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by Lester E. DUBINS and Gideon SCHWARZ

0. **Introduction.** As reported by Doob [Th. 3.4, 4], the second moment of the supremum of a sub-martingale with a last term, is not larger than 4 times the second moment of its last term. Since the expectation of the supremum is, in turn, bounded by the square-root of its second moment, Doob's bound yields the upper bound 2 for the ratio between the expectation of the supremum and the L_2-norm of the last term. In this note, the bound is replaced by $\sqrt{2}$, and this value is shown to be attained. An analoguous inequality for martingales is established first. Also, solutions to some optimal stopping problems for Brownian Motion and for the simple symmetric random walk are obtained, some as steps toward the inequality for sub-martingales, and some as a consequence of the inequality for martingales.

1. **The bound for Martingales and Brownian Motion.** What is the least upper bound over all mean-zero martingales X of variance v of the expectation of the essential supremum X^* of X? Answer: The least upper bound is $\sqrt{v}$; it is attained by a martingale in continuous time that is closed on the right by a random variable

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whose distribution is necessarily exponential, centered at 0. These facts are immediate consequences of two results:

First, the distribution of X^* is stochastically dominated by that of the Hardy-Littlewood maximal function [9] of the distribution of the last term X_∞ of X [Blackwell & Dubins, 3], and this bound is attained [Dubins & Gilat, 5].

Second, among all distributions with mean 0 and variance not exceeding v, the maximum of the expectation of their Hardy-Littlewood maximal functions is $\sqrt{v}$, and this maximum is attained by, and only by, the exponential distribution, centered at 0, as is established as follows: Let f be the unique nonincreasing function on the unit interval I that has the same distribution as X_∞. Then its Hardy-Littlewood maximal function is the function H(f) on I, whose value at t is the average of f on [0,t]. By a change of the order of integration, the integral of H(f) over I is $\int f(s)\log(1/s)ds$, or, since the mean of f is 0, $\int f(s)(\log(1/s)-1)ds$, which, by the Cauchy-Schwarz Inequality is at most the product of the L_2-norms of f and of $\log(1/s)-1$. The first is bounded by $\sqrt{v}$, and the second is 1. So, the expectation of H(f) is bounded by $\sqrt{v}$; the bound is attained if, and only if, f has variance v, and is proportional to $\log(1/s)-1$ or, equivalently, the distribution of f (and hence of X_∞), is exponential with parameter $1/\sqrt{v}$, shifted by $\sqrt{v}$ to the left (to give it expectation 0). For this extremal f, the distribution of

H(f) is exponential with the same parameter, in its usual (unshifted) location.

A martingale that attains the bound $\sqrt{v}$ can be realized as Brownian Motion B, stopped at a stopping-time, T, of expectation v, that maximizes, among all such T, the expectation of the maximum of B for $t \le T$ [Azema & Yor 1]. It is of interest to describe the extremal T explicitly. Let M_t be the maximum of B on [0,t], and let G_t be the "gap" $M_t - B_t$. An extremal T is the first t such that $G_t = k$, where k is so chosen that the expectation of T will be v, as will now be shown. Since $B_t^2 - t$ is a martingale, the variance of B_T and, hence, of M_T is v. Since $E(B_T)=0$, the expectation of M_T is the gap k. The distribution of M_T is exponential: The event $\{M_T \ge m\}$ happens if B reaches m at some time, say R, before G reaches k. Given that this event happened, the conditional probability of $\{M_T \ge m+b\}$, for some positive b, is the same as the (unconditional) probability of $\{M_T \ge b\}$, since the process $B_{R+t} - m$ also is Brownian motion. Consequently, M_T has an exponential distribution of variance v. For exponential distributions, the square of the expectation equals the variance. So, $(E[M_T])^2 = v$, that is, $E(M_T)$ attains the bound $\sqrt{v}$.

The set **D** of all pairs (x,y) in the plane, such that y is the expectation of X^* for a mean-zero martingale X whose variance is bounded by x, can be described explicitly. By the answer above, **D** includes the parabolic arc $\{(x,y): x=y^2, y \ge 0\}$ and this arc bounds **D** from above. The pairs (x,0) with x>0 are not attained.

Since the set of distributions of all mean-zero martingales is convex, so is **D**. Therefore $\mathbf{D}=\{(x,y):0<y^2\leq x, \text{ or } x=y=0\}$. By the previous paragraph, the parabolic arc is also attained by $(E(T),E(M_T))$ as T ranges over the constant gap stopping-times for Brownian Motion and, therefore, **D** is attained if T ranges over randomized stopping-times as well.

In view of this geometric description of **D**, these four optimal stopping problems for Brownian Motion are easily solved: the constant-gap stopping-times maximize $E[M_T]$ under the constraint $E[T]\leq v$, maximize $(E[M_T])^2/E[T]$, maximize linear combinations $E[M_T]-cE[T]$ for arbitrary positive c, and, minimize $E(T)$ under the constraint $E(M_T)\geq m$. The values of the gap k for these four problems are $\sqrt{v}$, arbitrary k, 1/2c and m respectively.

2. **Optimal stopping of the random walk.** When the class of mean-zero martingales is replaced by the class of nonnegative sub-martingales, there arises a natural counterpart to **D**. One functional of interest is again the expectation of the maximal term of the process. For the other functional, the variance and the second moment are no longer the same; choose the latter, and define the set **D** as the set of pairs (x,y) such that y is the expectation of the maximal term S^*, for a nonnegative sub-martingale S whose second moment is bounded by x. One such sub-martingale is the absolute value of Brownian Motion, stopped by a stopping-time of expectation x. If T is now defined as the

first time the process is g units below its current maximum, rather straightforward calculations as in Section 1, show $E(T) = 2g^2$ and the expectation $E(M_t)$ of the maximum of the process for $t \leq T$ is 2g. Therefore, D contains all points $(2g^2, 2g)$, that is, the parabolic arc $\{(x,y): y^2 = 2x, y \geq 0\}$. To show that D has no other extreme points and, hence, that $D=\{(x,y): 0 < y^2 \leq 2x \text{ or } x = y = 0\}$, we find it necessary to state and solve two stopping-problems for the simple random walk S, of interest in their own right.

First, for some positive c, find a stopping-time T that maximizes the expectation of the "return" $\text{Max}(S_n : n \leq T) - cT$. As is typical of optimization and gambling problems, to solve the problem, it is helpful to solve also a family of conditional problems, arising when the process has already completed a partial history. The optimal return, given such a partial history, of length n, last value $S_n = s$ and maximal term m, is denoted by $U = U(n,s,m;c)$. The program is to define a function Q and then show that Q equals U. For a nonnegative integer k, let $Q = Q_k(n,s,m;c)$ be the conditional expected return, given a partial history of length n, last term s and maximum m, when the following stopping-time T is used: if $m - s < k$, T is the first time after n that a gap of size k occurs; if $m-s \geq k$, $T=n$. Clearly, $U \geq Q_k$ holds for all k. For each c, a value of k will be found, such that the reverse inequality holds. The proof that $U \leq Q_k$ for those k will be an application of

Theorem 2.12 in [6], or a variant thereof as in [7] or [10], once the following two properties of Q_k are established:

Condition (1), Q_k is at least the return of stopping right at the end of the given partial history, that is, $Q_k \geq m-cn$.

Condition (2), Q_k is "excessive", that is, its value at each state is no less that its expectation at the (random) state that is reached by the process after one more step.

By a straightforward calculation, one finds:

$$Q_k = m - cn, \text{ for } m-s \geq k;$$

$$Q_k = k+s-c(n+(k+m-s+1)(k-m+s)), \text{ for } m-s \leq k.$$

For checking Condition (2) it is useful to record that both formulas agree when $m-s=k$, and that the second is of the form $c(s^2-n) + L(m,s)$, where L is linear in s.

Condition (1) holds trivially where $m-s \geq k$, since there, Q_k and the return of stopping agree. For $m-s<k$, the condition is $0 \leq Q_k-(m-cn) = (k-m+s)(1-c(k+m-s+1))$, which holds whenever $1/c \geq k+m-s+1$ for all s such that $m-s<k$, or equivalently, for the smallest such s, that is, for $s=m-k+1$. Condition (1) is therefore fulfilled if and only if $1/c \geq 2k$, or equivalently, $k \leq 1/2c$. Condition (2) can be checked to hold for all c and k, except when $m-s=k$. Beginning at such a state, after a step up, Q_k will be $m+1-c(n+1+2k)$, and after a step down, it will be $m-c(n+1)$. The excessivity condition, that is, that the average of these two values is at most $m-cn$, is equivalent to $c(2k+2) \geq 1$, that is, $k \geq 1/2c - 1$.

Combining Conditions (1) and (2), yields $1/2c - 1 \leq k \leq 1/2c$. If $1/2c$ is an integer, both it and its predecessor may be used for k. For all other c, there is a unique k such that the stopping-time T_k, the first time the gap is k, is optimal.

As T ranges over all T_k, the points $(E[T], E[M_T])$ that are attained are (0,0), (2,1), (6,2),..., (k(k+1),k),..., and the slope of the segment connecting the point with y-coordinate k-1 to the next point is $1/2k$. The interior points of this segment are attained by randomizing between T_{k-1} and T_k. Each of these randomized stopping-times maximizes the return $E[M_T - T/2k]$. The infinite polygon consisting of all the segments is the upper boundary of the attainable set, which is the convex hull of the polygon. As in the Brownian Motion case, the solutions to various optimal stopping-problems are easily described in terms of this boundary: for maximizing M_T under the constraint $E[T] \leq v$, if $v=k(k+1)$, T_k is optimal; for v strictly between successive values of k(k+1), the optimal T is a mixture, with appropriate weights, of the two corresponding T_ks. Similarly, for minimizing E[T] for a given lower bound m on the expected maximum, T_m is optimal if m is an integer and, for all other m, a mixture of the two constant-gap stopping-times with integer gaps closest to m attains the minimum.

3. **The maximal absolute value of the random walk.** The second problem for random walks arises from the first when the maximum of

S in the return is replaced by the maximum of its absolute value. The stopping times that are analoguous to the T_k of the former problem are the times W_k when the maximum of the absolute value first exceeds the absolute value of the current value by k. As will be seen, these stopping-times attain extreme points of the set

$$\{(E[T],E[\max|S_n|:n\leq T) : T \text{ is a stopping-time}\}$$

but they turn out not to be the only ones. Indeed, besides the W_k, the other stopping-times that attain extreme points of the set are W_k^+, defined as follows: stop the first time that the current absolute value is not 0, and is k units lower than the maximum of the absolute values till this time.

Consider first the performance of the stopping-times W_k. One can express W_k as the sum of two time-periods. The first is the time until S_n reaches either k or -k. The expectation of the number N of steps in this period is k^2. The second consists of the additional steps until the process $|S_{N+n}|$ reaches for the first time a value k units less than its maximum. For the second period, the expected number of steps and the expected increase of the maximum, beyond the value k reached at the end od the first period are, in view of Section 2, k(k+1) and k, respectively. The values reached by W_k are thus seen to be 2k for the expectation of the maximal absolute value, and k(2k+1) for the expected number of steps.

The calculation of the function Q_k for W_k, is again rather straightforward; when the decomposition of W_k into the two periods is used, and the observation that for the second period, the calculations for T_k are valid. From here on, let s denote the absolute value of the current position of the random walk, and m, the maximum of the absolute value. For W_k, different formulas describe Q_k in three different regions of the (s,m)-plane:

a. for $m \geq s+k$, $Q_k = m-cn$;

b. for $k \leq m \leq s+k$, $Q_k = k+s-c(n+(m+k-s+1)(s-m+k))$;

c. for $s \leq m \leq k-1$, $Q_k = 2k-c(n+k(2k+1)-s^2)$.

For checking excessivity below, note that the formulas in a and b agree on the boundary $m=s+k$, and that in region b and c, the difference $Q_k-c(s^2-n)$ is linear in s.

Condition (1) holds trivially in region a. In region b, the condition is the same as in this region for the problem without absolute values. So, Condition (1) holds in region b whenever $c \leq 1/2k$. In region c, the condition becomes $2k-m \geq c(k(2k+1)-s^2)$, which holds, if it holds for the largest m and the smallest s in the region, that is, for $m=k-1$ and $s=0$. This yields $k+1 \geq ck(2k+1)$, or equivalently, $c \leq (k+1)/(k(2k+1))$. This bound for c is at least 1/2k, the bound that has already been found for region b. Therefore, Condition (1) holds everywhere if and only if $c \leq 1/2k$.

For Condition (2), three cases are examined first:

I. If $s=m=k=0$, the only possible step is to $s=m=1$, which changes Q_0 from $-cn$ to $1-c(n+1)$. Q_0 is excessive there if and only if $c \geq 1$.

II. If $s=0$ and $m=k>0$, the only possible step is to $s=1$, $m=k$, which changes Q_k from $k-nc$ to $k+1-c(n+1+2k)$. This yields the inequality $c \geq 1/(2k+1)$ as equivalent to Condition (2) at this point. Case I is covered by this inequality.

III. At the remaining points of the common boundary of regions a and b, where $s = m-k > 0$, $Q_k = m-cn$. In one step up, s changes to $m-k+1$, and region b is entered, leading to $Q_k = m+1-c(n+1+2k)$, while a step down to $s = m-k-1$ leads into region a, and to $Q_k = m-c(n+1)$. The expected change in Q_k is therefore $1/2 - c(k+1)$, and Condition (2) will hold if and only if $c \geq 1/(2k+2)$.

At all other states, Q_k is excessive for all k. To verify this five cases have to be checked, which is a somewhat lengthy, but straightforward calculation, that is omitted here. In summary, the necessary and sufficient condition for excessivity of Q_k is the more stringent one of the two inequalities obtained in II and III, for $k>0$, and in I and III, for $k=0$, that is, Q_k is excessive if and only if $c \geq 1/(2k+1)$.

The two inequalities obtained for Conditions (1) and (2) show that the range of c for which W_k maximizes the linear combination $E[\max(|S_n|:n\leq T)] - cE[T]$, is the closed interval

$[1/(2k+1),\ 1/2k]$. Equivalently, for each c in this interval, a support line of slope c to the attainable set passes through the point $P_k=(k(2k+1),2k)$; all these points lie, therefore, on the boundary of the attainable set, In particular, for each k, the two lines through P_k whose slopes are the endpoints of the interval bound the set. Therefore, the intersection of the lower half-planes defined by the lines as k ranges over the positive integers, and by the line of slope 1 through the origin, is a closed convex set, that includes all attainable points. The points $(j(j+1)/2,j)$, $j=0,1,\cdots$ are its extreme points: the point corresponding to $j = 2k$ is the one attained by W_k; for $j = 2k+1$, the point is the intersection of one of the two lines through the point attained by W_k with one of the two lines through the point attained by W_{k-1}. The "new" points, those for odd j, are also attained: the point for $j = 2k+1$ is attained by W_k+. This is easily checked by expressing W_k+ as the sum of two periods: the first one is the time till the random walk reaches either k+1 or $-(k+1)$, and the second one is the additional time till a gap of size k occurs, that is, till $|S|$ reaches a value k units lower than its current maximum. All points $((j(j+1)/2,j)$ are therefore in the attainable set. Since their convex hull was seen to include the set, it is the set itself. The stopping times W_k+ are therefore optimal exactly for the values of c in the gaps between the intervals for which the W_k are optimal. That is, for each k, W_k+ is optimal for the slopes in the closed interval $[1/(2k+2),\ 1/(2k+1)]$. For the endpoints, that is for the reciprocals of integers, this yields two optimal stopping times; for all other values of the slope c, just one.

Now that the attainable set has been determined, it is easy to find stopping-times T that maximize $E[\max(|S_n|:n\leq T)]$ under the constraint $E[T]\leq v$: if $v=k(2k+1)$, $T=W_k$ and $E[\max|S|]=2k$; if $v=(k+1)(2k+1)$, $T=W_k^+$ and $E[\max|S|] =2k+1$; in all other cases, v is between two such values, and the appropriate mixture can be taken as T. Similarly, to minimize E[T] under the constraint $E[\max(|S_n|:n\leq T)]\geq m$, $T=W_k$ if $m=2k$, $T=W_k^+$ if $m=2k+1$, and if m is not an integer, a mixture is optimal.

4. From random walks to Brownian Motion. Since Brownian Motion can be approximated by a random walk on the integer multiples of a small number, it is not surprising that solutions to analoguous stopping-problems for Brownian Motion are now accessible. Consider Brownian Motion B restricted to the (random) times $t(j,n)$, $n=0,1,...$ when it completes n consecutive changes of size $1/j$. The expected time for each such change is j^{-2}. The maxima of the restricted and the unrestricted process in any interval $[0,t]$ are always less than $1/j$ apart. The restricted process is a rescaled version of the simple random walk. Now let T be a stopping-time for B, for which the expectation of the maximum, M, of the absolute value of B till T is at least 1. Let T^j be the smallest among the times $t(j,n)$ that are at least T.

Then $T^j \geq T$, so the maximum M' of $|B|$ till T^j is at least M, and the maximal absolute value M^r of the restricted process till T^j falls short of M by 1/j at most. Therefore $E[M^r]$ is at least 1 - 1/j. For the embedded random walk, rescaled to its usual steps of size 1, the expected maximum is at least $j(1-(1/j))=j-1$. Clearly, T^j is a stopping-time for B. For the restricted process, it is a stopping-time with respect to the filtration of B. It is therefore a mixture of stopping-times for the restricted process. By the result in the previous section, the lower bound j-1 on the expectation of the maximal absolute value of the embedded random walk implies that the expected number of steps of T^j, when regarded as a randomized stopping-time on the random walk, is at least $(j-1)((j-1)+1)/2 = j(j-1)/2$. Return to regard T^j as a stopping-time for the Brownian Motion, so each step is replaced by a time-interval of expectation j^{-2}; then $E(T^j) \geq j^{-2}(j-1)j/2 = 1/2 - 1/2j$. Since T^j exceeds T by at most the time needed to complete one more change of size 1/j, $E(T) \geq 1/2 - 1/2j - 1/j^2$. Therefore, $E(T) \geq 1/2$. Furthermore, this bound is attained: For T the first time the gap is 1/2, a caculation analoguous to the evaluation of W_k for the random walk yields $E[T]=1/2$ and $E[\max(|Bt|:t\leq T)]=1$. By the usual rescaling of Brownian Motion, this yields the minimum $m^2/2$ for the expectation of a stopping-time such that $E[(\max|Bt|:t\leq T)] \geq m$, and this minimum is attained by the first time the gap is m.

The set of pairs $(E[T],E[\max(|B_t|:t\leq T)])$ for all stopping-times T is thereby determined to be $B=\{(x,y):0<y^2\leq 2x \text{ or } x=y=0\}$. In terms of this set it is again easy to give solutions to related problems: The maximum of the expected maximal absolute value till stopping at time T, among all T such that $E[T]\leq v$, is $\sqrt{2v}$, and it is attained by the first time the gap is $\sqrt{2v}/2$. For the linear problem, $\max(|B_t|:t\leq T)-cT$ attains its maximal expectation when T is the first time the gap is $1/2c$, and the maximum attained is also $1/2c$.

As in the definition of the functions Q_k above, it is possible to answer also the question how to stop optimally, given that the process has been running for some initial period, reached a maximum of m, and a latest value of s. For maximizing the expected maximum of Brownian Motion, for example, when allowed additional expected time x, the optimal T is described as follows: continue until a gap of $g=((m-s)^2+x)^{1/2}$ is first reached. The expected return for this T, in terms of g, is s + g. For maximizing the expected maximum of the absolute value, given a partial history, not every additional expected time x can be attained by a constant-gap stopping-time: when the gap g approaches m from above, x can be shown to approach $2m^2-s^2$, but when g reaches m, x jumps down to $s(2m-s)$. The set of points attainable by all constant-gap stopping times is therefore disconnected. It consists of two parabolic arcs. The first arc is finite and closed, and is given by y = s + g, with g as above, for

$0 \le x \le s(2m-s)$; the second one is open and infinite, and is the graph of $y = (2(s^2+x))^{1/2}$, for $x > 2m^2-s$. The slope of the first arc at $x=s(2m-s)$, and of the second arc at $x=2m^2-s^2$ are both $1/2m$. The expected maxima attainable for $s(2m-s) < x \le 2m^2-s^2$ are bounded from above by the segment connecting the two arcs. The end-point $(2m^2-s^2, 2m)$ of the second arc, however, is not attained by a constant-gap stopping time; it is attained by T defined as follows: stop the first time the gap is m, <u>after</u> m or −m have been reached by the process. Mixtures between this T and the T that stops when the gap is first m will attain the interior points of the segment.

5. The bound for sub-martingales. By a theorem of Gilat [8] (see also Barlow [2], where further references to Protter & Sharpe, Barlow & Yor and Maisonneuve are found), every nonnegative sub-martingale is the absolute value of a mean-zero martingale. Therefore, **D** as defined in Section 2, is also the set of all (x,y), such that y is the expectation of $|X|^*$, the maximum of the absolute values, for a mean-zero martingale X whose variance is bounded by x. But, **D** cannot include more than the set **B** defined above, as is seen with the help of Skorokhod-type embeddings.

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Contemporaneously with our work, and without any knowledge of it, S. D. Jacka had undertaken a parallel, and in some ways more general, investigation. Without leaving the realm of the continuous time-parameter, he obtained the L_2-bound $\sqrt{2}$. Moreover, though he doesn't study the discrete random walk, he manages to obtain best L_p bounds for continuous-time submartingales. Thus, his methods as well as his results nicely supplement those of this paper. We thank him for having sent us an early handwritten manuscript. To Marc Yor, too, we express our thanks. It was his insights at the Colloque Paul Levy sur les Processus Stochastiques that enabled us to understand how our work overlapped with the interesting contributions of Jacka.

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