}, \tag{1.6}$$

and this can be greater than 2 in absolute value only if γ is short and β is long. In this case both terms of (1.6) are even, and the two terms must reinforce each other. Without loss of generality, suppose

$$\frac{2\langle \beta, \gamma \rangle}{|\gamma|^2} = - \frac{2\langle \beta, \bar{\gamma} \rangle}{|\gamma|^2} = +2 .$$

Then $\beta - \gamma$ is a short root, and $\gamma \perp \bar{\gamma}$ implies

$$\frac{2\langle \beta - \gamma, \bar{\gamma} \rangle}{|\gamma|^2} = -2 .$$

Since $\beta - \gamma$ and $\bar{\gamma}$ are both short, we conclude $\beta - \gamma = -\bar{\gamma}$. Thus $\beta = \gamma - \bar{\gamma}$. Since the difference of the orthogonal roots γ and $\bar{\gamma}$ is a root, so is the sum. But $\gamma + \bar{\gamma}$ is real, and we have a contradiction.

If $c = \frac{1}{4}$, then $2\gamma_I$ is a root, and (1.5) equals

$$2 \cdot \frac{2\langle \beta, 2\gamma_I \rangle}{|2\gamma_I|^2},$$

which is even. If this integer is greater than 2 in absolute value, one possibility is that $\beta = \pm 2\gamma_I$. Then $\beta_I = \beta$ is in Δ_B, hence is in $\Delta_{B,n}$. The other possibility is that β is long and $2\gamma_I$ is short (and hence γ is short). Lemma 1.2 gives

$$8|\gamma_I|^2 = 2|2\gamma_I|^2 = |\beta|^2 = (4 \text{ or } 2 \text{ or } 1)|\beta_I|^2,$$

so that

$$\frac{2|\beta_I|}{|\gamma_I|} = (2\sqrt{2} \text{ or } 4 \text{ or } 4\sqrt{2}).$$

Thus the Schwarz inequality gives

$$\left| \frac{2\langle \beta_I, \gamma_I \rangle}{|\gamma_I|^2} \right| \leq \frac{2|\beta_I|}{|\gamma_I|} = (2\sqrt{2} \text{ or } 4 \text{ or } 4\sqrt{2}). \tag{1.7}$$

If 4 is attained in (1.7) with $|\beta|^2 = 2|\beta_I|^2$, then the equality in the Schwarz inequality forces $\beta_I = d\gamma_I$ for some d. Since (1.5) is ± 4, d is ± 2. So $\beta_I = \pm 2\gamma_I$. Since $2\gamma_I$ is a root, it follows that $\beta_R = \beta - \beta_I$ is a root, in contradiction to the fact that there are no real roots. We conclude that 4 is attained in (1.7) with $|\beta|^2 = |\beta_I|^2$, i.e., with β imaginary. Then we have

$$\pm 4 = \frac{2\langle \beta_I, \gamma_I \rangle}{|\gamma_I|^2} = \frac{2\langle \beta, \gamma \rangle}{\frac{1}{4}|\gamma|^2} = 4 \cdot \frac{2\langle \beta, \gamma \rangle}{|\gamma|^2},$$

and $|2\langle \beta, \gamma \rangle / |\gamma|^2| = 1$. This equality forces $|\beta| \leq |\gamma|$, contradiction.

Lemma 1.4. If γ_I is in Δ_K and β_I is in Δ_n with $2\langle \beta_I, \gamma_I \rangle / |\gamma_I|^2 = -2$, then $\beta_I = -\gamma_I$ or $|\beta_I|^2 = 2|\beta_I + \gamma_I|^2 = 2|\gamma_I|^2$.

Proof. Suppose $\beta_I \neq -\gamma_I$, and let $\eta_I = \beta_I + \gamma_I$. This is a nonzero weight of $\mathfrak{p}^{\mathbb{C}}$ and hence is in Δ_n. Let β, γ, and η be extensions of β_I, γ_I, and η_I to members of Δ, not necessarily consistently. By Lemma 1.2,

$$|\beta_I|^2 = 2^a|\beta|^2, \quad |\gamma_I|^2 = 2^b|\gamma|^2, \quad |\eta_I|^2 = 2^c|\eta|^2$$

for integers a, b, and c. The norms squared of any two roots in Δ are in the ratio of 2, 1, or $\frac{1}{2}$, and thus $|\beta_I|^2 = 2^d|\gamma_I|^2$ and $|\eta_I|^2 = 2^e|\gamma_I|^2$. Expansion of $|\beta_I+\gamma_I|^2$ gives

$$\frac{|\eta_I|^2}{|\gamma_I|^2} = \frac{|\beta_I|^2}{|\gamma_I|^2} + \frac{2\langle\beta_I,\gamma_I\rangle}{|\gamma_I|^2} + 1 = \frac{|\beta_I|^2}{|\gamma_I|^2} - 1 .$$

Hence $2^e = 2^d - 1$, and the only possibility is that $d=1$ and $e=0$.

Starting in §2, we shall work with a specific choice of Δ^+, the set of positive roots within Δ, and we shall let Δ_K^+ be the set of restrictions of $\Delta^+ \cap [(\Delta - \Delta_B) \cup \Delta_{B,c}]$. But for now let us suppose that Δ_K^+ is any positive system for Δ_K. If W_K denotes the Weyl group of Δ_K and if μ' is a linear functional on $\mathfrak{b}^{\mathbb{C}}$ that is real-valued on $i\mathfrak{b}$, then there exists $w \in W_K$ such that $w\mu'$ is Δ_K^+ dominant, and we write $(\mu')^{\vee}$ for the dominant form.

We say that a linear form μ' on $\mathfrak{b}^{\mathbb{C}}$ is <u>integral</u> if $\exp \mu'$ is well defined on B. If μ' is integral, then $2\langle\mu',\gamma_I\rangle/|\gamma_I|^2$ is an integer for every $\gamma_I \in \Delta_K$. (Recall the argument: If $\gamma \in \Delta - \Delta_{B,n}$ restricts to γ_I, then γ gives us a copy of $\mathfrak{su}(2)$ within $\mathfrak{g}$. Since $SU(2)$ is simply connected, it maps into G, and our assertion follows from known properties of $SU(2)$.)

Because G is linear, μ' integral implies $2\langle\mu',\beta\rangle/|\beta|^2$ is an integer for every $\beta \in \Delta_{B,n}$. This assertion follows because the isomorphism

$$\mathfrak{sl}(2,\mathbb{R}) \cong (\mathbb{C}H_\beta + \mathbb{C}X_\beta + \mathbb{C}X_{-\beta}) \cap \mathfrak{g}$$

and the linearity of G give a homomorphism of $SL(2,\mathbb{R})$ into G.

If Λ' is integral and is Δ_K^+ dominant, we let $\tau_{\Lambda'}$ be an irreducible representation of K with highest weight Λ'. We shall regard $\mathfrak{p}^{\mathbb{C}}$ as a representation of K under $\mathrm{Ad}(K)$; we have observed that the nonzero weights of $\mathfrak{p}^{\mathbb{C}}$ are the members of Δ_n, each with multiplicity one. The weight 0 has multiplicity equal to $\dim \mathfrak{a}$.

<u>Proposition 1.5. Let Λ' be integral and Δ_K^+ dominant. Then</u>
(a) <u>every irreducible constituent of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ has highest weight of the form $\Lambda' + \beta$ with β in $\Delta_n \cup \{0\}$.</u>

(b) <u>every irreducible constituent of</u> $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ <u>other than</u> $\tau_{\Lambda'}$ <u>has multiplicity one</u>.

This follows from the description of the weights of $\mathfrak{p}^{\mathbb{C}}$ and from Problems 13 and 14 on p. 111 of [12]. We sharpen this result in Theorem 1.7 below.

If Λ' is dominant integral, we let $\Delta_{K,\Lambda'}$ be the subset of roots in Δ_K orthogonal to Λ'. This is a root system, and the simple roots of $\Delta^+_{K,\Lambda'} = \Delta_{K,\Lambda'} \cap \Delta^+_K$ are simple in Δ^+_K since Λ' is dominant. Let $W_{K,\Lambda'}$ be the Weyl group of $\Delta_{K,\Lambda'}$; this is the subgroup of W_K fixing Λ', by Chevalley's Lemma (p. 81 of [12]).

To simplify the notation, we shall drop the subscripts "I" from members of Δ_K and Δ_n for the remainder of this section.

<u>Lemma 1.6</u>. <u>Let</u> Λ' <u>be integral and</u> Δ^+_K <u>dominant, and let</u> β <u>be in</u> Δ_n. <u>Then</u> $(\Lambda'+\beta)^\vee$ <u>is of the form</u> $\Lambda'+\beta'$ <u>with</u> β' <u>in</u> $\Delta_n \cup \{0\}$, <u>and</u> β' <u>is obtained constructively as follows</u>: <u>Let</u> β_1 <u>be the result of making</u> β <u>dominant for</u> $\Delta^+_{K,\Lambda'}$ <u>(by means of</u> $W_{K,\Lambda'}$<u>)</u>. <u>Then exactly one of the following things happens</u>:

(a) $\Lambda'+\beta_1$ <u>is</u> Δ^+_K <u>dominant, and</u> $\beta'=\beta_1$.

(b) <u>There exists a</u> Δ^+_K <u>simple root</u> γ <u>with</u> $2\langle\Lambda',\gamma\rangle/|\gamma|^2=+1$, $2\langle\beta_1,\gamma\rangle/|\gamma|^2=-2$. <u>In this case</u> $\beta_2=\beta_1+\gamma$ <u>is in</u> $\Delta_n \cup \{0\}$. <u>Either</u> β' <u>is the result of making</u> β_2 <u>dominant for</u> $\Delta^+_{K,\Lambda'}$ <u>(by means of</u> $W_{K,\Lambda'}$<u>)</u>, <u>or</u> $\beta'=0$.

<u>Proof</u>. Lemma 1.2 of [5] handles the special case $\alpha=0$. In the general case, we argue as in that lemma, fixing the proof as necessary. If (a) fails, we are led to a Δ^+_K simple root γ as in (b) or else to a Δ^+_K simple root γ with

$$\frac{2\langle\beta_1,\gamma\rangle}{|\gamma|^2} < -2 \quad \text{and} \quad \frac{2\langle\Lambda'+\beta_1,\gamma\rangle}{|\gamma|^2} < 0 .$$

In the latter case, Lemma 1.3 says $\beta_1=-2\gamma$ and β_1 is in $\Delta_{B,n}$. <u>Since G is linear</u>, $2\langle\Lambda',\beta_1\rangle/|\beta_1|^2$ is an integer. Thus $2\langle\Lambda',\gamma\rangle/|\gamma|^2 = +2$, and we have

$$s_\gamma(\Lambda'+\beta_1) = (\Lambda'-2\gamma)-\beta_1 = \Lambda' .$$

Hence $\beta'=0$. (In any event $\beta_1+\gamma$ is in $\Delta_n \cup \{0\}$ by consideration of the γ weight string.)

So we may assume γ is as in (b). As in [5], $\Lambda'+\beta$ is conjugate via W_K to $\Lambda' + \beta_2$, where $\beta_2 = \beta_1 + \gamma$. If $\beta_2 = 0$, then $\beta' = 0$. Otherwise Lemma 1.4 gives

$$|\beta_1|^2 = 2|\beta_2|^2 = 2|\gamma|^2 . \tag{1.8a}$$

Let β_3 be the result of making β_2 dominant for $\Delta^+_{K,\Lambda'}$, say $\beta_2 = w\beta_3$ with $w \in W_{K,\Lambda'}$. We shall show that $\beta' = 0$ or $\beta' = \beta_3$. Notice that

$$|\beta_3|^2 = |\beta_2|^2 . \tag{1.8b}$$

If $\Lambda' + \beta_3$ is not Δ^+_K dominant, then we can repeat the argument in the first paragraph of the proof to find γ' simple for Δ^+_K with either

$$\frac{2\langle \Lambda', \gamma' \rangle}{|\gamma'|^2} = +2 \quad \text{and} \quad \beta_3 = -2\gamma' \in \Delta_{B,n}$$

or else

$$\frac{2\langle \Lambda', \gamma' \rangle}{|\gamma'|^2} = +1 \quad \text{and} \quad \frac{2\langle \beta_3, \gamma' \rangle}{|\gamma'|^2} = -2 .$$

In the first case, $s_{\gamma'}(\Lambda' + \beta_3) = \Lambda'$, so that $\beta' = 0$ and we are done. In the second case, $\beta_4 = \beta_3 + \gamma'$ is such that $\Lambda' + \beta_2$ is conjugate to $\Lambda' + \beta_4$. If $\beta_4 = 0$, then $\beta' = 0$ and we are done. Otherwise Lemma 1.4 gives

$$|\beta_3|^2 = 2|\beta_4|^2 = 2|\gamma'|^2 . \tag{1.8c}$$

From the equation $\beta_1 = \beta_2 - \gamma$, we have

$$\begin{aligned} \frac{2\langle \beta_1, w\gamma' \rangle}{|\gamma'|^2} &= \frac{2\langle \beta_2, w\gamma' \rangle}{|\gamma'|^2} - \frac{2\langle \gamma, w\gamma' \rangle}{|\gamma'|^2} \\ &= \frac{2\langle \beta_3, \gamma' \rangle}{|\gamma'|^2} - \frac{2\langle \gamma, w\gamma' \rangle}{|\gamma'|^2} = -2 - \frac{2\langle \gamma, w\gamma' \rangle}{|\gamma'|^2} . \end{aligned} \tag{1.9}$$

Lemma 1.3 thus implies

$$\beta_1 = -2w\gamma' \in \Delta_{B,n} \tag{1.10a}$$

or

$$\langle \gamma, w\gamma' \rangle \leq 0 . \tag{1.10b}$$

If (1.10a) holds, then the linearity of G forces $2\langle\Lambda',\beta_1\rangle/|\beta_1|^2$ to be an integer. This integer is

$$-\frac{2\langle\Lambda',2w\gamma'\rangle}{4|\gamma'|^2} = -\tfrac{1}{2}\frac{2\langle\Lambda',\gamma'\rangle}{|\gamma'|^2} = -\tfrac{1}{2},$$

contradiction. So (1.10b) holds.

In (1.10b), first suppose $\langle\gamma,w\gamma'\rangle = 0$. Then (1.9) is -2, and Lemma 1.4 says that either $\beta_1 = -w\gamma'$ or $|\beta_1|^2 = 2|\gamma'|^2$; in either case, (1.8) gives a contradiction.

We conclude that $\langle\gamma,w\gamma'\rangle < 0$, so that $\gamma + w\gamma'$ is in Δ_K. From (1.8) we have $|\gamma|^2 = 2|\gamma'|^2$, so that $|\gamma + w\gamma'|^2 = |\gamma'|^2$ and $2\langle\gamma,w\gamma'\rangle/|\gamma'|^2 = -2$. Substituting into (1.9), we obtain

$$\frac{2\langle\beta_1,\gamma+w\gamma'\rangle}{|\gamma+w\gamma'|^2} = \frac{2\langle\beta_1,\gamma\rangle}{\tfrac{1}{2}|\gamma|^2} + \frac{2\langle\beta_1,w\gamma'\rangle}{|\gamma'|^2} = -4+0 = -4,$$

and Lemma 1.3 gives $\beta_1 = -2(\gamma+w\gamma') \in \Delta_B$. Since G is linear, $2\langle\Lambda',\beta_1\rangle/|\beta_1|^2$ is an integer. This integer is

$$\frac{-2\langle\Lambda',2\gamma+2w\gamma'\rangle}{2|\gamma|^2} = -\frac{2\langle\Lambda',\gamma\rangle}{|\gamma|^2} - \tfrac{1}{2}\frac{2\langle\Lambda',w\gamma'\rangle}{|\gamma'|^2} = \frac{3}{2},$$

contradiction. This completes the proof of the lemma.

We come to the main theorem of this section, which goes in the direction of identifying the irreducible constituents of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. Our result will not handle every case, but we state it in enough generality so that it includes both the situation $\mathfrak{a} = 0$ and the cases that are needed for our applications in this paper.

Theorem 1.7. Suppose that the length squared of any two members of $\Delta_K \cup \Delta_n$ stand in the ratio $\tfrac{1}{2}$, 1, or 2. Let Λ' be integral and Δ_K^+ dominant, let β be in Δ_n, and suppose $\Lambda' + \beta$ is Δ_K^+ dominant. Then $\tau_{\Lambda'+\beta}$ fails to occur in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ if and only if there exists a Δ_K^+ simple root γ such that γ is in $\Delta_{K,\Lambda'}^+$, $\gamma \perp \beta$, and both $\gamma + \beta$ and $\gamma - \beta$ are in $\Delta_n \cup \{0\}$.

Proof. It is a routine exercise to take the proof of Theorem 1.3 of [5], which handles the case $\mathfrak{a} = 0$, and adapt it to the situation here. The formula that replaces (1.1) in [5] is

$$\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}} = (\dim \mathfrak{a})\tau_{\Lambda'} + \sum_{\beta' \in \Delta_n} \operatorname{sgn}(\Lambda'+\beta'+\delta_K)\tau_{(\Lambda'+\beta'+\delta_K)^\vee - \delta_K} ,$$

where δ_K is half the sum of the members of Δ_K^+.

Theorem 1.8. *Let μ' be integral, let $\Lambda' = (\mu')^\vee$, let β be in Δ_n, suppose $\tau_{(\mu'+\beta)^\vee}$ occurs in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$, and suppose $(\mu'+\beta)^\vee \neq \Lambda'$. Let $E = E_{(\mu'+\beta)^\vee}$ be the projection of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ on the $\tau_{(\mu'+\beta)^\vee}$ subspace (along the subspaces for the other K types). If v' is a nonzero weight vector for $\tau_{\Lambda'}$ with weight μ', then $E_{(\mu'+\beta)^\vee}(v' \otimes X_\beta)$ is nonzero.*

Remark. This theorem generalizes Theorem 1.5 of [5], which handles the case $\mathfrak{a} = 0$.

Proof. For much of the proof, we shall assume that $\mu' = \Lambda'$, i.e., that μ' is Δ_K^+ dominant. First suppose $\Lambda' + \beta$ is Δ_K^+ dominant. Then we can trace through the first part of the proof of Theorem 1.5 of [5], adapting the notation to allow for the 0 weight space in $\mathfrak{p}^{\mathbb{C}}$ to be nonzero, and see that $E(v' \otimes X_\beta) \neq 0$. The next case to consider is that $\Lambda' + s\beta$ is Δ_K^+ dominant for some s in $W_{K,\Lambda'}$, and the argument for Theorem 1.5 of [5] handles this case as well.

Next we consider general β. Choose s in $W_{K,\Lambda'}$ such that $s\beta$ is $\Delta_{K,\Lambda'}^+$ dominant. Since $(\Lambda'+\beta)^\vee \neq \Lambda'$ by assumption, the previous paragraph and Lemma 1.6 show that there is a Δ_K^+ simple root γ with $2\langle \Lambda', \gamma\rangle/|\gamma|^2 = +1$ and $2\langle s\beta, \gamma\rangle/|\gamma|^2 = -2$. Put $\beta_2 = s\beta + \gamma$. Then Lemma 1.6 shows that $\Lambda' + s'\beta_2$ is Δ_K^+ dominant for some s' in $W_{K,\Lambda'}$. By the result of the previous paragraph, $E(v' \otimes X_{\beta_2}) \neq 0$. Since v' is a highest weight vector for $\tau_{\Lambda'}$, we have

$$\begin{aligned}\tau_{(\Lambda'+\beta)^\vee}(X_\gamma)E(v' \otimes X_{s\beta}) &= E(\tau_{\Lambda'}(X_\gamma)v' \otimes X_{s\beta} + v' \otimes \operatorname{ad}(X_\gamma)X_{s\beta}) \\ &= E(v' \otimes \operatorname{ad}(X_\gamma)X_{s\beta}) .\end{aligned}$$

The right side is a nonzero multiple of $E(v' \otimes X_{s\beta+\gamma})$, which we have just seen is nonzero. Therefore $E(v' \otimes X_{s\beta})$ on the left side is nonzero. Applying s^{-1}, we see that $E(v' \otimes X_\beta)$ is nonzero.

Finally in the general case in which μ' is not necessarily Δ_K^+ dominant, we introduce a new positive system for Δ_K so that μ' is dominant, and then the theorem reduces to the case that has already been proved.

2. Representations to be studied

Our objective, as noted in the introduction, is to obtain explicit formulas for the effect of standard intertwining operators on certain K types of induced representations. In this section we introduce the representations to be studied, note how to compute their minimal K types, and establish some identities for half sums of roots.

We continue with the notation of §1. In particular, $\mathfrak{g}$ has a maximally compact Cartan subalgebra $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$, Δ is $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, and Δ_K and Δ_n are certain sets of linear functionals on $\mathfrak{b}^{\mathbb{C}}$.

We selected root vectors X_β in (1.1) and Proposition 1.1 with a certain normalization and found in (1.2) that $[X_\beta, X_{-\beta}] = 2|\beta|^{-2}H_\beta$. For β in $\Delta - \Delta_B$, put

$$Y_\beta = X_\beta - X_{\theta\beta}.$$

As in §1, we fix a lexicographic ordering on the linear functionals on $\mathfrak{a}$. Then we can use $\mathfrak{a}$ to form a parabolic subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{b}^{\mathbb{C}} \oplus \sum_{\beta \in \Delta_B} \mathbb{C}X_\beta$$

$$\mathfrak{n}^{\mathbb{C}} = \sum_{\substack{\beta \in \Delta - \Delta_B \\ \beta|_{\mathfrak{a}} > 0}} \mathbb{C}X_\beta .$$

This parabolic subalgebra is maximal cuspidal. We have an Iwasawa-like direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus (\mathfrak{m} \cap \mathfrak{p})^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}, \tag{2.1}$$

and we let $P_{\mathfrak{k}}$, $P_{\mathfrak{m}}$, and $P_{\mathfrak{a}}$ be the respective projections on the first three factors. These projections can be read off from the formulas

$$X_\beta = \begin{cases} 0 + 0 + 0 + X_\beta & \text{if } \beta|_{\mathfrak{a}} > 0 \\ (X_\beta + X_{\theta\beta}) + 0 + 0 - X_{\theta\beta} & \text{if } \beta|_{\mathfrak{a}} < 0 \\ X_\beta + 0 + 0 + 0 & \text{if } \beta \in \Delta_{B,c} \\ 0 + X_\beta + 0 + 0 & \text{if } \beta \in \Delta_{B,n} . \end{cases} \tag{2.2}$$

The Hermitian form

$$\langle X,Y\rangle = -B_0(X,\theta\overline{Y}) \tag{2.3}$$

is a positive definite inner product on $\mathfrak{g}^{\mathbb{C}}$ that is invariant under $\mathrm{Ad}(K)$. If (τ,V) is any finite-dimensional representation of K and if $\langle\cdot,\cdot\rangle$ is a positive definite K-invariant inner product on V, then $\tau(X)^* = -\tau(X)$ for X in $\mathfrak{k}$, and it follows that

$$\tau(X)^* = -\tau(\overline{X}) = -\tau(\theta\overline{X}) \quad \text{for } X \in \mathfrak{k}^{\mathbb{C}}.$$

From this identity and (1.1) we readily find that

$$\begin{aligned} \tau(H_\beta)^* &= \tau(H_{\theta\beta}) && \text{for } \beta \in \Delta \\ \tau(X_\beta)^* &= \tau(X_{-\beta}) && \text{for } \beta \in \Delta_{B,c} \\ \tau([Y_\beta,Y_{\beta'}])^* &= -\tau([Y_{-\beta},Y_{-\beta'}]) && \text{for } \beta \in \Delta-\Delta_B,\ \beta' \in \Delta-\Delta_B. \end{aligned} \tag{2.4}$$

Also (1.1) allows us to compute the norms of X_β and Y_β relative to (2.3) as

$$\begin{aligned} |X_\beta|^2 &= 2/|\beta|^2 && \text{for } \beta \in \Delta \\ |Y_\beta|^2 &= 4/|\beta|^2 && \text{for } \beta \in \Delta-\Delta_B. \end{aligned} \tag{2.5}$$

In view of (1.3b), an orthogonal basis of $\mathfrak{p}^{\mathbb{C}}$ consists of

$$\{Y_\beta \mid \beta \in \Delta-\Delta_B,\ \beta|_{\mathfrak{a}} > 0\} \cup \{X_\beta \mid \beta \in \Delta_{B,n}\} \cup \{\text{orthogonal basis of } \mathfrak{a}^{\mathbb{C}}\}. \tag{2.6}$$

From (2.2) we read off

$$\begin{aligned} P_{\mathfrak{k}} Y_\beta &= -(X_\beta + X_{\theta\beta}) \\ P_{\mathfrak{m}} Y_\beta &= 0 \\ P_{\mathfrak{a}} Y_\beta &= 0 \end{aligned} \tag{2.7}$$

for $\beta \in \Delta-\Delta_B$ with $\beta|_{\mathfrak{a}} > 0$. Also

$$\begin{aligned} P_{\mathfrak{k}} X_\beta &= 0 \\ P_{\mathfrak{m}} X_\beta &= X_\beta \\ P_{\mathfrak{a}} X_\beta &= 0 \end{aligned} \tag{2.8}$$

for $\beta \in \Delta_{B,n}$. We shall make use of the formula

$$[X_{-\beta}+X_{-\theta\beta},Y_\beta] = -4|\beta|^{-2}P_{\mathfrak{a}}H_\beta \quad \text{for } \beta \in \Delta-\Delta_B, \tag{2.9}$$

which is verified by direct calculation.

Starting in §4, we shall use vector field notation for differentiation of functions on G, letting

$$Xf(g) = \frac{d}{dt} f((\exp tX)^{-1}g)\Big|_{t=0}$$

if X is in $\mathfrak{g}$. If X and Y are in $\mathfrak{g}$ and if $Z = X + iY$, we let $Zf = Xf + iYf$. Then $Z\bar{f} = \overline{\bar{Z}f}$.

The representations that we study will be those in the "fundamental series" of G. (Here is where we specialize our situation so that we are no longer generalizing [5].) Namely we study certain representations induced from the parabolic subgroup MAN that corresponds to $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let ρ be half the sum, with multiplicities counted, of the roots of $(\mathfrak{g}, \mathfrak{a})$ that are positive relative to N.

We fix a discrete series or limit of discrete series representation σ of M. (In §5, we assume that σ is nondegenerate in the sense of [14]. In §6, M will be compact, and nondegeneracy will be automatic.) Let $M^{\#} = M_0 Z_M$, the product of the identity component and the center. By (12.82) and Proposition 12.32 of [12], σ is induced from a discrete series or limit $\sigma^{\#}$ of $M^{\#}$ acting in a Hilbert space $V^{\sigma^{\#}}$.

Now Lemma 12.30a of [12] shows that $M^{\#} = M_0$, since there are no real roots. Thus $\sigma^{\#}$ is determined by its Harish-Chandra parameter (λ_0, Δ_B^+).

Let λ be the minimal $(K \cap M^{\#})$ type of $\sigma^{\#}$ given on $\mathfrak{b}$ by

$$\lambda = \lambda_0 - \delta_{B,c} + \delta_{B,n},$$

where $\delta_{B,c}$ and $\delta_{B,n}$ are the respective half sums of the members of $\Delta_B^+ \cap \Delta_{B,c}$ and $\Delta_B^+ \cap \Delta_{B,n}$. Following the procedure of [11], we introduce a positive system Δ^+ containing Δ_B^+ and built from a lexicographic ordering in which $\mathfrak{b}$ comes before $\mathfrak{a}$. The subset

$$\Delta_K^+ = \{\beta\big|_{\mathfrak{b}} \quad \text{with } \beta \in \Delta^+, \beta \notin \Delta_{B,n}\}$$

is then a positive system for Δ_K. We let δ and δ_K be the half sums of the members of Δ^+ and Δ_K^+, respectively.

We shall study the family of induced representations

$$U(\nu) = U(MAN, \sigma, \nu) = \mathrm{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1), \tag{2.10}$$

where ν is a complex-valued linear functional on $\mathfrak{a}$ and the induction is normalized so that imaginary ν yields unitary $U(\nu)$. We regard the induced representation as acting on functions by the left regular representation.

By [11], $U(\nu)$ has a unique minimal K type given simply by

$$\Lambda = \lambda = \lambda_0 + \delta - 2\delta_K . \tag{2.11}$$

It is clear that $\Lambda\big|_{\mathfrak{b}} = \lambda$. The first part of the proof of the minimal K type formula that appears on pp. 629-631 of [12] shows that a highest weight vector for τ_Λ in $U(\nu)$ is highest of type τ_λ for $K \cap M_0 = K \cap M^{\#}$. The argument shows also that τ_Λ has multiplicity one in $U(\nu)$. This fact was shown originally by Vogan [17].

Theorem 2.1. Let μ' be an integral form on $\mathfrak{b}$, and define

$$\mathrm{I} = H_\rho - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle > 0}} P_{\mathfrak{a}} H_\beta , \qquad \mathrm{II} = H_\rho - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle \geq 0}} P_{\mathfrak{a}} H_\beta .$$

Then

(a) $\mathrm{I} + \mathrm{II} = 0$

(b) $\mu' = \Lambda'$ dominant for Δ_K^+ implies

$$\mathrm{I} = \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\Lambda',\beta\rangle = 0}} P_{\mathfrak{a}} H_\beta .$$

Proof of (a).

$$\begin{aligned}\mathrm{I} + \mathrm{II} &= H_{2\rho} - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle > 0}} P_{\mathfrak{a}} H_\beta - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle \geq 0}} P_{\mathfrak{a}} H_\beta \\ &= \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0}} P_{\mathfrak{a}} H_\beta - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle < 0}} P_{\mathfrak{a}} H_\beta - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\mu',\beta\rangle \geq 0}} P_{\mathfrak{a}} H_\beta = 0 .\end{aligned}$$

The middle equality holds because $P_{\mathfrak{a}} H_{-\theta\beta} = P_{\mathfrak{a}} H_\beta$.

Proof of (b). If $\beta|_{\mathfrak{a}} > 0$, then $\beta|_{\mathfrak{b}}$ is in Δ_K. If also $\langle \Lambda', \beta\rangle \neq 0$, then the sign of $\langle \Lambda', \beta\rangle$ determines whether β is in Δ^+ or $-\Delta^+$, since Λ' is Δ_K^+ dominant. Thus

$$I = \frac{1}{2} \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0}} P_{\mathfrak{a}} H_\beta - \sum_{\substack{\beta\in\Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle \Lambda', \beta\rangle > 0}} P_{\mathfrak{a}} H_\beta$$

$$= \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{a}} > 0}} P_{\mathfrak{a}} H_\beta - \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{a}} > 0 \\ \langle \Lambda', \beta\rangle \neq 0}} P_{\mathfrak{a}} H_\beta = \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{a}} > 0 \\ \langle \Lambda', \beta\rangle = 0}} P_{\mathfrak{a}} H_\beta$$

as required.

3. Necessary conditions for unitarity

Continuing with notation as in §2, we recall the techniques of [1] and [2] for proving nonunitarity. (Those papers assumed rank G = rank K, and we have to modify the techniques slightly in our current situation.) Fix an element w in K normalizing A such that w^2 centralizes A, and assume throughout that $w\sigma \cong \sigma$. Then we can deduce from [13] that there exists a unique family of intertwining operators $T(\nu)$ with the following properties:

(1) $T(\nu)$ is defined for ν's in the -1 eigenspace of $\mathrm{Ad}(w)$ such that $\mathrm{Re}\,\nu$ is in a suitable neighborhood of the closed positive Weyl chamber of the dual $\mathfrak{a}'$ of $\mathfrak{a}$.

(2) For each Λ', $T(\nu)$ carries the $\tau_{\Lambda'}$ K type for $U(\nu)$ into the $\tau_{\Lambda'}$ K type for $U(-\nu)$, varies holomorphically in ν, and satisfies

$$U(-\nu,X)T(\nu) = T(\nu)U(\nu,X)$$

for all X in $\mathfrak{g}^{\mathbb{C}}$.

(3) $T(\nu)$ is the identity on the minimal K type τ_Λ.

For $\mathrm{Re}\,\nu$ in the closed positive Weyl chamber (under our hypotheses), $U(\nu)$ has a unique irreducible quotient $J(\nu)$, and $J(\nu)$ contains the K type τ_Λ with multiplicity one. If ν is real-valued, then $J(\nu)$ admits an invariant Hermitian form, unique up to a real scalar; this form lifts to $U(\nu)$, where it is given by

$$\langle f,g\rangle = (T(\nu)f,g)_{L^2(K)} . \tag{3.1}$$

Since the normalization (3) makes $T(\nu)$ positive definite on the K type τ_Λ, (3.1) shows that $J(\nu)$ will fail to be infinitesimally unitary for some real ν lying in the -1 eigenspace of $\mathrm{Ad}(w)$ within the closed positive Weyl chamber if we can produce a K type $\tau_{\Lambda'}$ such that $T(\nu)$ fails to be positive semidefinite on that K type.

The papers [1] and [2] introduce two techniques for finding such a Λ'. Both use the following definitions. If τ_{Λ_1} is an irreducible representation of K, we let P_{Λ_1} be the projection of the induced space to the τ_{Λ_1} subspace given by

$$P_{\Lambda_1}f(k_0) = d_{\Lambda_1}\int_K \overline{\chi_{\Lambda_1}(k)}f(k^{-1}k_0)\,dk . \tag{3.2}$$

Here d_{Λ_1} is the degree of τ_{Λ_1}, and χ_{Λ_1} is the character.

Fix f_0 in the induced space to be a nonzero highest weight vector for the minimal K type τ_Λ. If v_0 denotes a nonzero highest weight vector in an abstract representation space V^Λ of K of type τ_Λ, then f_0 is necessarily of the form

$$f_0(k) = A\tau_\Lambda(k)^{-1}v_0 \tag{3.3}$$

for a unique operator A in $\mathrm{Hom}_{K\cap M^\#}(V^\Lambda, V^{\sigma^\#})$. It follows from the remarks after (2.11) that there exists a unique element u_0 in $V^{\sigma^\#}$ of weight λ in the τ_λ subspace such that

$$A^*u_0 = v_0 . \tag{3.4}$$

We fix this element u_0.

Let τ_{Λ_1} be an irreducible representation of K, and let X_1 be in $\mathfrak{g}^{\mathbb{C}}$. Define

$$a(\nu,k) = \langle P_{\Lambda_1}U(\nu,X_1)f_0(k),u_0\rangle , \tag{3.5}$$

the inner product being taken in $V^{\sigma^\#}$. Let

$$b(\nu,k) = \langle P_\Lambda U(\nu,\overline{X}_1)P_{\Lambda_1}U(\nu,X_1)f_0(k),u_0\rangle . \tag{3.6}$$

<u>Theorem 3.1.</u>
(a) <u>Suppose</u> τ_{Λ_1} <u>has multiplicity at most one in</u> $U(\nu)$ <u>and</u>

$a(\nu,k)$ is not identically 0 as a function of k in K. Then the quotient

$$c(\nu) = a(-\nu,k)/a(\nu,k)$$

is independent of k. If $Ad(w)\nu = -\nu$, if ν is real-valued, and if ν is in the closed positive Weyl chamber, then $c(\nu) < 0$ implies that $T(\nu)$ is not positive semidefinite on the K type τ_{Λ_1}.

(b) Regardless of whether τ_{Λ_1} has multiplicity one in $U(\nu)$, suppose $Ad(w)\nu = -\nu$, ν is real-valued, and ν is in the positive Weyl chamber. If $b(-\nu,1) > 0$, then $T(\nu)$ is not positive semidefinite on the K type τ_{Λ_1}.

A proof can be obtained by making slight adjustments to the arguments in [1] and [2].

4. General formula

The main result of this section, Theorem 4.1, will give formulas for the quantities $a(\nu,k)$ and $b(\nu,k)$ of §3 under certain hypotheses. At this stage the Weyl group representative w of §3 does not enter the computation, since $a(\nu,k)$ and $b(\nu,k)$ make perfectly good sense without it. We shall make particular choices of w in §§5-6.

Theorem 4.1. Fix a complex root $\alpha = \alpha_R + \alpha_I$, let X be any member of $\mathfrak{p}^{\mathbb{C}}$, and define $\Lambda_1 = (\Lambda+\alpha_I)^\vee$ and

$$H_0 = \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{a}} > 0 \\ \langle\Lambda,\beta\rangle = 0}} p_\alpha H_\beta .$$

Suppose that

(a) $\langle\Lambda,\alpha_I\rangle = 0$

(b) α_I is short among the members of Δ_K.

Then

(1) $\langle P_{\Lambda_1} U(\nu,X) f_0(k), u_0\rangle = \langle \tau_{\Lambda_1}(k)^{-1} E_{\Lambda_1}(v_0 \otimes X), E_{\Lambda_1}(v_0 \otimes (H_{\bar\nu}+H_0))\rangle'$,

and also

(2) real rank $G = \dim \mathfrak{a}$ implies

$$\langle P_\Lambda U(\nu,X)P_{\Lambda_1}U(\nu,Y_\alpha)f_0,u_0\rangle$$
$$= \langle \tau_\Lambda(k)^{-1}E_\Lambda(E_{\Lambda_1}(v_0\otimes Y_\alpha)\otimes X), E_\Lambda(E_{\Lambda_1}(v_0\otimes(H_{\bar\nu}+H_0))\otimes(H_{\bar\nu}-H_0))\rangle .$$

Remarks. Note from (a) that $|\Lambda_1| \neq |\Lambda|$; hence $\Lambda_1 \neq \Lambda$. Both (1) and (2) are trivial if τ_{Λ_1} does not occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$; so we may assume τ_{Λ_1} does occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

Preliminaries for the proof. In the proof we shall use the formulas (2.7) and (2.8) relating the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ (relative to $(\mathfrak{b}\oplus\mathfrak{a})^{\mathbb{C}}$) and the Iwasawa-like decomposition (2.1). Suppose that f is a member of the space of the induced representation and is given on K by the formula

$$f(k) = C(\nu)\tau_{\Lambda'}(k)^{-1}v$$

with $C(\nu)$ in $\mathrm{Hom}_{K\cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$. For X in $\mathfrak{g}^{\mathbb{C}}$ we compute $U(\nu,X)f(k)$ by the method of §5 of [5]. As in (5.7) of [5], the result is

$$U(\nu,X)f(k) = [(\nu+\rho)(P_\mathfrak{a}Y)]C(\nu)\tau_{\Lambda'}(k)^{-1}v + [\sigma(P_\mathfrak{m}Y)]C(\nu)\tau_{\Lambda'}(k)^{-1}v$$
$$+ C(\nu)\tau_{\Lambda'}(P_\mathfrak{k}Y)\tau_{\Lambda'}(k)^{-1}v ,$$

where $Y = \mathrm{Ad}(k)^{-1}X$. Let $\{H_j\}$ be an orthogonal basis of $\mathfrak{a}$. Using our basis (2.6) and arguing as in the first part of the proof of Theorem 5.1 of [5], we obtain

$$\langle P_{\Lambda''}U(\nu,X)f(k),u_0\rangle$$
$$= \langle E_{\Lambda''}(v\otimes X),\pi(k)(C(\nu)^*u_0\otimes \sum_j |H_j|^{-2}(\bar\nu+\rho)(H_j)H_j)\rangle$$
$$+ \sum_{\beta\in\Delta_{B,n}} \tfrac{1}{2}|\beta|^2\langle E_{\Lambda''}(v\otimes X),\pi(k)(C(\nu)^*\sigma^\#(P_\mathfrak{m}X_\beta)^*u_0\otimes X_\beta)\rangle$$
$$+ \sum_{\substack{\beta\in\Delta\\ \beta|_\mathfrak{a}>0}} \tfrac{1}{4}|\beta|^2\langle E_{\Lambda''}(v\otimes X),\pi(k)(\tau_{\Lambda'}(P_\mathfrak{k}Y_\beta)^*C(\nu)^*u_0\otimes Y_\beta)\rangle , \qquad (4.1)$$

where π refers to the representation of K on the tensor product. We refer to the three lines of the right side of (4.1) as the $\mathfrak{a}$

term, the $\mathfrak{m}$ terms, and the $\mathfrak{l}$ terms.

Proof of conclusion (1). We take $\Lambda' = \Lambda$, $\Lambda'' = \Lambda_1 = (\Lambda+\alpha_I)^\vee$, $v = v_0$, $C(\nu) = \Lambda$, and $f = f_0$, remembering that $A^* u_0 = v_0$. The $\mathfrak{a}$ term of (4.1) is simply

$$= \langle E_{\Lambda_1}(v_0 \otimes X), \pi(k)(v_0 \otimes H_{\bar\nu+\rho})\rangle .$$

Let us see that the $\mathfrak{m}$ terms are all 0. In fact, u_0 has $\mathfrak{b}$ weight λ, so that $\sigma^\#(P_\mathfrak{m} X_\beta)^* u_0$ has weight $\lambda - \beta$. Since β is a noncompact root for M, $\lambda - \beta$ is not a weight of τ_λ. But A^* annihilates all $K \cap M^\#$ types of $\sigma^\#$ other than τ_λ, and thus $A^* \sigma^\#(P_\mathfrak{m} X_\beta)^* u_0 = 0$. So the $\mathfrak{m}$ terms are 0.

Next we consider the $\mathfrak{l}$ term corresponding to β with $\beta|_\mathfrak{a} > 0$. Formula (2.7) gives $P_\mathfrak{l} Y_\beta = -(X_\beta + \theta X_\beta)$; hence (2.4) shows that the $\mathfrak{l}$ term corresponding to the root β is

$$= -\frac{1}{\pi}|\beta|^2 \langle E_{\Lambda_1}(v_0 \otimes X), \pi(k)(\tau_\Lambda(X_{-\beta}+X_{-\theta\beta})v_0 \otimes Y_\beta)\rangle . \tag{4.2}$$

The weight of $\tau_\Lambda(X_{-\beta}+X_{-\theta\beta})v_0$ in τ_Λ is $\Lambda - \beta_I$, and $\langle\Lambda,\beta\rangle \leq 0$ would imply

$$|\Lambda-\beta_I|^2 = |\Lambda|^2 - 2\langle\Lambda,\beta\rangle + |\beta_I|^2 \geq |\Lambda|^2 + |\beta_I|^2 > |\Lambda|^2 ;$$

thus (4.2) is 0 unless $\langle\Lambda,\beta\rangle > 0$.

When $\langle\Lambda,\beta\rangle > 0$, we can use (2.9) to write

$$\begin{aligned}\tau_\Lambda(X_{-\beta}+X_{-\theta\beta})v_0 \otimes Y_\beta &= \pi(X_{-\beta}+X_{-\theta\beta})(v_0 \otimes Y_\beta) - v_0 \otimes \operatorname{ad}(X_{-\beta}+X_{-\theta\beta})Y_\beta \\ &= \pi(X_{-\beta}+X_{-\theta\beta})(v_0 \otimes Y_\beta) + 4|\beta|^{-2}(v_0 \otimes P_\mathfrak{a} H_\beta) . \end{aligned} \tag{4.3}$$

In this expression, the first term on the right projects to 0 under E_{Λ_1} since $\Lambda + \beta_I$ cannot be a weight of Λ_1:

$$\begin{aligned}|\Lambda+\beta_I|^2 = |\Lambda|^2 + 2\langle\Lambda,\beta\rangle + |\beta_I|^2 &> |\Lambda|^2 + |\beta_I|^2 \\ &\geq |\Lambda|^2 + |\alpha_I|^2 = |\Lambda+\alpha_I|^2 = |\Lambda_1|^2 .\end{aligned}$$

(Here we have used that $\langle\Lambda,\beta\rangle > 0$, that α_I is short, and that $\langle\Lambda,\alpha\rangle = 0$.) Putting (4.3) into (4.2), we see that the $\mathfrak{l}$ term corresponding to β is

$$= -\langle E_{\Lambda_1}(v_0 \otimes X), \pi(k)(v_0 \otimes P_\mathfrak{a} H_\beta)\rangle .$$

Adding the contributions from all the terms, we obtain

$$\langle P_{\Lambda_1} U(\nu, X) f_0(k), u_0 \rangle = \langle E_{\Lambda_1}(v_0 \otimes X), \pi(k)(v_0 \otimes (H_{\bar{\nu}+\rho} - \sum_{\substack{\beta \in \Delta \\ \beta|_{\mathfrak{a}} > 0 \\ \langle \Lambda, \beta \rangle > 0}} P_\alpha H_\beta))\rangle ,$$

and this is

$$= \langle E_{\Lambda_1}(v_0 \otimes X), \pi(k)(v_0 \otimes (H_{\bar{\nu}} + H_0))\rangle$$

by Theorem 2.1b. This proves conclusion (1).

Proof of conclusion (2). By Frobenius reciprocity the map $V^{\Lambda_1} \otimes \mathrm{Hom}_{K \cap M^\#}(V^{\Lambda_1}, V^{\sigma^\#})$ into the induced space, given by

$$v \otimes B \to B\tau_{\Lambda_1}(k)^{-1} v , \tag{4.4}$$

is one-one onto the K type τ_{Λ_1} of the induced space. Put

$$f_1 = P_{\Lambda_1} U(\nu, Y_\alpha) f_0 .$$

This is a member of the K type τ_{Λ_1}, and it has weight $\Lambda + \alpha_I$, which is extreme for τ_{Λ_1}. Since $\Lambda + \alpha_I$ is extreme, it has multiplicity one as a weight, and multiples of $v' = E_{\Lambda_1}(v_0 \otimes Y_\alpha)$ are the only v's that can contribute to the realization of f_1 via (4.4), as a consequence of Theorem 1.8. Thus

$$f_1(k) = B(\nu)\tau_{\Lambda_1}(k)^{-1} v'$$

for unique members $B(\nu)$ of $\mathrm{Hom}_{K \cap M^\#}(V^{\Lambda_1}, V^{\sigma^\#})$.

In (4.1) we take $\Lambda' = \Lambda_1$, $\Lambda'' = \Lambda$, $v = v'$, $C(\nu) = B(\nu)$, and $f = f_1$. The $\mathfrak{m}$ terms are absent since the assumption real rank $G = \dim \mathfrak{a}$ means that $\Delta_{B,n}$ is empty. Let us compute $B(\nu)^* u_0$. This is some vector of weight $\lambda = \Lambda$ in V^{Λ_1}. Thus if $\{v_i\}$ is an orthonormal basis of the Λ weight space of τ_{Λ_1}, we can write

$$B(\nu)^* u_0 = \sum \overline{b_i(\nu)} v_i \quad \text{with} \quad b_i(\nu) = \langle v_i, B(\nu)^* u_0 \rangle . \tag{4.5}$$

Then we have

$$\begin{aligned} \sum b_i(\nu)\langle \tau_{\Lambda_1}(k)^{-1} v', v_i \rangle &= \langle \tau_{\Lambda_1}(k)^{-1} v', B(\nu)^* u_0 \rangle \\ &= \langle B(\nu)\tau_{\Lambda_1}(k)^{-1} v', u_0 \rangle \\ &= \langle P_{\Lambda_1} U(\nu, Y_\alpha) f_0(k), u_0 \rangle \\ &= \langle \tau_{\Lambda_1}(k)^{-1} v', E_{\Lambda_1}(v_0 \otimes (H_{\bar{\nu}} + H_0))\rangle \quad \text{by conclusion (1)} \end{aligned}$$

$$= \Sigma \langle v_i, E_{\Lambda_1}(v_0 \otimes (H_{\bar\nu} + H_0))\rangle \langle \tau_{\Lambda_1}(k)^{-1} v', v_i\rangle ,$$

and the irreducibility of τ_{Λ_1} allows us to conclude

$$b_i(\nu) = \langle v_i, E_{\Lambda_1}(v_0 \otimes (H_{\bar\nu} + H_0))\rangle .$$

Hence

$$\Sigma \, \overline{b_i(\nu)} v_i = E_{\Lambda_1}(v_0 \otimes (H_{\bar\nu} + H_0)) . \tag{4.6}$$

We substitute from (4.5) into (4.1) and obtain

$$\langle P_\Lambda U(\nu,X) f_1(k), u_0\rangle$$

$$= \sum_i b_i(\nu) \{ \langle E_\Lambda(v' \otimes X), \pi(k)(v_i \otimes H_{\bar\nu+\rho})\rangle$$

$$- \sum_{\substack{\beta \in \Delta \\ \beta|_\alpha > 0}} \tfrac{1}{4} |\beta|^2 \langle E_\Lambda(v' \otimes X), \pi(k)(\tau_{\Lambda_1}(X_{-\beta} + X_{-\theta\beta}) v_i \otimes Y_\beta)\rangle \} . \tag{4.7}$$

Here v_i has $\mathfrak{b}$ weight Λ and $\tau_{\Lambda_1}(X_{-\beta}+X_{-\theta\beta})$ pulls down $\mathfrak{b}$ weights by β_I. Since α_I is short, we have

$$|\Lambda-\beta_I|^2 - |\Lambda+\alpha_I|^2 = -2\langle\Lambda,\beta\rangle + |\beta_I|^2 - |\alpha_I|^2 \geq -2\langle\Lambda,\beta\rangle ,$$

and $\Lambda - \beta_I$ cannot be a weight of τ_{Λ_1} if $\langle\Lambda,\beta\rangle < 0$. Thus the β^{th} term is 0 unless $\langle\Lambda,\beta\rangle$ is ≥ 0. If $\langle\Lambda,\beta\rangle \geq 0$, then $\Lambda + \beta_I$ is not a weight of τ_Λ since

$$|\Lambda+\beta_I|^2 - |\Lambda|^2 = 2\langle\Lambda,\beta\rangle + |\beta_I|^2 > 2\langle\Lambda,\beta\rangle .$$

So in this case

$$E_\Lambda(\tau_{\Lambda_1}(X_{-\beta}+X_{-\theta\beta}) v_i \otimes Y_\beta) = -E_\Lambda(v_i \otimes \operatorname{ad}(X_{-\beta}+X_{-\theta\beta}) Y_\beta)$$

$$= 4|\beta|^{-2} E_\Lambda(v_i \otimes P_\alpha H_\beta) .$$

Substituting into (4.7), we obtain

$$\langle P_\Lambda U(\nu,X) f_1(k), u_0\rangle = \sum_i b_i(\nu) \langle E_\Lambda(v' \otimes X), \pi(k)(v_i \otimes (H_{\bar\nu+\rho} - \sum_{\substack{\beta\in\Lambda \\ \beta|_\alpha > 0 \\ \langle\Lambda,\beta\rangle \geq 0}} P_\alpha H_\beta))\rangle$$

$$= \sum_i b_i(\nu) \langle E_\Lambda(v' \otimes X), \pi(k)(v_i \otimes (H_{\bar\nu} - H_0))\rangle$$

by Theorem 2.1. Taking (4.6) into account completes the proof of conclusion (2).

5. Application to $\mathfrak{so}$(odd,odd)

We shall apply conclusion (1) of Theorem 4.1 to a group G with Lie algebra $\mathfrak{so}$(odd,odd) in order to supply a proof of Lemma 14.3 of [4].

Let us recall the notation of § 14 of [4]. The root system is of type D_N, which has roots $\pm e_i \pm e_j$ $(i \neq j)$ in standard notation. We take $\mathfrak{b}$ to correspond to indices $1, \ldots, N-1$ and $\mathfrak{a}$ to correspond to index N. The infinitesimal character for a representation σ of interest is

$$\lambda_0 = (n_1, \ldots, n_{N-2}, 0, 0)$$

with $n_1 \geq \ldots \geq n_{N-2} \geq 0$ and with integer entries. The positive system Δ^+ is the usual one, whose simple roots are

$$e_1 - e_2, \ldots, e_{N-2} - e_{N-1}, e_{N-1} - e_N, e_{N-1} + e_N .$$

We take $\alpha = e_{N-1} + e_N$. Then $\alpha = \alpha_R + \alpha_I$ has

$$\alpha_I = e_{N-1} \quad \text{and} \quad \alpha_R = e_N . \tag{5.1}$$

By convention, e_N is positive as a root of $\mathfrak{a}$.

As is noted on p. 24 of [4], the reflection s_{α_R} in the Weyl group of $\mathfrak{a}$ acts on λ_0 by reflection in α_I and thus fixes λ_0; hence s_{α_R} fixes the class of the representation σ of M that corresponds to λ_0. Consequently we can take w in § 3 to be a representative in K of s_{α_R}, and Theorem 3.1 will be applicable when $\nu = \frac{1}{2} c\alpha_R$ with $c \geq 0$.

We shall see that the multiplicity assumption in part (a) of Theorem 3.1 is actually satisfied and that Theorem 4.1 can be used to compute the quantity $a(\nu, k)$ in the theorem. Let

$$\nu_0 = 2\#\{\beta \in \Delta^+ \mid \beta\big|_{\mathfrak{a}} > 0 \text{ and } \langle \Lambda, \beta \rangle = 0\} . \tag{5.2}$$

<u>Theorem 5.1.</u> <u>With notation as above, suppose that</u> σ <u>is nondegenerate in the sense of [14].</u> <u>Put</u> $\Lambda_1 = (\Lambda + e_{N-1})^\vee$. <u>Normalize the standard Hermitian form for</u> $U(\frac{1}{2} c\alpha_R)$ <u>so that it is positive on</u>

τ_Λ. Then τ_{Λ_1} has multiplicity one in $U(\frac{1}{2}c\alpha_R)$, and the signature of the standard form on τ_{Λ_1} is $\operatorname{sgn}(\nu_0 - c)$.

Proof. From the top of p. 116 of [4], we know that τ_{Λ_1} has multiplicity at most one. Thus Theorem 3.1a is applicable, and we are to compute a certain quotient $a(-\nu,k)/a(\nu,k)$. We shall use conclusion (1) of Theorem 4.1 with $X = Y_\alpha$ in order to make the computation.

From (5.1), we have $\Lambda_1 = (\Lambda + e_{N-1})^\vee = (\Lambda + \alpha_I)^\vee$, and Lemma 14.1 of [4] shows that $\langle \Lambda, \alpha_I \rangle = 0$. Also it is apparent that α_I is short among the members of Δ_K. Thus Theorem 4.1 applies. The conclusion for ν real is that

$$a(\nu,k) = \langle \tau_{\Lambda_1}(k)^{-1} E_{\Lambda_1}(v_0 \otimes Y_\alpha), E_{\Lambda_1}(v_0 \otimes (H_\nu + H_0)) \rangle .$$

Since $\mathfrak{a}$ has dimension 1, H_0 is a multiple of H_{α_R}, the multiple being given by

$$|\alpha_R|^{-2} \sum_{\substack{\beta \in \Delta^+ \\ \beta|_{\mathfrak{a}} > 0 \\ \langle \Lambda, \beta \rangle = 0}} \langle P_{\mathfrak{a}} H_\beta, H_{\alpha_R} \rangle .$$

Each of the β's in the sum has $P_{\mathfrak{a}} H_\beta = \beta_R = \alpha_R$, and the number of such β's is $\frac{1}{2}\nu_0$, with ν_0 as in (5.2). Thus for $\nu = \frac{1}{2}c\alpha_R$, we have

$$H_\nu + H_0 = \tfrac{1}{2}(c + \nu_0) H_{\alpha_R} .$$

Hence

$$a(\tfrac{1}{2}c\alpha_R, k) = \tfrac{1}{2}(\nu_0 + c) \langle \tau_{\Lambda_1}(k)^{-1} E_{\Lambda_1}(v_0 \otimes Y_\alpha), E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \rangle .$$

If the inner product in this expansion is not identically zero, then

$$\frac{a(-\frac{1}{2}c\alpha_R, k)}{a(\frac{1}{2}c\alpha_R, k)} = \frac{\nu_0 - c}{\nu_0 + c} ,$$

and Theorem 3.1a will finish the proof.

First we check that τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$. Define β_I' in $\Delta_n \cup \{0\}$ by $\Lambda + \beta_I' = (\Lambda + \alpha_I)^\vee = \Lambda_1$. (Recall Proposition 1.5a.) We shall prove that β_I' is conjugate to α_I by $W_{K,\Lambda}$. In fact, first notice that $\beta_I' \neq 0$ since $|\Lambda_1|^2 = |\Lambda|^2 + |\alpha_I|^2 \neq |\Lambda|^2$. Let $\beta_{1,I}$ be the result of making α_I dominant for $\Delta_{K,\Lambda}^+$. If $\Lambda + \beta_{1,I}$ is not

Δ_K^+ dominant, then Lemma 1.6 produces a Δ_K^+ simple γ_I with $2\langle\Lambda,\gamma_I\rangle/|\gamma_I|^2 = +1$ and $2\langle\beta_{1,I},\gamma_I\rangle/|\gamma_I|^2 = -2$. The latter formula shows that γ_I is short. Since the entries of Λ are integers, $2\langle\Lambda,\gamma_I\rangle/|\gamma_I|^2$ is then an even integer and cannot be $+1$. We conclude that $\Lambda+\beta_{1,I}$ is Δ_K^+ dominant, so that $\beta_I' = \beta_{1,I}$.

Now we apply Theorem 1.7 with $\Lambda' = \Lambda$ and $\beta = \beta_I'$. The length condition is clearly satisfied. If τ_{Λ_1} does not occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$, then the theorem gives us a Δ_K^+ simple root γ_I' such that γ_I' is in $\Delta_{K,\Lambda}^+$, $\langle\gamma_I',\beta_I'\rangle = 0$, and both $\gamma_I'+\beta_I'$ and $\gamma_I'-\beta_I'$ are in $\Delta_n \cup \{0\}$. Since $\gamma_I'\pm\beta_I'$ are in $\Delta_n \cup \{0\}$, γ_I' is short. Since γ_I' is in $\Delta_{K,\Lambda}^+$, Lemma 14.1 of [4] shows that $\gamma_I' = e_{N-1} = \alpha_I$. From the previous paragraph, β_I' is W_K conjugate to $\alpha_I = \gamma_I'$. Taking into account that Δ_K is a root system of type $B_p + B_q$, we see that $\gamma_I'\pm\beta_I'$ are in Δ_K. But they are in Δ_B also, since they are long. Hence they are in $\Delta_{B,c}$. But then they cannot be in $\Delta_n \cup \{0\}$, and we have a contradiction. We conclude that τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

Now let us return to proving that

$$\langle\tau_{\Lambda_1}(k)^{-1}E_{\Lambda_1}(v_0 \otimes Y_\alpha), E_{\Lambda_1}(v_0 \otimes H_{\alpha_R})\rangle$$

is not identically 0. Since $\Lambda_1 \neq \Lambda$, τ_{Λ_1} occurs irreducibly in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$, by Proposition 1.5b. Thus it is enough to prove that $E_{\Lambda_1}(v_0 \otimes Y_\alpha)$ and $E_{\Lambda_1}(v_0 \otimes H_{\alpha_R})$ are nonzero. The first of these vectors is nonzero by Theorem 1.8, and we examine the second. Since $E_{\Lambda_1}(v_0 \otimes Y_\alpha)$ has weight $\Lambda + \alpha_I$ and since

$$\langle\Lambda+\alpha_I,\alpha_I\rangle = |\alpha_I|^2 > 0,$$

$\tau_{\Lambda_1}(X_{-\alpha}+X_{-\theta\alpha})E_{\Lambda_1}(v_0 \otimes Y_\alpha)$ is not zero. Thus

$$\begin{aligned} 0 &\neq \tau_{\Lambda_1}(X_{-\alpha}+X_{-\theta\alpha})E_{\Lambda_1}(v_0 \otimes Y_\alpha) \\ &= E_{\Lambda_1}(v_0 \otimes [X_{-\alpha}+X_{-\theta\alpha}, Y_\alpha]) \qquad \text{since } \langle\Lambda,\alpha_I\rangle = 0 \\ &= -4|\alpha|^{-2}E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \qquad \text{by (2.9)}, \end{aligned}$$

and the proof is complete.

6. Application to certain groups of real rank two

We shall apply conclusion (2) of Theorem 4.1 to a group G with restricted root system of type A_2 and with just one conjugacy class of Cartan subgroups. These groups are the subject of [3], and our objective in this section is to supply a proof of Proposition 4.1 of that paper.

Let us recall the setting in [3]: All complex roots have the same length and are orthogonal to their complex conjugates, by (1.1) of [3]. We denote the positive restricted roots by e_1-e_2, e_1-e_3, e_2-e_3. We take $\alpha_R = e_1-e_3$ and define α_I as in Lemma 3.8a of [10] to make $\alpha = \alpha_R+\alpha_I$ be a root; since $M^{\#}$ is connected, this choice makes it so that the action of s_{α_R} on the equivalence class of σ is mirrored by the action of s_{α_I} on the infinitesimal character λ_0. We let w be a representative in K of s_{α_R}, and we assume that $w\sigma \cong \sigma$, so that the material in §3 applies. Let Λ be the minimal K type of the induced representations, and define

$$\Delta_L = \{\beta \in \Delta \mid \langle \Lambda, \beta\rangle = 0\} .$$

Proposition 1.4 of [3] says that Δ_L is generated by simple roots of Δ^+. Let L be the analytic subgroup of G corresponding to $\mathfrak{b} \oplus \mathfrak{a}$ and Δ_L, and let L_{ss} be the commutator subgroup. Then Proposition 1.2 of [3] says that L_{ss} has real rank one or two, and our interest is in the case that it has real rank two. Let ρ_L be the functional ρ for L_{ss}.

<u>Theorem 6.1.</u> <u>Let notation be as above, in particular with</u> s_{α_R} <u>fixing the equivalence class of</u> σ. <u>Suppose that the standard invariant form is normalized so that</u> f_0 <u>has</u> $\langle f_0, f_0\rangle = 1$. <u>Put</u> $\Lambda_1 = (\Lambda+\alpha_I)^\vee$. <u>Then the function</u>

$$f_1 = P_{\Lambda_1} U(c\rho_L, Y_\alpha) f_0$$

<u>is a nonzero member of the induced space, and</u> $\langle f_1, f_1\rangle$ <u>is a positive multiple of</u> $1-c^2$.

<u>Remarks.</u> The parabolic subgroup MAN is minimal under our assumptions, and M is thus compact. Hence σ is finite-dimensional.

<u>Proof.</u> We are going to apply Theorem 3.1b with $\nu = c\rho_L$, $c \geq 0$. Since ρ_L is a positive multiple of α_R, the theorem is applicable. We are to compute a certain quantity $b(\nu,k)$, which we take to be

$$b(\nu,k) = \langle P_\Lambda U(\nu, Y_{-\alpha}) P_{\Lambda_1} U(\nu, Y_\alpha) f_0(k), u_0\rangle ,$$

and show that $b(-\nu,1)$ is a positive multiple of c^2-1 when $\nu = c\rho_L$.

To compute $b(\nu,k)$, we use conclusion (2) of Theorem 4.1. By (1.2c) of [3], we have $\langle\Lambda,\alpha_I\rangle = 0$. Let us check that α_I is short among the members of Δ_K. In doing so, we shall assume that $\mathfrak{g}$ is simple, as we may for the proof of Theorem 6.1 without loss of generality. If α_I is not short, then there exists $\beta = \beta_R + \beta_I$ in Λ such that $|\beta_I| < |\alpha_I|$ and $\langle\alpha_I,\beta_I\rangle \neq 0$. Since all complex roots in Λ are orthogonal to their conjugates and have the same length, $\beta_R = 0$. Thus β_I is in Δ. But then

$$|\beta_I|^2 = \tfrac{1}{2}|\alpha_I|^2 = \tfrac{1}{4}|\alpha|^2$$

gives an illegal length relation among the roots of Δ. We conclude that α_I is short.

Since there is just one conjugacy class of Cartan subgroups in G, we have real rank $G = \dim \mathfrak{a}$. Thus Theorem 4.1b applies and gives us

$$b(\nu,k) = \langle\tau_\Lambda(k)^{-1}E_\Lambda(E_{\Lambda_1}(v_0\otimes Y_{-\alpha})\otimes Y_\alpha), E_\Lambda(E_{\Lambda_1}(v_0\otimes(H_\nu+H_0))\otimes(H_\nu-H_0))\rangle$$

for ν real. Now

$$H_0 = \sum_{\substack{\beta\in\Delta' \\ \beta|_\mathfrak{a}>0 \\ \langle\Lambda,\beta\rangle=0}} P_\mathfrak{a} H_\beta = \sum_{\substack{\beta\in\Delta'_L \\ \beta|_\mathfrak{a}>0}} P_\mathfrak{a} H_\beta = \tfrac{1}{2}\sum_{\substack{\beta\in\Delta_L \\ \beta|_\mathfrak{a}>0}} P_\mathfrak{a} H_\beta = \tfrac{1}{2}H_{2\rho_L} = H_{\rho_L}.$$

For $\nu = c\rho_L$, $b(\nu,k)$ therefore reduces to

$$b(\nu,k) = (c^2-1)\langle\tau_\Lambda(k)^{-1}E_\Lambda(E_{\Lambda_1}(v_0\otimes Y_\alpha)\otimes Y_{-\alpha}), E_\Lambda(E_{\Lambda_1}(v_0\otimes H_{\rho_L})\otimes H_{\rho_L})\rangle. \tag{6.1}$$

Since the expression of interest for Theorem 3.1b is $b(-\nu,1)$, the proof will be complete if we show that the inner product in (6.1) is positive for $k=1$. Actually it is enough to prove that the inner product is nonzero, since it is constant in ν and since $U(\nu)$ is unitary for $\nu = 0$.

First we check that τ_{Λ_1} occurs in $\tau_\Lambda\otimes\mathfrak{p}^{\mathbb{C}}$. Define β_I' in $\Delta_n\cup\{0\}$ by $\Lambda+\beta_I' = (\Lambda+\alpha_I)^\vee = \Lambda_1$. (Recall Proposition 1.5a.) We shall prove that β_I' is conjugate to α_I by $W_{K,\Lambda}$. In fact, first notice that $\beta_I'\neq 0$ since $|\Lambda_1|^2 = |\Lambda|^2+|\alpha_I|^2 \neq |\Lambda|^2$. Let $\beta_{1,I}$ be the result of making α_I dominant for $\Delta^+_{K,\Lambda}$. If $\Lambda+\beta_{1,I}$ is not Δ^+_K dominant, then Lemma 1.6 produces a Δ^+_K simple γ_I with

$$2\langle\Lambda,\gamma_I\rangle/|\gamma_I|^2 = +1 \tag{6.2a}$$

and

$$2\langle\beta_{1,I},\gamma_I\rangle/|\gamma_I|^2 = -2\,. \tag{6.2b}$$

Since real rank $\mathfrak{g} = \dim \mathfrak{a}$, $\Delta_{B,n}$ is empty. Thus $\Delta_n \subseteq \Delta_K$, and $\beta_{1,I}$ is in Δ_K. Since $|\alpha_I| = |\beta_{1,I}|$ and since α_I is short within Δ_K, (6.2b) says $\beta_{1,I} = -\gamma_I$. But then (6.2a) gives

$$0 = \langle\Lambda,\alpha_I\rangle = \langle\Lambda,\beta_{1,I}\rangle = -\langle\Lambda,\gamma_I\rangle \neq 0\,,$$

and we have a contradiction. We conclude that $\beta'_I = \beta_{1,I}$.

Now we apply Theorem 1.7 with $\Lambda' = \Lambda$ and $\beta = \beta'_I$. The length condition is satisfied since $\Delta_n \subseteq \Delta_K$ and since Δ_K is a root system. If τ_{Λ_1} does not occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$, then the theorem gives us a Δ^+_K simple root γ'_I such that γ'_I is in $\Delta^+_{K,\Lambda}$, $\langle\gamma'_I,\beta'_I\rangle = 0$, and both $\gamma'_I+\beta'_I$ and $\gamma'_I-\beta'_I$ are in $\Delta_n \cup \{0\}$. Now

$$|\alpha_I|^2 = |\beta'_I|^2 < |\gamma'_I|^2 + |\beta'_I|^2 = |\gamma'_I \pm \beta'_I|^2$$

implies that $\gamma'_I \pm \beta'_I$ are in Δ_B, hence in $\Delta_{B,n}$ (since all complex roots have equal lengths and are orthogonal to their conjugates). But $\Delta_{B,n}$ is empty, and we have a contradiction. We conclude that τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

Arguing with characters, we see from the occurrence of τ_{Λ_1} in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$ that τ_Λ occurs in $\tau_{\Lambda_1} \otimes \mathfrak{p}^{\mathbb{C}}$. Note that $\Lambda_1 \neq \Lambda$ since $|\Lambda_1| \neq |\Lambda|$.

Now let us return to proving that the inner product in (6.1) is nonzero at $k=1$. Since τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$ and τ_Λ occurs in $\tau_{\Lambda_1} \otimes \mathfrak{p}^{\mathbb{C}}$, Theorem 1.8 shows that

$$E_\Lambda(E_{\Lambda_1}(v_0 \otimes Y_\alpha) \otimes Y_{-\alpha}) \tag{6.3}$$

is nonzero. If we can prove that the vector

$$E_\Lambda(E_{\Lambda_1}(v_0 \otimes H_{\rho_L}) \otimes H_{\rho_L}) \tag{6.4}$$

is a nonzero multiple of (6.3), then the inner product in (6.1) will be nonzero at $k=1$, and the proof will be complete. In place of (6.4), we may as well consider

$$E_\Lambda(E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \otimes H_{\alpha_R}) . \tag{6.5}$$

We start from the identity

$$E_\Lambda(E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \otimes Y_{-\alpha}) = 0 ,$$

which holds since $\Lambda + \alpha_I$ is too long to be a weight of τ_Λ. Applying $\tau_\Lambda(X_\alpha + X_{\theta\alpha})$, we get

$$\begin{aligned} 0 &= E_\Lambda(E_{\Lambda_1}(v_0 \otimes [X_\alpha + X_{\theta\alpha}, H_{\alpha_R}]) \otimes Y_{-\alpha}) \\ &\quad + E_\Lambda(E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \otimes [X_\alpha + X_{\theta\alpha}, Y_{-\alpha}]) \\ &= |\alpha_R|^2 E_\Lambda(E_{\Lambda_1}(v_0 \otimes Y_\alpha) \otimes Y_{-\alpha}) \\ &\quad + 4|\alpha|^2 E_\Lambda(E_{\Lambda_1}(v_0 \otimes H_{\alpha_R}) \otimes H_{\alpha_R}) \qquad \text{by (2.9).} \end{aligned}$$

This relation exhibits (6.5) as a nonzero multiple of (6.3), and the proof is complete.

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