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by Pal RÉVÉSZ

1. INTRODUCTION

Let $\mathcal{E} = \{\dots, E_{-2}, E_{-1}, E_0, E_1, E_2, \dots\}$ be a sequence of i.i.d.r.v.'s with

$$\mathbb{P}\{E_0 < x\} = \begin{cases} 0 & \text{if } x \leq 0, \\ F(x) & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

The sequence $\mathcal{E}$ is called a <u>random environment</u>. (The random sequence $\{\dots, E_{-2}, E_{-1}, E_0, E_1, E_2, \dots\}$ and a realization of it will be denoted by the same letter $\mathcal{E}$). For any fixed sample sequence of this environment define a random walk $R_0, R_1, \dots$ by $R_0 = 0$ and

$$\mathbb{P}_{\mathcal{E}}\{R_{n+1} = i+1 \mid R_n = i\} = 1 - \mathbb{P}_{\mathcal{E}}\{R_{n+1} = i-1 \mid R_n = i\} = E_i \quad (n = 0, 1, 2, \dots, \ i = 0, \pm 1, \pm 2, \dots).$$

In this paper the following conditions will be always assumed:

(i) there exists a $0 < a < 1/2$ such that $\mathbb{P}(a < E_0 < 1-a) = 1$,

(ii) $\mathbb{E} \log \frac{1-E_0}{E_0} = 0$,

(iii) $0 < \sigma^2 = \mathbb{E}\left(\log \frac{1-E_0}{E_0}\right)^2 < \infty$.

REMARK 1.1. In case of a simple symmetric random walk (i.e. if $\mathbb{P}(E_0 = 1/2) = 1$) we have $\sigma^2 = 0$. (i) clearly implies that $\sigma^2 < \infty$. We also mention that if (i) and (ii) hold and $\sigma^2 = 0$ then $\mathbb{P}(E_0 = 1/2) = 1$.

REMARK 1.2. Many of the following results can be proved replacing (i) by weaker conditions. We do not intend to discuss this question in the present paper.

Introduce the following notations:

$\xi(x,n) = \#\{k: 0 \leq k \leq n,\ R_k = x\}$,

$\xi(n) = \max_x \xi(x,n)$,

$\rho_0 = 0,\ \rho_1 = \min\{k: k > 0,\ R_k = 0\}$,

$\rho_2 = \min\{k: k > \rho_1,\ R_k = 0\},\ldots,$

$\rho_{j+1} = \min\{k: k > \rho_j,\ R_k = 0\},\ldots,$

$M(n) = \max_{0 \leq k \leq n} |R_k|,\ M^+(n) = \max_{0 \leq k \leq n} R_k,\ M^-(n) = -\min_{0 \leq k \leq n} R_k$

We recall a few known results.

THEOREM A. (Deheuvels, P.-Révész, P.(1986) and Révész, P.(1987)). For any $\varepsilon > 0$ there exists a r.v. $n_o = n_o(\varepsilon)$ such that

$$(1.1)\quad (\log n)^2(\log_2 n)^{-2-\varepsilon} \leq M(n) \leq (\log n)^2(\log_2 n)^{2+\varepsilon} \quad \text{a.s. if} \quad n \geq n_o,$$

$$(1.2)\quad M(n) \leq \frac{1+\varepsilon}{2\sigma^2}\frac{(\log n)^2}{\log_3 n} \quad \text{i.o. a.s.},$$

$$(1.3)\quad \xi(0,n) \geq \exp(\log n\ (\log_2 n)^{-1-\varepsilon}) \quad \text{a.s. if} \quad n \geq n_o,$$

$$(1.4)\quad \xi(0,n) \leq \exp(\log n\ (\log_2 n)^{-1+\varepsilon}) \quad \text{i.o. a.s.}$$

There exists $C = C(a) > 0$ such that

$$(1.5)\quad \xi(0,n) \leq \exp((1-C(\log_3 n)^{-1})\log n) \quad \text{i.o. a.s.}$$

where $\log_p n$ is the p-th iterated of log and the meaning of a.s. is: for almost all environment $\mathcal{E}$ the stated inequality holds with probability one.

The inequalities (1.3) and (1.4) describe how small can be $\xi(0,n)$. In fact they say that $\xi(0,n)$ can be and will be as small as n^{ε_n} where $\varepsilon_n \approx (\log_2 n)^{-1}$. (1.5) says that $\xi(0,n)$ will be i.o. very big. In fact $\xi(0,n)$ will be for some n as big as $n^{1-\sigma_n}$ where $\sigma_n = C(\log_3 n)^{-1}$. Our first result will give an upper bound of $\xi(0,n)$. In fact we prove our

THEOREM 1. <u>There exists a</u> $C > 0$ <u>such that</u>

$$\xi(0,n) \leq \exp((1-\theta_n)\log n) \qquad \underline{a.s.} \tag{1.6}$$

<u>for all but finitely many</u> n <u>where</u>

$$\theta_n = \exp(-C(\log_2 n)(\log_3 n)^{-1/2} \log_4 n).$$

REMARK 1.3. Note that $\theta_n \log n \to \infty$ but since $\theta_n << C(\log_3 n)^{-1}$ there is an essential gap between (1.5) and (1.6).

We are also interested to study the behaviour of $\xi(n)$. (1.1) and (1.2) clearly imply: <u>for any</u> $\varepsilon>0$ <u>we have</u>

$$\lim_{n\to\infty} (\log n)^2 (\log_2 n)^{2+\varepsilon} \frac{\xi(n)}{n} = \infty \qquad \underline{a.s.} \tag{1.7}$$

<u>and</u>

$$\limsup_{n\to\infty} \frac{\log^2 n}{(2\sigma^2 \log_3 n)} \cdot \frac{\xi(n)}{n} \geq 1 \qquad \underline{a.s.} \tag{1.8}$$

It looks obvious that much stronger lower bounds than those of (1.7) and (1.8) should exist. I do

CONYECTURE: there exists a $0 < C=C(F) < 1$ such that

$$\limsup_{n\to\infty} n^{-1} \xi(n) = C \qquad \underline{a.s.}$$

In fact the following much weaker result will be proved

THEOREM 2. <u>Let</u>

$$\mathbb{P}\{E_i=p\} = \mathbb{P}\{E_i=1-p\}=1/2 \qquad (0<p<1/2).$$

<u>Then</u>

$$\limsup_{n\to\infty} n^{-1} \xi(n) \geq g(p) \qquad \underline{a.s.}$$

<u>where</u>

$$1/g(p) = \frac{16}{p} f(x)+1,$$

$$f(x) = \frac{2x^2-x+1}{(1-x)^3} \quad \text{and} \quad x=\frac{p}{1-p}.$$

2. PROOF OF THEOREM 1.

Introduce the following notations

$$U_j = \frac{1-E_j}{E_j} \qquad (j=0,\pm 1,\pm 2,\ldots),$$

$$V_j = \log U_j \qquad (j=0,\pm 1,\pm 2,\ldots),$$

$$T_0=0,\ T_n=T(n) = V_1+V_2+\ldots+V_n,\ T_{-n}=V_{-1}+V_{-2}+\ldots+V_{-n},$$

$$D(a,b) = \begin{cases} 0 & \text{if } b=a \\ 1 & \text{if } b=a+1 \\ 1+U_{a+1}+U_{a+1}U_{a+2}+\ldots+U_{a+1}U_{a+2}\ldots U_{b-1}, & \end{cases}$$

$D(b)=D(0,b)$.

Observe that

$$(2.1)\quad \exp\Big(\max_{0\le k\le n-1} T(k)\Big) \le D(n) = \sum_{k=0}^{n-1} \exp(T_k) \le n\exp\Big(\max_{0\le k\le n-1} T(k)\Big).$$

The proof is based on the following three lemmas.

LEMMA 1. _For all but finitely many_ n _and for any_ $\varepsilon>0$ _with probability one at most one of the following two inequalities can hold_

$$T(n) \le -(2\sigma n\log_2 n)^{1/2}, \qquad \max_{0\le k\le n} T(k) \ge \varepsilon(n\log_2 n)^{1/2}.$$

PROOF. It is a trivial consequence of the Strassen's law of iterated logarithm.

LEMMA 2. _Let_

$$p(a,b,c) = \mathbb{P}_{\mathcal{E}}\{\min\{j:\ j>m,\ R_j=a\} < \min\{j:\ j>m,\ R_j=c\} \mid S_m=b\}$$

$(a \leq b \leq c)$ i.e. $p(a,b,c) = p(a,b,c,\ )$ is the probability that a particle starting from b hits a before c given the environment $\mathcal{E}$. Then

$$p(a,b,c) = 1-\frac{D(a,b)}{D(a,c)}.$$

PROOF. This lemma was proved by Deheuvels-Révész(1986) in the case a=0, b=1. The proof of the general case is going on the same line.

LEMMA 3. (Erdös, P.-Révész, P.(1987)). Let

$$\psi(N) = \max\{n:\ 0\leq n\leq N,\ T(n)\leq -\sigma(2n\log_2 n)^{1/2}\}.$$

Then there exists a $C>0$ such that

$$\psi(N) \geq \exp((1-C(\log_3 N)(\log_2 N)^{-1/2})\log N) \qquad \text{a.s.}$$

for all but finitely many N.

PROOF OF THEOREM 1. Introduce the following notations

$$A_n = \{\xi(0,n)\geq n^{1/2}\} \qquad (n=1,2,\ldots),$$

$$N=N(n) = [(\log n)^2(\log_3 n)^{-1}],$$

$$M^+(\rho_j,\rho_{j+1}) = \max_{\rho_j\leq k\leq \rho_{j+1}} R_k \qquad (j=0,1,2,\ldots),$$

$$\tilde{\xi}(x,n)=\#\{j:\ 1\leq j\leq \xi(0,n)-1,\ M^+(\rho_{j-1},\rho_j)\geq x\},$$

$$\tilde{\xi}(\psi(N),n)=\xi_n \qquad (N=N(n)),$$

$$B_n = \{\xi_n \leq E_0(\xi(0,n)-1)\frac{1}{2D(\psi(N))}\}.$$

Note that by Lemma 1 for any $\varepsilon>0$

$$\max_{0\leq k\leq \psi(N)} T(k) \leq \varepsilon\,(\psi(N)\log_2\psi(N))^{1/2} \qquad \text{a.s.}$$

for all but finitely many N(i.e. n). Hence by Lemma 2 and (2.1) (since $\psi(N)\leq N$) we have

$$\mathbb{P}\{M^+(\rho_1) \geq \psi(N)\} = \frac{E_0}{D(\psi(N))} \geq \frac{E_0}{\psi(N)} \exp(-\max_{0 \leq k \leq \psi(N)-1} T(k)) \geq$$

$$\geq \frac{E_0}{\psi(N)} \exp(-\varepsilon(\psi(N)\log_2\psi(N))^{1/2}) \geq \frac{E_0}{N} \exp(-\varepsilon(N\log_2 N)^{1/2}) =$$

$$= O(\frac{\log_3 n}{(\log n)^2} n^{-\varepsilon})$$

and by an elementary calculation one gets

$$\mathbb{P}(B_n \mid \xi(0,n)) \leq \exp(-\frac{1}{8}(\xi(0,n)-1)\frac{E_0}{D(\psi(N))}) \leq$$

$$\leq \exp(-\xi(0,n)\ O(\frac{\log_3 n}{(\log n)^2} n^{-\varepsilon}))$$

Consequently

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n \mid A_n) < \infty .$$

Hence

$$(2.2) \qquad \xi(0,n) \leq n^{1/2} \quad \text{or} \quad \xi_n \geq E_0(\xi(0,n)-1)\frac{1}{2D(\psi(N))} \qquad \text{a.s.}$$

for all but finitely many n. Applying Lemma 2 again we have

$$(2.3) \qquad p(0,\psi(N)-1,\ \psi(N)) = 1-\frac{D(\psi(N)-1)}{D(\psi(N))} \leq \exp(-\sigma(2\psi(N)\log_2\psi(N))^{1/2}).$$

In case if ξ_n satisfies the second inequality of (2.2) then by (2.3) and Lemma 3 we obtain

$$n \geq \xi(\psi(N),n) \geq \frac{(1-E_{\psi(N)})\xi_n}{2p(0,\psi(N)-1,\ \psi(N))} \geq O(1)\frac{\xi(0,n)}{D(\psi(N))-D(\psi(N)-1)} =$$

$$=O(1)\,\xi(0,n)\exp(-T(\psi(N)-1)) \geqq O(1)\,\xi(0,n)\exp(\sigma(2\,\psi(N)\log_2\psi(N))^{1/2}) \geqq$$

$$\geqq O(1)\,\xi(0,n)\exp(\sigma(2\exp((1-C\log_3 N(\log_2 N)^{-1/2})\log N)\log_2 N)^{1/2}) \geqq$$

$$\geqq O(1)\,\xi(0,n)\exp(\sigma 2^{1/2}\log n\ \exp(-2c(\log N)(\log_3 N)(\log_2 N)^{-1/2}))\quad \text{a.s.}$$

for all but finitely many n. Hence we have Theorem 1.

3. A LEMMA ON SIMPLE SYMMETRIC RANDOM WALK.

In order to formulate our lemma we introduce the following definitions.

Let $\ldots X_{-2}, X_{-1}, X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with

$$\mathbb{P}(X_1=+1) = \mathbb{P}(X_1=-1) = 1/2,$$

$S_0=0$, $S_n=X_1+X_2+\ldots+X_n$, $S_{-n}=X_{-1}+X_{-2}+\ldots+X_{-n}$ $(n=1,2,\ldots)$.

Let N be a positive integer and define

$$\nu_N^+ = \min\{k:\ k>0,\ S_k=N\},$$

$$\nu_N^- = \max\{k:\ k>0,\ S_k=N\},$$

$$\mu_N = -\min\{S_k:\ \nu_N^- \leqq k \leqq \nu_N^+\},$$

$$\alpha_N = \max\{k:\ \nu_N^- \leqq k \leqq \nu_N^+,\ S_k=-\mu_N\},$$

$$\tau_N^- = \max\{k:\ \nu_N^- \leqq k \leqq \alpha_N,\ S_k+\mu_N=N\},$$

$$\tau_N^+ = \min\{k:\ \alpha_N \leqq k \leqq \nu_N^+,\ S_k+\mu_N=N\},$$

$$L_N(j)=\#\{k:\ \tau_N^- \leqq k \leqq \tau_N^+,\ S_k=-\mu_N+j\}\qquad (j=0,1,2,\ldots,N-1),$$

$$U_N = \max\{S_j-S_i:\ \tau_N^- \leqq i<j \leqq \alpha_N\},$$

$$V_N = \max\{S_i-S_j:\ \alpha_N \leqq i<j \leqq \tau_N^+\},$$

$$\rho = \min\{k:\ k>0,\ S_k=0\}.$$

LEMMA 4. *Assume that* $X_1=1$. *Let*

$$N = \max_{0\le k\le\rho} S_k,$$

$$x(j)=\#\{k:\ 0<k<\rho,\ S_k=N-j\}\qquad (j=0,1,2,\dots,N-1)$$

and

$$q_j = q_j(N) = \mathbb{P}\{x(j)\ge 4j^2+4 \mid N\}.$$

Then there exists an absolute constant $0<\theta<1$ *such that*

$$\sum_{j=0}^{N-1} q_j(N) \le \theta \qquad (N=2,3,\dots).$$

PROOF. A simple combinatorial argument gives

$$\mathbb{P}(x(0)=k,\ N=N_o) = \frac{1}{2N_o^2}\left(\frac{N_o-1}{2N_o}\right)^{k-1}, \qquad \mathbb{P}(N=N_o) = \frac{1}{N_o(N_o+1)}$$

and

$$\mathbb{P}(x(0)=k \mid N) = \frac{N+1}{2N}\left(\frac{N-1}{2N}\right)^{k-1}. \tag{3.1}$$

Consequently

$$q_o = \mathbb{P}(x(0)\ge 4 \mid N) = \left(\frac{N-1}{2N}\right)^3 < \frac{1}{8}. \tag{3.2}$$

Let

$$\nu=\nu(N_o) = \min\{k:\ S_k=N_o\}$$

and

$$\lambda(m) = \#\{k:\ 0<k<\nu,\ S_k=N_o-m\}\qquad (m=1,2,\dots,N_o-1).$$

Then again by a simple combinatorial argument we have

$$\mathbb{P}\{\lambda(m)=k,\ N\ge N_o \mid N\ge N_o-m\} = \frac{1}{2m}\left(1-\frac{N_o}{2m(N_o-m)}\right)^{k-1}.$$

Consequently

$$\mathbb{P}\{\lambda(m)=k \mid N=N_0\} = \mathbb{P}\{\lambda(m)=k \mid N \geq N_0\} = \frac{N_0}{2m(N_0-m)} \left(1-\frac{N_0}{2m(N_0-m)}\right)^{k-1},$$

$$\mathbb{P}\{\lambda(m) \geq 2m^2+2 \mid N=N_0\} = \left(1-\frac{N_0}{2m(N_0-m)}\right)^{2m^2+1}.$$

Observe that

$$(3.3) \qquad \sum_{m=1}^{N_0-1} \left(1-\frac{N_0}{2m(N_0-m)}\right)^{2m^2+1} \leq 1/4 \qquad (N_0=2,3,\dots)$$

Because of symmetry we have

$$\mathbb{P}\{x(m) \geq 4m^2+4 \mid x(0)=1,\ N\} \leq 2\mathbb{P}\{\lambda(m) \geq 2m^2+2 \mid N\}$$

and by (3.1)

$$\mathbb{P}\{x(m) \geq 4m^2+4 \mid N\} = \mathbb{P}\{x(m) \geq 4m^2+4 \mid N, x(0)=1\}\ \mathbb{P}\{x(0)=1 \mid N\} +$$

$$+ \mathbb{P}\{x(m) \geq 4m^2+4 \mid N,\ x(0)>1\}\ \mathbb{P}(x(0)>1 \mid N) \leq$$

$$\leq 2\mathbb{P}\{\lambda(m) \geq 2m^2+2 \mid N\} \cdot \frac{N+1}{2N} + \frac{1}{2}.$$

Similarly by (3.3)

$$(3.4) \qquad \mathbb{P}\{\bigcup_{m=1}^{N-1} \{x(m) \geq 4m^2+4\} \mid N\} \leq \frac{1}{2} + \frac{N+1}{N} \cdot \frac{1}{4}.$$

Lemma 4 follows from (3.2) and (3.4).

LEMMA 5. _There exists an absolute constant_ Θ $(0 < \Theta < 1)$ _such that_

$$\mathbb{P}\{L_N(j) \leq 4j^2+4,\ j=0,1,2,\dots,N-1;\ U_N \leq \frac{N}{2},\ V_N \leq \frac{N}{2}\} \geq \Theta \qquad (N=2,3,\dots).$$

PROOF. Lemma 4 implies that

$$(3.5) \qquad \mathbb{P}\{L_N(j) \leq 4j^2+4,\ j=0,1,2,\dots,N-2\}$$

is larger than an absolute positive constant independent from N. It is easy to see that

(3.6) $$\mathbb{P}\{U_N \leq \frac{N}{2},\ V_N \leq \frac{N}{2}\}$$

is also larger than an absolute positive constant independent from N and the events involved in (3.5) and (3.6) are asymptotically independent as $N \to \infty$. Hence we have Lemma 5.

4. A FEW LEMMAS

Observe that replacing the sequence $\ldots, X_{-2}, X_{-1}, X_1, X_2, \ldots$ by the sequence $\ldots, U_{-2}, U_{-1}, U_1, U_2, \ldots$ and the definition of $L_N(j)$ by the following definition

$$L_N(j) = \#\{k:\ \tau_N^- \leq k \leq \tau_N^+,\ T_k = S_k = -\mu_N + j\,|\log \frac{p}{1-p}|\}$$

$$(j = 0, 1, 2, \ldots, (N-1)\,(\,|\log \frac{p}{1-p}|\,)^{-1})$$

then Lemma 5 remains true as it is.

For sake of simplicity from now on we assume that $\alpha_N > 0$ and introduce the following notations:

let $N = N_k(\mathcal{E})$ be a sequence of positive integers for which

$L_N(j) \leq 4j^2 + 4\ (j = 0, 1, 2, \ldots, N-1)$, $U_N \leq \frac{N}{2}$ and $V_N \leq \frac{N}{2}$.
(by Lemma 5 for almost all $\mathcal{E}$ there exists such an infinite sequence),

$$F_N = \min\{k:\ k > 0,\ R_k = \alpha_N\},$$

$$G_N = \min\{k:\ k > 0,\ R_k = \nu_N^-\},$$

$$H_N = \min\{k:\ k > F_N,\ R_k = \tau_N^- \text{ or } \tau_N^+\} - F_N.$$

LEMMA 6. <u>Let $\mathcal{E}$ be fixed. Then</u>

(4.1) $$\xi(0, F_N) \leq e^{2/3N},\quad G_n \geq \xi(0, G_N) \geq e^{3/4N}$$

<u>and</u>

(4.2) $H_N > \xi(\alpha_N, H_N+F_N) \geq e^{3/4N}, \quad F_N \leq e^{2/3N}$

($N=N_k$) <u>a.s. for all but finitely many</u> k. <u>Consequently</u>

(4.3) $F_N = o(G_N)$ <u>a.s.</u>

PROOF. Since

$$\mathbb{P}\{M^+(\rho_1)\geq\alpha_N\} = E_0(1-p(0,1,\alpha_N)) = \frac{E_0}{D(\alpha_N)} \geq \frac{E_0}{\alpha_N}\exp(-\max_{0\leq k\leq\alpha_N-1} T(k)) \geq$$

$$\geq \frac{E_0}{\alpha_N}\exp(-U_N) \geq \frac{E_0}{\alpha_N}\exp(-\frac{N}{2})$$

and

$$\mathbb{P}\{M^-(\rho_1) \geq -\nu_N^-\} = (1-E_0)\ p(\nu_N^-,-1,0) \leq (1-E_0)\exp(-\max_{\nu_N^-\leq k\leq 0} T(k)) =$$

$$= (1-E_0)\ e^{-N}.$$

Hence by Borel-Cantelli lemma one can easily obtain (4.1) and (4.3). Similarly one can obtain the first inequality of (4.2). In order to prove the second inequality of (4.2) observe that (by (4.3))

$$F_N = \sum_{k=-\infty}^{\alpha_N-1} \xi(k,F_N) = \sum_{k=\nu_N^-}^{\alpha_N-1} \xi(k,F_N) \leq (\alpha_N-1-\nu_N^-)\exp((1+\varepsilon)\frac{N}{2})$$

what proves Lemma 6 completely.

Introduce the following notations:

$$\frac{1}{D^*(n)} = p(0,n-1,n) = 1-\frac{D(n-1)}{D(n)}$$

i.e.

$$D^*(n) = e^0+\exp(-(T_{n-1}-T_{n-2}))+\exp(-(T_{n-1}-T_{n-3}))+\ldots+\exp(-(T_{n-1}-T_0)) =$$

$$=D(n)\exp(-T_{n-1}),$$

$$m_k = \mathbb{E}_{\mathcal{E}}\, \xi(k,\rho_1), \qquad \sigma_k^2 = \mathbb{E}_{\mathcal{E}}(\xi(k,\rho_1)-m_k)^2,$$

$$\varphi(\lambda) = \varphi_k(\lambda) = \mathbb{E}_{\mathcal{E}}\ \exp(\lambda\xi(k,\rho_1)).$$

Then we have

LEMMA 7. (Csörgö, M.-Horváth, L.-Révész, P. (1987)). *We have*

$$(4.4)\qquad m_k = \frac{E_0}{1-E_k}\cdot\frac{D^*(k)}{D(k)} = \frac{E_0}{1-E_k}\ \exp(-T_{k-1}) \qquad (k=1,2,\ldots),$$

$$(4.5)\qquad \sigma_k^2 = \frac{E_0}{(1-E_k)^2}\,\frac{(D^*(k))^2}{D(k)}\ \left(2-\frac{1-E_k}{D^*(k)} - \frac{E_0}{D(k)}\right) \qquad (k=1,2,\ldots),$$

$$(4.6)\qquad \varphi(\lambda) = 1-\frac{E_0}{D(k)} + \frac{E_0(1-E_k)}{D(k)D^*(k)}\ \frac{e^{\lambda}}{1-e^{\lambda}\left(1-\frac{1-E_k}{D^*(k)}\right)}$$

for any $k=1,2,\ldots$ *and* $\lambda < -\log\left(1-\frac{1-E_k}{D^*(k)}\right)$. *Especially*

$$(4.7)\qquad \varphi\left(\frac{1-E_k}{2D^*(k)}\right) = 1+\frac{E_0}{D(k)}\ \left(\frac{2\lambda e^{\lambda}}{1-e^{\lambda}(1-2\lambda)} -1\right).$$

Observe that

$$(4.8)\qquad 0<\lambda=\lambda_k = \frac{1-E_k}{2D^*(k)} < 1/2$$

and

$$(4.9)\qquad \sqrt{e} \le \frac{2\lambda e^{\lambda}}{1-e^{\lambda}(1-2\lambda)} \le 2 \qquad \textit{if} \qquad 0<\lambda<1/2.$$

LEMMA 8. *For any* $C_1 \ge 4a^{-1}$ *we have*

$$\mathbb{P}_{\mathcal{E}}\{\xi(k,\rho_n) \ge C_1 n\,\frac{D^*(k)}{D(k)}\} \le \exp\left(-\frac{n}{D(k)}\right) \qquad (k=1,2,\ldots;\ n=1,2,\ldots).$$

PROOF. By (4.7), (4.8) and (4.9)

$$\mathbf{P}_{\mathcal{E}}\{\xi(k,\rho_n) \geq C_1 n \frac{D^*(k)}{D(k)}\} =$$

$$= \mathbf{P}_{\mathcal{E}}\{\exp(\lambda\xi(k,\rho_n)) \geq \exp(\lambda C_1 n \frac{D^*(k)}{D(k)})\} \leq$$

$$\leq \exp(-\lambda C_1 n \frac{D^*(k)}{D(k)}) \, \mathbf{E}_{\mathcal{E}}(\exp \lambda\xi(k,\rho_n)) =$$

$$= [\exp(-\lambda C_1 \frac{D^*(k)}{D(k)}) \, \mathbf{E}_{\mathcal{E}}(\exp \lambda\xi(k,\rho_1))]^n =$$

$$= [\exp(-\lambda C_1 \frac{D^*(k)}{D(k)})(1+ \frac{E_0}{D(k)} (\frac{2\lambda e^{\lambda}}{1-e^{\lambda}(1-2\lambda)} -1))]^n \leq \exp(-\frac{n}{D(k)})$$

where $\lambda=\frac{1-E_k}{2D^*(k)}$. Hence we have Lemma 8.

In case when k can be very big it is worth while to formulate

LEMMA 9. <u>For any</u> $K>0$ <u>there exists a</u> $C=C(K)>0$ <u>such that</u>

$$\mathbf{P}_{\mathcal{E}}\{\xi(k,\rho_n) \geq 2nm_k + C\, D^*(k)\log n\} \leq n^{-K}$$

$(k=1,2,\ldots;\ n=1,2,\ldots)$.

PROOF. By (4.7), (4.8) and (4.9) we have

$$\mathbf{P}_{\mathcal{E}}\{\xi(k,\rho_n) \geq 2nm_k + C\, D^*(k)\log n\} =$$

$$= \mathbf{P}_{\mathcal{E}}\{\exp(\lambda\xi(k,\rho_n)) \geq \exp(2\lambda nm_k + \lambda C D^*(k)\log n)\} \leq$$

$$\leq (\mathbf{E}_{\mathcal{E}} \exp(\lambda\xi(k,\rho_1)))^n \exp(-2\lambda nm_k - \lambda\, CD^*(k)\log n) \leq$$

$$\leq \exp(n\frac{E_0}{D(k)} - 2n\frac{1-E_k}{2D^*(k)}\frac{E_0}{1-E_k}\frac{D^*(k)}{D(k)} - \frac{1-E_k}{2D^*(k)} CD^*(k)\log n) =$$

$$= \exp(-\frac{1-E_k}{2} C\log n)$$

where $\lambda=\frac{1-E_k}{2D^*(k)}$. Hence we have Lemma 9.

Introduce the following further notations

$\hat{\rho}_1=\hat{\rho}_1(N) = \min\{n: n > 0, \quad R_{F_N+n} = \alpha_N\},$

$\hat{\rho}_2=\hat{\rho}_2(N) = \min\{n: n > \hat{\rho}_1, \quad R_{F_N+n} = \alpha_N\},$

....

$\hat{\rho}_{m+1}= \hat{\rho}_{m+1}(N) = \min\{n: n > \hat{\rho}_m, \quad R_{F_N+n} = \alpha_N\},\ldots$

$$D(j,N) = (p(j, \alpha_N-1, \alpha_N))^{-1} = \frac{D(j,\alpha_N)}{D(j,\alpha_N)-D(j, \alpha_N-1)} =$$

$$= 1-\exp(-(T(\alpha_N-1)-T(\alpha_N-2)))+\ldots+\exp(-(T(\alpha_N-1)-T(j)))$$

and

$$\hat{D}^*(j,N) = (1-p(j,j+1,\alpha_N))^{-1} = D(j, \alpha_N).$$

Observe that

$$(4.10) \qquad \frac{\hat{D}^*(j,N)}{\hat{D}(j,N)} = \frac{p(j, \alpha_N-1, \alpha_N)}{1-p(j,j+1, \alpha_N)} = D(j,\alpha_N) - D(j,\alpha_N-1) =$$

$$= U_{j+1} \ U_{j+2} \cdots U_{\alpha_N-1}.$$

In the same way as Lemmas 8 and 9 were proved one can prove

LEMMA 10. <u>For any</u> $j < \alpha_N$ <u>we have</u>

$$(4.11) \qquad \mathbb{P}_{\mathcal{C}}\{\hat{\xi}(j, \hat{\rho}_n) \geq C_1 n \frac{\hat{D}^*(j,N)}{\hat{D}(j,N)}\} \leq \exp(-\frac{n}{\hat{D}(j,N)})$$

<u>where</u> $C_1 \geq 4p^{-1}$ <u>and</u>

$$\hat{\xi}(j, \hat{\rho}_n) = \xi(j, F_N+ \hat{\rho}_n)-\xi(j, F_n)$$

<u>further for any</u> $K > 0$ <u>there exists a</u> $C=C(K)>0$ <u>such that</u>

$$(4.12)\quad \mathbf{P}_{\mathcal{E}}\{\hat{\xi}(j,\rho_n) \geq 2n\,\frac{1-E_{\alpha_N}}{E_j}\cdot\frac{\hat{D}^*(j,N)}{\hat{D}(j,N)} + C\hat{D}^*(j,N)\log n\} \leq n^{-K}.$$

5. PROOF OF THEOREM 2.

In order to simplify the notations from now on we assume that $\tau_n^- > 0$. (The case $\tau_n^- \leq 0$ can be treated similarly).

Let $2/3 < \psi_1 < \psi_2 < 3/4$ and introduce the following notations

$$n=\exp(\psi_2 N) \qquad\qquad (\text{where } N=N_k=N_k(\mathcal{E}))$$

and

$$\gamma(j) = \min\{k:\ \tau_N^- < k < \alpha_N,\ T_k=-\mu_N+j\,|\log\frac{p}{1-p}|\}.$$

Consider any integer $k\in(\gamma(\psi_1 N),\ \alpha_N)$. Then

$$\hat{D}(k,n) = \alpha_N \exp(\max_{\gamma(\psi_1 N)<j<\alpha_N}(T_j-T_{\alpha_N-1})) = \alpha_N\exp(\psi_1 N)$$

and by (4.11)

$$\mathbf{P}_{\mathcal{E}}\{\hat{\xi}(k,\rho_n^*) \geq C_1 n\,\frac{\hat{D}^*(k,N)}{\hat{D}(k,N)}\} \leq \exp(-\exp(\psi_2-\psi_1)N).$$

Consequently by (4.10)

$$(5.1)\quad \sum_{k=\gamma(\psi_1 N)}^{\alpha_N-1}\hat{\xi}(k,\rho_n) \leq C_1 n\sum_{k=\gamma(\psi_1 N)}^{\alpha_N-1}\frac{\hat{D}^*(k,N)}{\hat{D}(k,N)} =$$

$$= C_1 n\sum_{k=\gamma(\psi_1 N)}^{\alpha_N-1}\exp(T(\alpha_N-1)-T(k)) \leq$$

$$\leq C_1 n\sum_{j=0}^{\infty}(4j^2+4)\exp(-j\,|\log\frac{p}{1-p}|) =$$

$$= 4C_1 n\left(\frac{2x^2}{(1-x)^3}+\frac{x}{(1-x)^2}+\frac{1}{1-x}\right) = 4C_1 n\,\frac{2x^2-x+1}{(1-x)^3} \qquad \text{a.s.}$$

if N is big enough where $x=\exp(-|\log \frac{p}{1-p}|)$.

Let $k\in(\tau_N^-,\ \gamma(\psi_1 N))$. Then by (4.12)

$$P_{\mathcal{C}}\{\hat{\xi}(k,\hat{\rho}_n) \geq 2n\frac{1-E_{\alpha_N}}{E_k}\frac{\hat{D}^*(k,N)}{\hat{D}(k,N)} + C\hat{D}^*(k,N)\log n\} \leq n^{-K}.$$

Consequently

$$(5.2)\qquad \sum_{k=\tau_N^-}^{\gamma(\psi_1 N)} \hat{\xi}(k,\hat{\rho}_n) \leq 2(1-E_{\alpha_N})n \sum_{k=\tau_N^-}^{\gamma(\psi_1 N)} \frac{\hat{D}^*(k,N)}{E_k\hat{D}(k,n)} +$$

$$+ C\log n \sum_{k=\tau_N^-}^{\gamma(\psi_1 N)} \hat{D}^*(k,N) \leq$$

$$\leq \frac{2(1-a)}{a} n\ \alpha_N \exp(-\psi_1 N) + C(\log n)\,\alpha_N \exp(\tfrac{N}{2}) = o(n) \qquad \text{a.s.}$$

(5.1) and (5.2) combined imply

$$(5.3)\qquad \sum_{k=\tau_N^-}^{\alpha_N-1} \hat{\xi}(k,\hat{\rho}_n) \leq 4C_1 n\ f(x) \qquad \text{a.s.}$$

where

$$f(x) = \frac{2x^2-x+1}{(1-x)^3}$$

and

$$x = \exp(-|\log \frac{p}{1-p}|) = \frac{p}{1-p}.$$

Similarly one can see that

$$\sum_{k=\alpha_N+1}^{\tau_N^+} \hat{\xi}(k,\hat{\rho}_N) \leq 4C_1 n\ f(x) \qquad \text{a.s.}$$

Hence

$$\rho_n = \sum_{j=-\infty}^{\infty} \hat{\xi}(j,\hat{\rho}_n) = \sum_{j=\tau_N^-}^{\tau_N^+} \hat{\xi}(j,\hat{\rho}_n) \leqq$$

$$\leqq (4C_1 f(x)+1)n.$$

Let $(4C_1 f(x)+1)n=m$ then for any $\varepsilon > 0$ we have

$$\xi((1+\varepsilon)m) \geqq \xi(F_N+m) \geqq \xi(F_N+\hat{\rho}_n) \geqq \xi(\alpha_N, F_N+\hat{\rho}_n)=n = \frac{m}{4C_1 f(x)+1}$$

what proves the Theorem.

REMARK 5.1. In fact we have proved a stronger result than Theorem 2. It can be formulated as follows:

THEOREM 2^*. For almost all environment ξ there exists a sequence of positive integers $n_1=n_1(\xi) < n_2=n_2(\xi) < \ldots$ such that

$$\xi(n_k) \geq (1-\varepsilon)\frac{n_k}{4C_1 \ f(x)+1} \qquad \text{a.s.}$$

for any $\varepsilon > 0$ and for all but finitely many k.

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