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Sharp inequalities for martingales and stochastic integrals

by Donald L. BURKHOLDER

This paper contains sharp inequalities for differentially subordinate martingales taking values in a real or complex Hilbert space H. These sharp inequalities, new even for $H = \mathbb{C}$, lead to the best constants for some inequalities between stochastic integrals in which either the martingale integrators or the predictable integrands are H-valued. In addition, they yield new information about the square-function inequality for H-valued martingales even in the case $H = \mathbb{R}$.

It will be convenient to denote the norm of $x \in H$ by $|x|$, the inner product of x and y by $\langle x,y \rangle$, and the real part of $\langle x,y \rangle$ by (x,y).

1. AN INEQUALITY FOR H-VALUED MARTINGALES

Let $(\Omega, \mathcal{F}_\infty, P)$ be a probability space and $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ a nondecreasing sequence of sub-σ-fields of $\mathcal{F}_\infty$. Suppose that $f = (f_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ are H-valued martingales with respect to $\mathcal{F}$. Denote their difference sequences by d and e: $f_n = \sum_{k=0}^n d_k$ and $g_n = \sum_{k=0}^n e_k$, $n \geq 0$. Set $\|f\|_p = \sup_n \|f_n\|_p$.

THEOREM 1.1. *Let p^* be the maximum of p and q where $1 < p < \infty$ and $1/p + 1/q = 1$. If*

$$|e_k(\omega)| \leq |d_k(\omega)| \tag{1.1}$$

for all $\omega \in \Omega$ and $k \geq 0$, then

$$\|g\|_p \leq (p^* - 1)\|f\|_p \ . \tag{1.2}$$

<u>The constant</u> $p^* - 1$ <u>is best possible</u>. <u>In the nontrivial case</u> $0 < \|f\|_p < \infty$, <u>there is equality in</u> (1.2) <u>if and only if</u> $p = 2$ <u>and equality holds in</u> (1.1) <u>for almost all</u> ω <u>and all</u> $k \geq 0$.

Observe that $p^* - 1 = \max\{p - 1, 1/(p - 1)\}$. Here "best possible" is to mean that if $\beta < p^* - 1$ then, for some probability space $(\Omega, \mathfrak{F}_\infty, P)$ and filtration $\mathfrak{F}$, there exist H-valued martingales f and g as above such that $\|g\|_p > \beta\|f\|_p$.

For a proof of Theorem 1.1 in the case $H = \mathbb{R}$, see [7], where it is also noted that the martingale condition can be relaxed.

Condition (1.1) may be described by saying that g is differentially subordinate to f.

PROOF. To prove the inequality (1.2), we can assume that $\|f\|_p$ is finite. By (1.1),

$$\|g_n\|_p \leq \sum_{k=0}^{n} \|e_k\|_p \leq \sum_{k=0}^{n} \|d_k\|_p \leq (2n + 1)\|f\|_p \ .$$

So both $\|f_n\|_p$ and $\|g_n\|_p$ are finite. Define $v\colon H \times H \to \mathbb{R}$ by

$$v(x,y) = |y|^p - (p^* - 1)^p|x|^p \ . \tag{1.3}$$

Then $Ev(f_n,g_n) = \|g_n\|_p^p - (p^* - 1)^p\|f_n\|_p^p$ and (1.2) would follow if, for all n,

$$Ev(f_n,g_n) \leq 0 \ . \tag{1.4}$$

Rather than proving (1.4) directly, we shall prove the analogous inequality

$$Eu(f_n,g_n) \leq 0 \tag{1.5}$$

for a more convenient function u, one that majorizes v on $H \times H$. The function $u\colon H \times H \to \mathbb{R}$ is defined by

$$u(x,y) = \alpha_p(|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1} \tag{1.6}$$

where $\alpha_p = p(1 - 1/p^*)^{p-1}$. As we shall prove,

$$\text{(1.7)} \qquad Ev(f_n, g_n) \leq Eu(f_n, g_n) ,$$

$$\text{(1.8)} \qquad Eu(f_n, g_n) \leq Eu(f_{n-1}, g_{n-1}) , \quad n \geq 1 ,$$

$$\text{(1.9)} \qquad Eu(f_0, g_0) \leq 0 .$$

These inequalities imply (1.2).

Inequality (1.7) follows from

$$\text{(1.10)} \qquad v(x,y) \leq u(x,y) .$$

To prove this we shall use the elementary inequality

$$\text{(1.11)} \qquad (p - 1)^{p-1}/p^{p-2} \leq 1 \quad \text{if} \quad 1 < p \leq 2 ,$$
$$\geq 1 \quad \text{if} \quad p > 2 ,$$

which follows at once from the fact that if $\varphi(p)$ is the left-hand side, then $\varphi(1+) = \varphi(2) = 1$, $\lim_{p \to \infty} \varphi(p) = \infty$, $\log \varphi$ is convex on $(1,2]$ and is concave on $[2,\infty)$. If $|x| + |y| = 0$, then (1.10) is satisfied. Therefore, to prove (1.10), we can assume that $|x| + |y| > 0$ and, by homogeneity, that $|x| + |y| = 1$. Letting $s = |x|$, we see that the proof of (1.10) reduces to showing that

$$F(s) = \alpha_p(1 - p^*s) - (1 - s)^p + (p^* - 1)^p s^p$$

is nonnegative for $0 \leq s \leq 1$. If $p = 2$, then $F \equiv 0$. If $p > 2$, then there is a number s_0 satisfying $0 < s_0 < 1/p^*$ such that F is concave on $[0,s_0]$ and is convex on $[s_0,1]$. Therefore, it follows from $F(1/p^*) = F'(1/p^*) = 0$ that F is nonnegative on $[s_0,1]$. By (1.11), $F(0) \geq 0$, so concavity implies that F is also nonnegative on $(0,s_0)$. If $1 < p < 2$, the argument is similar: F is convex on an interval containing $[0,1/p^*]$, F is concave on its complement, and $F(1) \geq 0$.

Before proving (1.8), we shall show that if $x,y,h,k \in H$, $|k| \leq |h|$, and

$|x + ht|$ and $|x + kt|$ are strictly positive for all $t \in \mathbb{R}$, then

$$u(x + h, y + k) \leq u(x,y) + (\varphi(x,y), h) + (\psi(x,y), k) \tag{1.12}$$

(recall that $(\cdot,\cdot) = \operatorname{Re}\langle\cdot,\cdot\rangle$) where

$$\varphi(x,y) = \alpha_p x'[(p - p^*)|y| - p(p^* - 1)|x|](|x| + |y|)^{p-2} ,$$

$$\psi(x,y) = \alpha_p y'[p|y| + (p + p^* - pp^*)|x|](|x| + |y|)^{p-2} ,$$

and $x' = x/|x|$, $y' = y/|y|$. To prove (1.12), let

$$G(t) = u(x + ht, y + kt) .$$

The conditions on x, y, h, k imply that G is infinitely differentiable on $\mathbb{R}$, and a little calculation shows that the inequality (1.12) is the same as

$$G(1) \leq G(0) + G'(0) . \tag{1.13}$$

Since G is concave on $\mathbb{R}$, as we shall see, (1.13) follows. To show the concavity of G, we can reduce the problem by translation to showing that $G''(0) \leq 0$. If $p \geq 2$,

$$G''(0) = - \alpha_p(A + B + C) \tag{1.14}$$

where

$$A = p(p - 1)(|h|^2 - |k|^2)(|x| + |y|)^{p-2}$$

is nonnegative since $|h| \geq |k|$,

$$B = p(p - 2)[|k|^2 - (y',k)^2]|y|^{-1}(|x| + |y|)^{p-1}$$

is nonnegative by the Cauchy-Schwarz inequality, and

$$C = p(p - 1)(p - 2)[(x',h) + (y',k)]^2|x|(|x| + |y|)^{p-3}$$

is also nonnegative. If $1 < p \leq 2$, a similar expression for $(p - 1)G''(0)$ can be obtained from (1.14) by interchanging x and y, h and k, and then multiplying the right-hand side by -1. This follows from

$$(p-1)u(x,y) = -\alpha_p(|x| - (p-1)|y|)(|y| + |x|)^{p-1}$$

and uses the identity $(p-1)(p^*-1) = 1$, which is valid for $1 < p \leq 2$. This completes the proof of (1.12).

Now let H_0 be a closed subspace of H such that $f_n(\omega), g_n(\omega) \in H_0$ for all $\omega \in \Omega$, $n \geq 0$. We can and do assume that H_0 is a proper subspace of H (otherwise enlarge H slightly). Let $a \in H$ be in the orthogonal complement of H_0 with $0 < |a| \leq 1$ and write $F_n = a + f_n$ and $G_n = a + g_n$. Then, by (1.12),

$$u(F_n,G_n) \leq u(F_{n-1},G_{n-1}) + (\varphi(F_{n-1},G_{n-1}),d_n) + (\psi(F_{n-1},G_{n-1}),e_n) \; . \tag{1.15}$$

These four terms are integrable:

$$|u(F_n,G_n)| \leq c_p[|F_n| + |G_n|]^p \leq c_p[2 + |f_n| + |g_n|]^p \; ,$$

$$|\varphi(F_n,G_n)| \leq c_p[2 + |f_n| + |g_n|]^{p-1} \; , \; \ldots \; .$$

By the martingale condition, the last two terms of (1.15) have zero expectation. Therefore,

$$Eu(F_n,G_n) \leq Eu(F_{n-1},G_{n-1}) \; .$$

Now let $a \to 0$ and use the continuity of u and the dominated convergence theorem to obtain (1.8).

To prove (1.9), observe that under the condition $|y| \leq |x|$,

$$|y| - (p^* - 1)|x| \leq (2 - p^*)|x| \leq 0 \; .$$

Therefore,

$$u(x,y) \leq \alpha_p(2 - p^*)|x|^p \leq 0 \quad \text{if} \quad |y| \leq |x| \; . \tag{1.16}$$

Accordingly, $u(f_0,g_0) \leq 0$, and (1.9) follows. This completes the proof of (1.2).

Suppose that $0 < \|f\|_p < \infty$. In the case $p = 2$, the statement of the theorem about equality in (1.2) follows from orthogonality: $\|f\|_2^2 = \sum_{k=0}^{\infty} \|d_k\|_2^2$. If $p \neq 2$, strict inequality holds in (1.2), which can be seen as follows. Let m be the least integer n such that $\|f_n\|_p > 0$. Then, with probability one, $|g_m| = |e_m| \leq |d_m| = |f_m|$ so that, by (1.8) and (1.16),

$$\mathrm{E}u(f_n,g_n) \leq \mathrm{E}u(f_m,g_m) \leq \alpha_p(2 - p^*)\|f_m\|_p^p < 0$$

provided $n \geq m$, in which case

$$\|g_n\|_p^p \leq (p^* - 1)^p\|f_n\|_p^p + \alpha_p(2 - p^*)\|f_m\|_p^p .$$

This implies that strict inequality holds in (1.2).

The constant $p^* - 1$ is best possible since it is best possible for $H = \mathbb{R}$; see [6] or [7].

This completes the proof of Theorem 1.1.

REMARK 1.1. Let $1 < p < \infty$. In [21], Pełczyński conjectured that the complex unconditional constant of the Haar basis $(h_n)_{n \geq 0}$ of $L^p_{\mathbb{C}}[0,1)$ is the same as the unconditional constant for the real case. To be precise, let β_p be the least number β such that

$$\|\sum_{k=0}^{n} \epsilon_k a_k h_k\|_p \leq \beta\|\sum_{k=0}^{n} a_k h_k\|_p$$

for all $a_k \in \mathbb{R}$, $\epsilon_k \in \{1,-1\}$, and $n \geq 0$. Let γ_p be the least number γ such that

$$\|\sum_{k=0}^{n} e^{i\theta_k} c_k h_k\|_p \leq \gamma\|\sum_{k=0}^{n} c_k h_k\|_p$$

for all $c_k \in \mathbb{C}$, $\theta_k \in \mathbb{R}$, and $n \geq 0$. Clearly, $\beta_p \leq \gamma_p$ and Pełczyński's conjecture is that equality holds. By [7], $\beta_p = p^* - 1$, so to prove Pełczynski's conjecture we need to show only that $\gamma_p \leq p^* - 1$. But this follows at once from Theorem 1.1: the martingale difference sequences defined by

$d_k = c_k h_k$ and $e_k = e^{i\theta_k} c_k h_k$ satisfy (1.1). For a slightly more direct proof, see [10].

REMARK 1.2. The function u defined by (1.6) has the property that, for all $x,y,h,k \in H$ with $|k| \leq |h|$, the mapping

$$t \mapsto u(x + ht, y + kt) \tag{1.17}$$

is concave on $\mathbb{R}$. But it is not the least majorant of v with this property. If $1 < p \leq 2$, let

$$\begin{aligned} U(x,y) &= u(x,y) \quad \text{if} \quad |y| \leq (p^* - 1)|x| \ , \\ &= v(x,y) \quad \text{if} \quad |y| > (p^* - 1)|x| \ , \end{aligned}$$

where u is given by (1.6) and v by (1.3). If $p > 2$, interchange u and v. For $1 < p < \infty$, the function U is the least majorant of v satisfying the concavity condition (1.17). It has the following martingale interpretation. Let $x,y \in H$. If f and g are H-valued martingales with respect to the same filtration, $f_0 \equiv x$, $g_0 \equiv y$, and $|e_k| \leq |d_k|$ for all $k \geq 1$, then

$$\|g\|_p^p \leq (p^* - 1)^p \|f\|_p^p + U(x,y)$$

and no number strictly smaller than $U(x,y)$ has this property.

The function U, which is defined on $H \times H$, is closely related to the function $u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of [8]: $U(x,y) = u(|y| + |x|, |y| - |x|)$. This formula could be used to define a function that majorizes v on $B \times B$ where B is any Banach space. But the concavity condition is less adaptable. It is easy to see that if $p = 2$, then $U(x,y) = |y|^2 - |x|^2$ and U satisfies the concavity condition if and only if $B = H$. In fact, if B is not isomorphic to a Hilbert space, then there is no constant that can be substituted for $p^* - 1$ in (1.2) that would make the inequality valid. See Section 5 of [5] where the proof is based on Kwapień's characterization of spaces isomorphic to Hilbert space [20].

2. EXPONENTIAL AND WEAK-TYPE INEQUALITIES

Let Φ be an increasing convex function on $[0,\infty)$ with $\Phi(0) = 0$ and $\int_0^\infty \Phi(t)e^{-t}dt < \infty$. Assume that Φ is twice differentiable on $(0,\infty)$ with a strictly convex first derivative satisfying $\Phi'(0+) = 0$.

THEOREM 2.1. <u>Suppose, as in Section</u> 1, <u>that</u> f <u>and</u> g <u>are</u> H-<u>valued martingales with respect to the same filtration and</u> g <u>is differentially subordinate to</u> f. <u>If</u> $\|f\|_\infty \le 1$, <u>then</u>

$$\sup_n E\Phi(|g_n|) < \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt \tag{2.1}$$

<u>and the constant on the right is best possible</u>.

For example, if $p > 2$ and $\Phi(t) = t^p$, then

$$\|g\|_p^p < \frac{1}{2}\Gamma(p+1) \ . \tag{2.2}$$

If $0 < \alpha < 1$ and $\Phi(t) = e^{\alpha t} - 1 - \alpha t$, then Theorem 2.1 gives the exponential inequality (2.14) below.

See [7] for (2.1) and (2.2) in the case $H = \mathbb{R}$.

PROOF OF THEOREM 2.1. Let $S = \{(x,y) \in H \times H: |x| \le 1\}$. By the assumption $\|f\|_\infty \le 1$, we can assume that $(f_n(\omega),g_n(\omega)) \in S$ for all $\omega \in \Omega$ and $n \ge 0$. We shall exhibit a function $u: S \to \mathbb{R}$ with the following properties:

$$E\Phi(|g_n|) \le Eu(f_n,g_n) \ , \tag{2.3}$$

$$Eu(f_n,g_n) \le Eu(f_{n-1},g_{n-1}) \ , \quad n \ge 1 \ , \tag{2.4}$$

$$Eu(f_0,g_0) \le \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt \ . \tag{2.5}$$

These inequalities give (2.1) but without the strictness, which will be proved below.

To define u let A and B be the functions on $[1,\infty)$ given by $B(t) = \Phi(t-1)$ and

$$A(t) = e^{t}\int_{t}^{\infty} B(s)e^{-s}ds = e^{t-1}\int_{t-1}^{\infty} \Phi(s)e^{-s}ds \,.$$

Now define $u\colon S \to \mathbb{R}$ by

$$(2.6) \qquad \begin{aligned} u(x,y) &= (1 + |y|^2 - |x|^2)A(1)/2 \quad \text{if} \quad |x| + |y| \le 1 \,, \\ &= (1 - |x|)A(|x| + |y|) + |x|B(|x| + |y|) \quad \text{if} \quad |x| + |y| > 1 \,. \end{aligned}$$

This is a continuous function: if $|x| + |y| = 1$, both expressions on the right-hand side are equal to $|y|A(1)$. In fact,

$$(2.7) \qquad u(x,y) = u^0(|y| + |x|, |y| - |x|)$$

where u^0 is the function u of Section 6 of [7] and this is one way to see that $\Phi(|y|) \le u(x,y)$ for $(x,y) \in S$. This majorization gives (2.3).

Here define φ and ψ on S by

$$\begin{aligned} \varphi(x,y) &= -A(1)x \quad \text{if} \quad |x| + |y| \le 1 \,, \\ &= [-A(|x| + |y|) + B(|x| + |y|) + B'(|x| + |y|)]x \quad \text{if} \quad |x| + |y| > 1 \,, \end{aligned}$$

$$\begin{aligned} \psi(x,y) &= A(1)y \quad \text{if} \quad |x| + |y| \le 1 \,, \\ &= [(1 - |x|)A(|x| + |y|) + (2|x| - 1)B(|x| + |y|)]y' \quad \text{if} \quad |x| + |y| > 1 \,, \end{aligned}$$

where $y' = y/|y|$ as before. Note that, for $(x,y) \in S$, the condition $|x| + |y| > 1$ entails $|y| > 1 - |x| \ge 0$. The functions φ and ψ are continuous on S.

Now suppose that $(x,y) \in S$, $(x + h, y + k) \in S$, and $|k| \le |h|$. Then

$$(2.8) \qquad u(x + h, y + k) \le u(x,y) + (\varphi(x,y),h) + (\psi(x,y),k)$$

and strict inequality holds if, in addition,

$$(2.9) \qquad |x| + |y| \le 1 < |x + h| + |y + k| \,.$$

By (2.9) and the condition $|k| \le |h|$, we have that $h \ne 0$, which we can assume in the proof of (2.8). If $t \in [0,1]$, then $|x + ht| \le 1$ so $G(t) = u(x + ht, y + kt)$ defines a continuous function G on $[0,1]$. Inequality (2.8) is equivalent to

$$G(1) \le G(0) + G'(0+) .$$

This follows from the mean-value theorem and the fact that G' is nonincreasing on $(0,1)$. To see that G' is nonincreasing, let $M(t) = |x + ht|$, $N(t) = |y + kt|$, and $F = A - B$. If $s > 1$, then, by the strict convexity of Φ', we have that $F(s) > 0$, $F'(s) > 0$, $F''(s) > 0$, and $sF'(s) - F(s) > 0$. Let

$$I = \{t \in (0,1) : M(t) + N(t) < 1\} ,$$

$$J = \{t \in (0,1) : M(t) > 0 \text{ and } M(t) + N(t) > 1\} .$$

By the strict convexity of M and the convexity of N, the complement of $I \cup J$ relative to $(0,1)$ is finite. Both M and N are strictly positive on J. Therefore, G is infinitely differentiable on $I \cup J$ and it is easy to check that G' exists and is continuous on $(0,1)$. Accordingly, to prove that G' is nonincreasing on $(0,1)$, we need to prove only that G'' is nonpositive on $I \cup J$. If I is nonempty, then $G'' \le 0$ on I: $G'' = -(|h|^2 - |k|^2)A(1)$. If J is nonempty, as it is under the condition (2.9), then $G'' \le 0$ on J:

$$\begin{aligned} (2.10) \qquad G'' = &-(|h|^2 - |k|^2)F'(M + N) \\ &- N''[(M + N)F'(M + N) - F(M + N)] \\ &- M(M' + N')^2F''(M + N) . \end{aligned}$$

This proves (2.8). The assertion about strict inequality follows from the fact that, under (2.9), the function $M(M' + N')^2$ has only a finite number of zeros in J so G' is strictly decreasing on J.

By (2.8) and the assumption $|e_n| \le |d_n|$,

(2.11) $$u(f_n,g_n) \le u(f_{n-1},g_{n-1}) + (\varphi(f_{n-1},g_{n-1}),d_n) + (\psi(f_{n-1},g_{n-1}),e_n)$$

with strict inequality holding with positive probability if

(2.12) $$P(|f_{n-1}| + |g_{n-1}| \le 1 < |f_n| + |g_n|) > 0 .$$

Each term in (2.11) is integrable since $\|g_n\|_\infty \le 2n + 1$. Therefore, taking expectations and using the martingale condition, we obtain (2.4), with strict inequality under (2.12).

Also, by (2.8), $u(f_0,g_0) \le u(0,0)$ which implies (2.5), with strict inequality if

(2.13) $$P(|f_0| + |g_0| > 1) > 0 .$$

If $\|g\|_\infty \le 1$, then $\sup_n E\Phi(|g_n|) \le \Phi(1)$. Since Φ' is strictly convex, $\Phi(1) = \int_0^1 \Phi'(t)dt < \frac{1}{2}\Phi'(1)$ and, by Jensen's inequality,

$$\Phi'(1) < \int_0^\infty \Phi'(t)e^{-t}dt = \int_0^\infty \Phi(t)e^{-t}dt .$$

So, in this case, (2.1) holds with strict inequality. If $\|g\|_\infty > 1$, then either (2.13) holds or (2.12) holds for some $n \ge 1$. So strict inequality holds in this case also.

That the constant on the right-hand side of (2.1) is best possible follows from the case $H = \mathbb{R}$; see [7]. This completes the proof of Theorem 2.1.

THEOREM 2.2. *If f and g are H-valued martingales with respect to the same filtration, g is differentially subordinate to f, and $\|f\|_\infty \le 1$, then, for $0 < \alpha < 1$,*

(2.14) $$\sup_n Ee^{\alpha|g_n|} + \alpha(1 - \|g\|_1) < \frac{1 - \alpha^2/2}{1 - \alpha}$$

and the constant on the right is best possible.

Note that $\|g\|_1 \le \|g_2\| \le \|f\|_2 \le \|f\|_\infty \le 1$. It can happen that $\sup_n Ee^{|g_n|} = \infty$.

PROOF. Let $\Phi(t) = e^{\alpha t} - 1 - \alpha t$ and $1 < p < 1/\alpha$. Then Φ^p satisfies the conditions of Theorem 2.1 so $(\Phi(|g_n|))_{n \ge 0}$ is an L^p-bounded, hence uniformly integrable, submartingale. Therefore, $\sup_n E\Phi(|g_n|) = E\Phi(|g_\infty|)$ where g_∞ is the almost sure limit of g. Accordingly, by Theorem 2.1,

$$E(e^{\alpha|g_\infty|} - 1 - \alpha|g_\infty|) < \frac{\alpha^2/2}{1 - \alpha}$$

and the constant on the right is best possible. The theorem follows.

REMARK 2.1. Let $g^*(\omega) = \sup_n |g_n(\omega)|$. Then, under the conditions of the above theorem,

$$Ee^{\alpha g^*} < \frac{e}{1 - \alpha}, \quad 0 < \alpha < 1 . \tag{2.15}$$

To prove (2.15), use $e^{\alpha t} = 1 + \alpha t + \alpha^2 t^2/2! + \ldots$, Doob's [17] inequality $\|g^*\|_p \le q\|g\|_p$ for $p = 2,3,\ldots$, and the inequality (2.2).

REMARK 2.2. There is an inequality that is dual to (2.1). Suppose that Φ is as above but with Φ' strictly concave. Then $g^* \ge 1$ a.s. implies that

$$\frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt < \sup_n E\Phi(|f_n|) . \tag{2.16}$$

The proof is similar to that of (2.1) and, as before, the inequality is sharp. For example, if $1 < p < 2$, then $\frac{1}{2}\Gamma(p + 1) < \|f\|_p^p$. The following theorem is closely related.

THEOREM 2.3. *Suppose that* f *and* g *are* H-*valued martingales with respect to the same filtration and* g *is differentially subordinate to* f. *Let* $1 \le p \le 2$. *Then*

$$\sup_{\lambda > 0} \lambda^p P(g^* \ge \lambda) \le 2\|f\|_p^p/\Gamma(p + 1) \tag{2.17}$$

and the constant $2/\Gamma(p + 1)$ *is best possible. Strict inequality holds if* $0 < \|f\|_p < \infty$ *and* $1 < p < 2$ *but equality can hold if* $p = 1$ *or* 2.

See [7] for (2.16) and (2.17) in the case $H = \mathbb{R}$ as well as related inequalities that carry over to the present setting.

3. APPLICATIONS TO THE MARTINGALE SQUARE FUNCTION

The inequalities of Sections 1 and 2 throw new light on the martingale square function. Let $f = (f_n)_{n \geq 0}$ be an H-valued martingale with difference sequence d. Its square function $S(f)$ is given by $S(f)(\omega) = (\sum_{k=0}^{\infty} |d_k(\omega)|^2)^{1/2}$.

THEOREM 3.1. *Let* $1 < p < \infty$. *Then*

$$(p^* - 1)^{-1}\|S(f)\|_p \leq \|f\|_p \leq (p^* - 1)\|S(f)\|_p \ . \tag{3.1}$$

In particular,

$$\|f\|_p \geq (p - 1)\|S(f)\|_p \quad \text{if} \quad 1 < p \leq 2 \ , \tag{3.2}$$

$$\|f\|_p \leq (p - 1)\|S(f)\|_p \quad \text{if} \quad 2 \leq p < \infty \ , \tag{3.3}$$

and the constant $p - 1$ *is best possible. If* $0 < \|f\|_p < \infty$, *then equality holds if and only if* $p = 2$.

This improves one of the inequalities of [2]. For further information about the cases not covered by (3.2) and (3.3), see Remark 3.1 below.

For the maximal function f^*, we have the inequality

$$\|f^*\|_p \leq p\|S(f)\|_p \ , \quad 2 \leq p < \infty \ , \tag{3.4}$$

in which strict inequality holds if $0 < \|f\|_p < \infty$. This follows from (3.3) and Doob's inequality $\|f^*\|_p \leq q\|f\|_p$, $1 < p < \infty$, which is a strict inequality when $0 < \|f\|_p < \infty$ as is obvious from his proof [17]. In (3.4), the constant p is best possible.

Klincsek [19] proved the inequality in (3.4) for $p = 3,4,5,\ldots$ and conjectured that it holds for $p \geq 2$. Pittenger [22] proved part of (3.3), namely, the case $p \geq 3$. In both [19] and [22], the proofs are given for $H = \mathbb{R}$ but can be carried over to any Hilbert space. Our approach is quite different and yields (3.3) and (3.4) for the full range $2 \leq p < \infty$ and, in addition, (3.2) for $1 < p \leq 2$.

PROOF OF THEOREM 3.1. Let $K = \ell^2_H$, the Hilbert space of sequences $x = (x_0, x_1, \ldots)$ with $x_j \in H$ and

$$|x|_K = \left(\sum_{j=0}^{\infty} |x_j|^2\right)^{1/2} < \infty .$$

Let $F = (F_n)_{n \geq 0}$ be the K-valued martingale with difference sequence $D = (D_k)_{k \geq 0}$ defined by $D_k(\omega) = (d_k(\omega), 0, 0, \ldots)$. Let G and E be defined similarly but with $E_k(\omega) = (0, \ldots, 0, d_k(\omega), 0, \ldots)$ where $d_k(\omega)$ is the k-th term in this sequence. Note that

$$|D_k(\omega)|_K = |E_k(\omega)|_K . \tag{3.5}$$

Also, $F_n = (f_n, 0, 0, \ldots)$ and $G_n = (d_0, \ldots, d_n, 0, \ldots)$ so

$$|F_n|_K = |f_n| \quad \text{and} \quad |G_n|_K = \left(\sum_{k=0}^{n} |d_k|^2\right)^{\frac{1}{2}} . \tag{3.6}$$

Therefore, $\|F\|_p = \|f\|_p$ and $\|G\|_p = \|S(f)\|_p$. If f is a martingale relative to a filtration $\mathfrak{F}$, then F and G are martingales with respect to $\mathfrak{F}$. By (3.5), G is differentially subordinate to F and F is differentially subordinate to G. Consequently, by Theorem 1.1,

$$(p^* - 1)^{-1}\|G\|_p \leq \|F\|_p \leq (p^* - 1)\|G\|_p ,$$

which is the same as (3.1). The assertion about equality also follows at once from Theorem 1.1.

The martingale f of (5.79) in [7] shows that $p - 1$ is best possible for (3.2) and (3.3) and that p is best possible for (3.4).

REMARK 3.1. For the cases not covered by (3.2) and (3.3), the best constants are not yet known. However, their orders of magnitude are known. If $H = \mathbb{R}$, then as $p \to \infty$ on the left-hand side of (3.1), the order of magnitude of the best constant is the same as $p^{-1/2}$ (see [3]) and, as $p \to 1$ on the right-hand side, the order of magnitude is the same as the constant function 1 (see [11], [14], and [12]). These results can be carried over to a general Hilbert space H. In fact, Garsia's proof [18] of $\|S(f)\|_p \leq (2p)^{1/2}\|f^*\|_p$, $p \geq 2$, carries over without change. This inequality and the inequality (3.1) give, for $p \geq 2$,

$$\|S(f)\|_p \leq \min\{p - 1, (2p)^{1/2}q\}\|f\|_p \leq 2p^{1/2}\|f\|_p .$$

REMARK 3.2. As in the proof of (3.1), inequality (2.1), its dual (2.16), the exponential inequalities (2.14) and (2.15), and the weak-type inequality (2.17) lead to similar inequalities for $S(f)$. However, sharpness need not carry over. For example, the sharp inequality (2.17) implies that

$$\lambda P(S(f) \geq \lambda) \leq 2\|f\|_1 , \quad \lambda > 0 . \tag{3.7}$$

However, for $H = \mathbb{R}$, Cox [13] has shown that the best constant is not 2 but $e^{1/2}$.

4. QUADRATIC VARIATION IN CONTINUOUS TIME

Here and in the following section, we assume for simplicity that the Hilbert space H is separable. Let $(\Omega, \mathfrak{F}_\infty, P)$ be a complete probability space and $\mathfrak{F} = (\mathfrak{F}_t)_{t \geq 0}$ a filtration that is continuous on the right. Suppose that $\mathfrak{F}_0$ contains all $A \in \mathfrak{F}_\infty$ with $P(A) = 0$. In this setting, we consider an H-valued martingale $M = (M_t)_{t \geq 0}$ that is continuous on the right. The space H may be real or complex, but to study the quadratic variation of M we can assume that $H = \ell^2_{\mathbb{R}}$. So

$$M_t(\omega) = (M_t^1(\omega), M_t^2(\omega), \ldots) \in \ell^2_{\mathbb{R}}$$

and the square bracket [15] of M can be written as

$$[M,M]_t = \sum_{j=1}^{\infty} [M^j,M^j]_t \ , \quad 0 \le t \le \infty \ .$$

Let $S(M) = [M,M]_\infty^{1/2}$ and $\|M\|_p = \sup_t \|M_t\|_p$. Then, for example,

$$(4.1) \qquad (p^* - 1)^{-1}\|S(M)\|_p \le \|M\|_p \le (p^* - 1)\|S(M)\|_p \ ,$$

an inequality that follows from (3.1) by approximation (see Doléans [16]). By Theorem 3.1, this inequality is sharp on the left-hand side for $1 < p \le 2$ and on the right-hand side for $2 \le p < \infty$.

5. INEQUALITIES FOR H-VALUED STOCHASTIC INTEGRALS

Let $(\Omega,\mathcal{F}_\infty,P)$ and $\mathcal{F}$ be as in Section 4. We assume here that $M = (M_t)_{t \ge 0}$ is a martingale relative to $\mathcal{F}$ with right-continuous paths on $[0,\infty)$ and limits from the left on $(0,\infty)$. Let $U = (U_t)_{t \ge 0}$ and $V = (V_t)_{t \ge 0}$ be predictable processes. Consider, for H a separable Hilbert space, the following two cases: (i) the case in which U and V are H-valued and M is scalar-valued, and (ii) the case in which U and V are scalar-valued and M is H-valued. If $\int_{[0,\infty)} |U_t|^2 d[M,M]_t$ is finite a.s., as we shall always assume, then the stochastic integral $U \cdot M$ exists (see [15]) and

$$S(U \cdot M) = [U \cdot M, U \cdot M]_\infty^{1/2} = \Big[\int_{[0,\infty)} |U_t|^2 d[M,M]_t\Big]^{1/2} \ .$$

THEOREM 5.1. _Let_ $1 < p < \infty$. _If_, _for all_ $\omega \in \Omega$ _and_ $t \ge 0$,

$$(5.1) \qquad |V_t(\omega)| \le |U_t(\omega)| \ ,$$

then, _for both_ (i) _and_ (ii),

$$(5.2) \qquad \|V \cdot M\|_p \le (p^* - 1)\|U \cdot M\|_p \ .$$

The constant $p^* - 1$ _is best possible_.

The special case in which U, V, and M are all real-valued is proved in [7] and, by a different method, in [9].

In the proof of (5.2), we can assume that $\|U \cdot M\|_p$ is finite and that H is either $\ell^2_{\mathbb{R}}$ or $\ell^2_{\mathbb{C}}$. Also, we can assume (see Section 2 of [7]) that $M_0 \equiv 0$. Let $\mathcal{E}$ (for elementary) consist of all U of the form

$$\begin{aligned} U_t(\omega) &= a_k \quad \text{if} \quad T_{k-1}(\omega) < t \le T_k(\omega) \quad \text{and} \quad 1 \le k \le n \ , \\ &= 0 \quad \text{if} \quad t \notin (0, T_n(\omega)] \ , \end{aligned}$$

where $0 \equiv T_0 \le T_1 \le \ldots \le T_n$ are bounded stopping times taking only a finite number of values. In case (i), the constants $a_k \in H$. In case (ii), $a_k \in \mathbb{R}$ or $\mathbb{C}$. Theorem 5.1 follows at once from Theorem 1.1 and the following lemma.

LEMMA 5.1. <u>For both</u> (i) <u>and</u> (ii), <u>there exist</u> U^n <u>and</u> V^n <u>in</u> $\mathcal{E}$ <u>such that</u>

$$|V^n_t(\omega)| \le |U^n_t(\omega)| \ , \tag{5.3}$$

$$\lim_{n \to \infty} \|U^n \cdot M - U \cdot M\|_p = 0 \ , \tag{5.4}$$

$$\lim_{n \to \infty} \|V^n \cdot M - V \cdot M\|_p = 0 \ . \tag{5.5}$$

Note that if n is a positive integer, the same stopping times $T^n_0, T^n_1, \ldots, T^n_{\nu_n}$ can be used for defining U^n and V^n.

Consider the proof of (5.4). The case $\mathbb{R}$ - H (that is, the integrand is $\mathbb{R}$-valued and the integrator is H-valued) is proved in the same way that Bichteler [1] proves the case $\mathbb{R}$ - $\mathbb{R}$. The cases $\mathbb{C}$ - $\ell^2_{\mathbb{C}}$ and $\ell^2_{\mathbb{C}}$ - $\mathbb{C}$ then follow.

So (5.4) and (5.5) can be satisfied. Can (5.3) also be satisfied? By (5.1),

$$|V^n_t| - |U^n_t| \le |V^n_t| - |V_t| - |U^n_t| + |U_t| \le |V^n_t - V_t| + |U^n_t - U_t| \ . \tag{5.6}$$

Define $W^n \in \mathcal{B}$ by

$$W_t^n(\omega) = V_t^n(\omega) \quad \text{if} \quad |V_t^n(\omega)| \le |U_t^n(\omega)| \ ,$$
$$= |U_t^n(\omega)|V_t^n(\omega)/|V_t^n(\omega)| \quad \text{if} \quad |V_t^n(\omega)| > |U_t^n(\omega)| \ .$$

Note that $|W_t^n(\omega)| \le |U_t^n(\omega)|$. By (5.6),

$$|W_t^n - V_t| \le |U_t^n - U_t| + 2|V_t^n - V_t| \ .$$

Therefore, by (4.1),

$$\begin{aligned}\|W^n \cdot M - V \cdot M\|_p &\le (p^* - 1)\|S((W^n - V) \cdot M)\|_p \\ &\le (p^* - 1)[\|S((U^n - U) \cdot M)\|_p + 2\|S((V^n - V) \cdot M)\|_p] \\ &\le (p^* - 1)^2[\|U^n \cdot M - U \cdot M\|_p + 2\|V^n \cdot M - V \cdot M\|_p]\end{aligned}$$

and the right-hand side approaches 0 as $n \to \infty$. Therefore, we can replace V^n by W^n in (5.3) and (5.5) to obtain the desired result.

Other inequalities and applications will be presented elsewhere. But there are many natural questions that are as yet unanswered. We shall mention several in the next section.

6. SQUARE-FUNCTION SUBORDINATION

Can differential subordination be replaced by square-function subordination without changing the values of the best constants? To be specific, let M and N be H-valued right-continuous martingales with respect to the same filtration. Let $S_t(M) = [M,M]_t^{1/2}$ and $S(M) = S_\infty(M)$ as before. Consider the condition

(6.1) $$S_t(N) \le S_t(M) \ , \quad t \ge 0 \ ,$$

and the less restrictive condition

(6.2) $$S(N) \le S(M) \ .$$

Note that under the conditions of Theorem 5.1, $S_t(V \cdot M) \leq S_t(U \cdot M)$, $t \geq 0$. Does (6.1) imply that $\|N\|_p \leq (p^* - 1)\|M\|_p$? Does (6.2)? We conjecture that, at least for (6.1), the answer is positive. There are similar questions for exponential and weak-type inequalities and all of these questions are open even for $H = \mathbb{R}$. A positive result for Itô integrals $U \cdot B$ and $V \cdot B$, where B is Brownian motion, is given in [4]. But, for these problems, discontinuous and discrete-time martingales are more challenging.

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