Astérisque

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Astérisque, tome 157-158 (1988), p. 217-231
<http://www.numdam.org/item?id=AST_1988__157-158__217_0>

Société Mathématique de France
Astérisque 157-158 (1988)

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Abstract: We discuss some recent results in (Bernoulli) percolation. In particular these include

(i) $p_T = p_H$. This equality (proved in [1] and [21]) between two differently defined critical probabilities solves one of the major problems of the subject.

(ii) Uniqueness of infinite clusters. With probability one there exists at most one infinite open cluster; this holds in independent models in any dimension.

(iii) Scaling relations for two-dimensional percolation. Some relations between the singularities of various quantities near the critical probability can be proven; these show that in two dimensions most of the conjectured scaling relations between critical exponents have to be true, provided the critical exponents exist.

1. The simplest model. The simplest percolation model is bond percolation on $\mathbb{Z}^d$. In this model each bond (also called edge) of $\mathbb{Z}^d$ is open with probability p and closed with probability

$q := 1-p$; all bonds are independent of each other. P_p will denote the corresponding product measure on configurations of open and closed bonds, and E_p will denote expectation with respect to P_p. An <u>open path</u> is a path on $\mathbb{Z}^d$ all of whose edges are open. An <u>open cluster</u> is a maximal set of vertices of $\mathbb{Z}^d$ any two of which are connected by an open path. We shall write W for the open cluster containing $\underline{0}$, the origin, and $|W|$ for the number of vertices in C.

This model is of interest, especially to statistical physicists, because it exhibits a so-called <u>phase transition</u>. For small values of p there are no infinite open clusters, while for large values of p such infinite open clusters do occur. In the latter case we also say that the system percolates or that percolation occurs. The value of p where the transition between these two phases takes place is called the <u>critical probability</u>, p_c. Thus, if we define the <u>percolation probability</u> as

$$\theta(p) = P_p\{|W| = \infty\},$$

then

$$p_c = p_c(\mathbb{Z}^d) = \sup\{p : \theta(p) = 0\}.$$

$\theta(\cdot)$ is believed to have a graph as shown in Fig. 1.

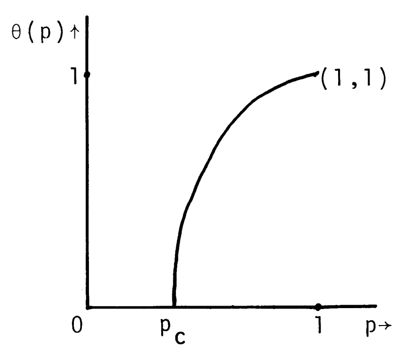

Fig. 1. General features of the graph of $\theta(\cdot)$.

Broadbent and Hammersley ([5], [15]), the originators of the subject, proved already in 1959 that $0 < p_c < 1$ so that both phases indeed occur. Much of the early work in percolation (see for instance [16], [28], [23], [25], [17], [32]) consisted of attempts to find p_c exactly. Sykes and Essam, [28], gave an incomplete argument for

$$(1) \qquad p_c(\mathbb{Z}^2) = \frac{1}{2} ,$$

but this was rigorously proved in 1980 ([17]).

A key step in the proof of (1) was to show that $p_c(\mathbb{Z}^2)$, equals another critical probability $p_T(\mathbb{Z}^2)$ which is defined as follows (for general dimension d):

$$\chi(p) = E_p\{|W|\},$$

$$p_T = p_T(\mathbb{Z}^d) = \sup\{p:\chi(p) = \infty\}.$$

Note that for $p > p_c$

$$\chi(p) \geq \infty.P_p\{|W| = \infty\} = \infty.\theta(p) = \infty$$

so that always $p_T \leq p_c$. In addition the proof of (1) relied heavily on graph duality, which restricted the method very much to two dimensions. $p_c(\mathbb{Z}^d)$ is still unknown for $d \geq 3$.

2. Generalizations and equality of critical probabilities. It is easy to replace $\mathbb{Z}^d$ by a general graph $\mathcal{G}$ in the above. Here we deal only with periodic graphs, i.e., graphs which can be imbedded in $\mathbb{R}^d$, such that they are (i) invariant under translations by any of the coordinate vectors, (ii) have only finitely many vertices and (iii) the maximal degree of the vertices is bounded. See [18, Sec. 2.1] for details. In bond percolation on $\mathcal{G}$ the edges of $\mathcal{G}$ are independently open with probability p. In site percolation on $\mathcal{G}$ the sites of $\mathcal{G}$ are independently open with probability p. In the latter case one calls a path on $\mathcal{G}$ open if all its sites are open. With this modification all the previous definitions carry over without difficulty. It is not hard to see ([18, Sect. 2.5, 3.1] and its references) that bond percolation on $\mathcal{G}$ is equivalent to site percolation on the so-called covering graph or line graph of $\mathcal{G}$. However, site percolation on $\mathcal{G}$ is in general not equivalent to bond percolation with independent bonds on another graph. Thus, as long as we stay in the class of independent percolation problems, as we do here, then site percolation is more general than bond percolation.

For very few examples is p_c known. It has been proven (cf. [18, Sect. 3.4] and the references cited there)

$$p_c(\mathbb{Z}^d,\text{bond}) = p_c(\text{triangular lattice,site}) = \frac{1}{2}, \tag{2}$$

$$p_c(\text{triangular lattice,bond}) = 2 \sin \frac{\pi}{18}, \tag{3}$$

$$p_c(\text{hexagonal lattice,bond}) = 1 - 2 \sin \frac{\pi}{18}. \tag{4}$$

(The triangular and hexagonal lattice are represented in Fig. 2). However $p_c(\mathbb{Z}^2, \text{site})$ is unknown (it is ≥ 0.503478; [30]) and

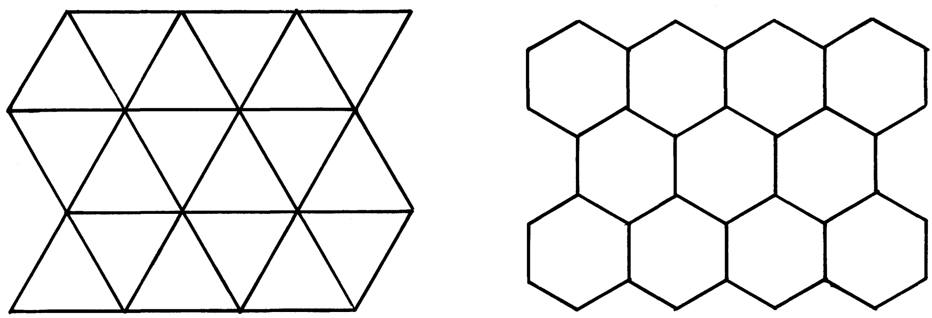

Fig. 2. Illustrations of the triangular and hexagonal lattices.

$p_c(\mathbb{Z}^d)$ is unknown for the bond and site problem when $d \geq 3$. In fact (2)-(4) rely heavily on duality for two dimensional graphs, as well as special symmetry properties of the graphs in question, and we do not know how to find p_c for any graph in dimension ≥ 3. Even though there is little hope of finding general methods to evaluate p_c one has been able to prove the first basic ingredient for the results (2)-(4) in complete generality. The following theorem, proved independently by Menshikov, Molchanov and Sidorenko [21] and Aizenman and Barsky [1], represents the most important progress in percolation in many years. It uses new tools which do not depend on properties of the plane, but hold in all dimensions.

(5) Theorem. For any periodic graph $\mathcal{G}$

$$p_T(\mathcal{G}, \text{bond}) = p_c(\mathcal{G}, \text{bond})$$

and

$$p_T(\mathscr{G}, \text{site}) = p_c(\mathscr{G}, \text{site}).$$

In fact [21] and [1] prove a more general result in which not all bonds or sites have the same probability of being open, and for the bond problem one may even allow the degrees of the vertices to be infinite.

A remarkable consequence of Theorem 5 and earlier work ([3], [18, Theorem 5.1]) is that for $p < p_c$ the distribution of $|W|$ must decay exponentially, i.e., for $p < p_c$ there exist constants $0 < C_i(p) < \infty$ such that

$$(6) \qquad P_p\{|W| \geq n\} \leq C_1 \exp(-C_2 n).$$

For the experts we point out that the results of [1] and [21] even give us something new when $d = 2$. This is so because [1] and [21] make no symmetry requirements on the percolation problem (in contrast to [18, Ch. 3]). In particular it can be used to prove that for the three parameter bond percolation problem on the triangular lattice - with probabilities p_1, p_2 and p_3 for bonds in the three directions to be open, respectively (cf. [18], pp. 58-62 and 380-381) - percolation will occur if and only if

$$(7) \qquad p_1 + p_2 + p_3 - p_1p_2p_3 > 1.$$

This condition was already conjectured by Sykes and Essam [28]. In [18, Ch. 12] this was listed as the first open problem and it was shown that (7) is a necessary condition for infinite open clusters to occur. (The proof used Theorem 12.1 which was given without proof in [18], but a proof of a more general result will appear in [13].) To see that (7) is also sufficient, we note that if $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$ lies in the surface $\mathcal{S} := \{p_1, p_2, p_3 : 0 < p_i < 1, p_1+p_2+p_3-p_1p_2p_3 = 1\}$, then by Theorem 12.1 of [18] infinitely many open circuits surround the origin with $P_{\bar{p}}$ -probability one. Here, $P_{\bar{p}}$ is the obvious product measure on open and closed bonds; see [18, Sect. 3.2] for details. But if there is an open circuit through the point $(n,0)$ which surrounds $\underline{0}$, then the open cluster of $(n,0)$ contains at least n edges. Thus, by the Borel-Cantelli lemma the occurrence of infinitely many open circuits implies

$$\Sigma\, P_{\bar{p}}\{|W| \geq n\} = \infty,$$

and a fortiori (6) with p replaced by $\bar{p}$ cannot hold (compare proof of Prop. 1 in [23]). By the multiparameter version of Theorem 5 tells us that if the distribution of $|W|$ does not fall of exponentially for a given $\bar{p}$ then for any $\epsilon > 0$ and $p(\epsilon) = (1+\epsilon)\bar{p} = ((1+\epsilon)\bar{p}_1, (1+\epsilon)\bar{p}_2, (1+\epsilon)\bar{p}_3)$ there do exist infinite open clusters under the measure $P_{p(\epsilon)}$. In particular, when (7) holds, we can write $p = (p_1, p_2, p_3) = (1+\epsilon)\bar{p}$ for some $\epsilon > 0$ and $\bar{p} \in \mathcal{S}$, so that there is percolation when (7) is satisfied.

3. <u>Uniqueness of infinite clusters and continuity of the percolation probability</u>. Harris [16] and Fisher [10] proved early on that on many two dimensional graphs there occurs w.p.1 at most one infinite open cluster. Again the proof was tied to properties of the plane and it was not clear whether this result also held in higher dimensions. The next theorem of [2] proves this uniqueness in general. The proof has been considerably simplified in [12].

(8) Theorem. For any periodic graph $\mathscr{G}$ and any p there exist (a.e. $[P_p]$) at most one infinite open cluster.

As with (5) the theorem holds more generally than for periodic $\mathscr{G}$, and one can even allow some dependence (see [11]).

By an argument of van den Berg and Keane [31], (8) immediately implies the following

(9) Theorem. For any periodic graph $\mathscr{G}$ the function $p \to \theta(p)$ is continuous except possibly at the point p_c.

It is easy to see ([23]) that $\theta(\cdot)$ is always right continuous, so that continuity of $\theta(\cdot)$ at p_c is equivalent to $\theta(p_c) = 0$ (as we have drawn the graph in Fig. 1). For some two dimensional graphs this is known to hold (cf. [24], [18, Sect. 3.3]) but it is still an open problem whether $\theta(p_c) = 0$ in general (this is even unknown for site or bond percolation on $\mathbb{Z}^d$ when $d \geq 3$). (It is known that $\theta(\cdot)$ can have a jump discontinuity at p_c in so-called long range percolation problems in dimension one; cf. [4].)

4. Power laws and scaling relations. The most important open problems concern the singularity of various functions at or near p_c. It is believed that many quantities behave like powers of $(p-p_c)$ or n (cf. [14], [26], [27]). As a specific example, we mention the conjecture

$$\theta(p) = P_p\{|W| = \infty\} \approx (p-p_c)^{\beta}, \ p \downarrow p_c, \tag{10}$$

for some $0 < \beta \leq 1$. Here and in the relations below $A(p) \approx B(p)$, respectively $A(n) \approx B(n)$, means

$$\frac{\log A(p)}{\log B(p)} \to 1 \quad \text{as} \quad p \to p_c, \quad \text{respectively} \quad \frac{\log A(n)}{\log B(n)} \to 1 \text{ as } n \to \infty.$$

A further conjecture is that for some $\gamma > 0$

$$(11) \qquad \chi^f(p) := E_p\{|W|; |W| < \infty\} = \sum_{n<\infty} n\, P_p\{|W| = n\}$$

$$\approx |p-p_c|^{-\gamma}, \; p \to p_c.$$

It is still debated whether the behavior of $\chi^f(p)$ is different on the two sides of p_c. Some people believe that two exponents γ_+ and γ_- should be used in (11), the first for $p \downarrow p_c$ and the second for $p \uparrow p_c$. For brevity we list here only the simpler versions of such "power laws", without distinguishing the approach to p_c from the right and from the left. Sykes and Essam [28] introduced the <u>number of clusters per site</u>

$$\kappa(p) := \sum_1^{\infty} \frac{1}{n} P_p\{|W| = n\}.$$

This function plays a role very similar to the free energy in statistical mechanics, and it is believed that

$$(12) \qquad \kappa'''(p) \approx |p-p_c|^{-1-\alpha}, \; p \to p_c, \text{ for some } -1 \leq \alpha < 0.$$

A crucial role is played by the <u>correlation length</u> $\xi(p)$. It is believed that this is the single length scale of importance. When everything is measured in this length scale, then most quantities should become independent of the detailed structure of $\mathcal{G}$, but should depend only on the dimension of $\mathcal{G}$. Alternative heuristic descriptions of $\xi(p)$ are "the diameter of a typical finite cluster" or "the minimal length at which one can see a difference in the connectivity picture at p from that at p_c" (see for instance [6], [14], [22], [26]). The last description can be justified a bit more. Define

$$\tau(p,x,y) = P_p\{\exists \text{ open path from } x \text{ to } y \text{ but } |W| < \infty\}.$$

Subadditivity arguments (see [6, Prop. 2.9] and [9]) show that for all p there exists a $C(p) < \infty$ such that

$$\tau(p,0,(n,0,\ldots,0)) \approx \exp(-n\, C(p)).$$

Thus, in first order, the connectivity probability τ does not become very small until n is much larger than $[C(p)]^{-1}$. Unfortunately we can only prove $C(p) > 0$ when $p < p_c$ or p close to 1. (For $p < p_c$ this is immediate from (6).) In this case one can take $[C(p)]^{-1}$ as the definition of $\xi(p)$. Whatever definition one adopts for $\xi(p)$, it is believed that

$$\xi(p) \approx |p-p_c|^{-\nu} \text{ for some } \nu > 0. \tag{13}$$

(10)-(13) are so-called <u>power laws</u> in $(p-p_c)$. Other conjectured power laws deal with behavior <u>at</u> p_c. For instance it is conjectured that for suitable $0 < \delta < \infty$ and η

$$P_{p_c}\{|W| \geq n\} \approx n^{-1/\delta}, \tag{14}$$

(15) $$\tau(p_c, 0, x) \approx |x|^{2-d-\eta}.$$

The α, β, γ, ... appearing in these relations are called critical exponents.

None of the above power laws have been proven for periodic graphs, but they are supported by simulations and experimental evidence for other systems which undergo phase transitions. It is perhaps the principal open problem to establish such power laws. So far only power bounds have been established for many of the above quantities (see [1], [3], [7], [18, Ch. 8]).

Another important feature of the conjectures here is the universality hypothesis. As mentioned, it is believed that on a certain scale only the dimension of $\mathcal{G}$, and not its fine structure, is important. Correspondingly, it is believed that the critical exponents depend on the dimension of $\mathcal{G}$ only. For instance for the bond and site problems on the square, triangular and hexagonal lattice we should find the same values for $\alpha, \beta, \ldots$. If correct, this property should make the critical exponents more significant than the value of p_c (which is not universal, cf. (2)-(4)), and perhaps also easier to find. In fact specific values have been proposed for the two-dimensional critical exponents (see [27] and its references).

The attempts to justify the power laws and universality go by the name of scaling theory. One of the predictions of this theory are the so-called scaling relations

(16) $$2-\alpha = \gamma+2\beta = \beta(\delta+1).$$

Somewhat more controversial are the hyperscaling relations which say that for d less than or equal to the critical dimension $d_c = 6$ all members of (16) equal $d\nu$ and

$$\eta = 2 - d \frac{\delta-1}{\delta+1} . \tag{17}$$

For $d \geq 6$ the exponents are supposed to become independent of d as well, and then their values are the so-called mean field values, which are the values of these exponents when $\mathcal{G}$ is a homogeneous tree, or Bethe tree (η is somewhat exceptional; its mean field value is not defined by means of percolation on the tree). A Bethe tree is not a periodic graph in $\mathbb{Z}^d$ for any finite d, but it plays the role of an infinite dimensional graph. On such a tree one can prove the power laws and calculate the critical exponents (cf. [14, Ch. 7] and its references). The values of the critical exponents on the tree have been proven to be one-sided bounds for their values on periodic graphs (provided the latter exist) in a number of cases ([1], [3], [7]).

H. Tasaki ([29]), and independently J. T. Chayes and L. Chayes [8], have shown that the critical dimension d_c has to be at least 6. More specifically they prove various inequalities which show that if the various power laws hold with the mean field exponents, and if the scaling and hyperscaling relations hold, then one cannot have $d < 6$.

Despite the conjectured state of much of this section we have made progress on the scaling relations when $d = 2$. In [19] and [20] we have proved the following result.

(18) Theorem. For bond and site percolation on $\mathbb{Z}^2$, if the critical exponents exist, then all the scaling relations (16) and the hyperscaling relations, including (17), hold, with the possible exception of relations involving α.

In fact [19] and [20] derive various relations between θ, χ^f, τ, etc. without any hypotheses on the existence of critical exponents. If one adds to this the assumptions (13) and (15) then (for $d = 2$) the other power laws with the right exponent follow, with the exception of (12). We have essentially no results for α, except that κ is twice differentiable with κ'' Lipschitz continuous on all of $[0,1]$, including at p_c. (This corresponds to $\alpha < 0$.)

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