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METRIC THEOREMS ON UNIFORM DISTRIBUTION AND APPROXIMATION THEORY

by

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A sequence (x_n) , $n = 1, 2, \dots$, of points in $[0, 1]^k$ is called uniformly distributed (u.d.) if

$$\lim_{N \rightarrow \infty} N^{-1} \text{card} \{1 \leq n \leq N: x_n \in J\} = \text{vol}(J)$$

for all subintervals J of $[0, 1]^k$. A sequence (x_n) of numbers in $[0, 1)$ is called completely uniformly distributed (c.u.d.) if for all $k \geq 1$ the sequence of points $x_n = (x_n, x_{n+1}, \dots, x_{n+k-1})$, $n=1, 2, \dots$, is u.d., and a sequence (x_n) of real numbers is called c.u.d. mod 1 if the sequence $(\{x_n\})$ of fractional parts is c.u.d.

Franklin [3] proved that the sequence (x^n) is c.u.d. mod 1 for almost all $x > 1$ (in the sense of Lebesgue measure). Knuth [4] defined a sequence (x_n) in $[0, 1)$ to be random if, for every sequence (b_n) of distinct positive integers obtained by an effective algorithm, the sequence (x_{b_n}) is c.u.d. and raised the problem of whether $(\{x^n\})$ satisfies this definition of randomness for almost all $x > 1$. This problem was recently settled in the affirmative by the following result.

THEOREM 1 (Niederreiter and Tichy [7]). - If (a_n) is any sequence of distinct positive integers, then the sequence (x^{a_n}) is c.u.d. mod 1 for almost all $x > 1$.

The following lemmas are the key ingredients in the proof.

LEMMA 1 (Davenport, Erdős and LeVeque [2]). - Let (u_n) be a sequence of Lebesgue-measurable functions on the interval $[a, b]$ and put

$$I(N) = \int_a^b \left| \frac{1}{N} \sum_{n=1}^N e(u_n(x)) \right|^2 dx, \text{ where } e(t) = e^{2\pi i t}.$$

If $\sum_{N=1}^{\infty} I(N)/N$ converges, then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(u_n(x)) = 0 \text{ for almost all } x \in [a, b].$$

LEMMA 2 (van der Corput [8]). - Let the function u on [a,b] have a continuous and monotone first derivative that satisfies $|u'(x)| \geq L > 0$ for all $x \in [a,b]$. Then

$$\left| \int_a^b e(u(x)) dx \right| \leq \frac{1}{L}.$$

LEMMA 3. - If P is a monic polynomial of degree $d \geq 1$, then for the supremum norm $\|P\|_\infty$ on [a,b] we have

$$(1) \quad \|P\|_\infty \geq 2^{1-2d} (b-a)^d.$$

Lemma 3 is a standard result in approximation theory (see [6,p.39]). The lower bound (1) is in fact best possible, as it is attained by the normalized Chebyshev polynomial on [a,b] of degree d. The question of whether the integer exponents in Theorem 1 can be replaced by real exponents leads to another problem in approximation theory, namely to establish a lower bound for $\|P\|_\infty$ when

$$(2) \quad P(x) = x^{\lambda_s} + \sum_{j=1}^{s-1} c_j x^{\lambda_j}$$

is a monic Müntz polynomial with real coefficients c_j and real exponents λ_j satisfying $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_s$, $s \geq 2$. The lower bound should not depend on the c_j . For $[a,b] = [0,1]$ this is a classical problem considered by Müntz and Szász. By applying the transformation $x \rightarrow bx$, intervals $[0,b]$ can also be handled. For $[a,b]$ with $a > 0$ we cannot use the transformation trick since $P(a + (b-a)x)$ is not necessarily a Müntz polynomial. We now have the following result. We are grateful to Prof. J. Korevaar for providing the important reference [5].

LEMMA 4. - For any monic Müntz polynomial P given by (2) and any interval [a,b] with $0 < a < b$ we have

$$\|P\|_\infty \geq C_s a^{\lambda_s} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)},$$

where $\tau = \frac{1}{2} \log \frac{b}{a}$ and the constant $C_s > 0$ depends only on s.

Proof.- We first assume that $\lambda_1 = 0$. By a result in [5] there exists a sequence (γ_n) , $n = 0, 1, \dots$, with $\gamma_n > 0$ and

$$\sum_{n=0}^{\infty} \gamma_n = 1 \text{ such that } \prod_{n=0}^{\infty} \cos(\gamma_n z) \text{ is entire and}$$

$$(3) \quad \prod_{n=0}^{\infty} |\cos(\gamma_n u)| \leq C e^{-\sqrt{u}} \text{ for all } u \geq 0,$$

where C is an absolute constant. Then the complex function

$$F(z) = \prod_{j=1}^{s-1} (z - i\lambda_j) \prod_{n=0}^{\infty} \cos(\gamma_n \tau z)$$

is entire and of exponential type τ , i.e. $|F(z)| = O_{\epsilon}(e^{(\tau+\epsilon)|z|})$ for all $\epsilon > 0$, and the restriction of F to \mathbb{R} is in $L^2(\mathbb{R})$. By the Paley-Wiener theorem (see [1,p.103]) we have

$$F(z) = \int_{-\tau}^{\tau} f(t)e^{-izt} dt$$

for some $f \in L^2[-\tau, \tau]$. With $\sigma = \frac{1}{2} \log(ab)$ we get

$$G(z) := e^{-i\sigma z} F(z) = \int_{-\tau+\sigma}^{\tau+\sigma} f(t-\sigma)e^{-izt} dt = \int_{\log a}^{\log b} g(t)e^{-izt} dt$$

with $g \in L^2[\log a, \log b]$. By Fourier inversion we obtain

$$(4) \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u)e^{itu} du \text{ for } t \in [\log a, \log b].$$

Put $h(x) = x^{-1}g(\log x)$ for $x \in [a, b]$. Then with $x = e^t$,

$$(5) \quad \int_a^b x^{\lambda_j} h(x) dx = \int_{\log a}^{\log b} g(t)e^{\lambda_j t} dt = G(i\lambda_j) \text{ for } 1 \leq j \leq s.$$

For P in (2) we use (5) and $G(i\lambda_j) = 0$ for $1 \leq j \leq s-1$ to get

$$(6) \quad \left| \int_a^b x^{\lambda_s} h(x) dx \right| = \left| \int_a^b P(x)h(x) dx \right| \leq \|P\|_{\infty} \int_a^b |h(x)| dx.$$

Then by (4), (5) and (6),

$$(7) \quad |G(i\lambda_s)| \leq \|P\|_{\infty} \int_a^b \frac{dx}{x} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(u)| du = \frac{\tau}{\pi} \|P\|_{\infty} \int_{-\infty}^{\infty} |G(u)| du.$$

Now

$$\begin{aligned} |G(i\lambda_s)| &= e^{\sigma\lambda_s} |F(i\lambda_s)| = e^{\sigma\lambda_s} \prod_{j=1}^{s-1} (\lambda_s - \lambda_j) \prod_{n=0}^{\infty} \cosh(\gamma_n \tau \lambda_s) \\ (8) &= b^{\lambda_s} \prod_{j=1}^{s-1} (\lambda_s - \lambda_j) \prod_{n=0}^{\infty} \frac{1}{2} (1 + e^{-2\gamma_n \tau \lambda_s}) \geq b^{\lambda_s} \prod_{j=1}^{s-1} (\lambda_s - \lambda_j) \prod_{n=0}^{\infty} e^{-2\gamma_n \tau \lambda_s} \\ &= a^{\lambda_s} \prod_{j=1}^{s-1} (\lambda_s - \lambda_j). \end{aligned}$$

Using (3) we get

$$\int_{-\infty}^{\infty} |G(u)| du \leq 2C \int_0^{\infty} \prod_{j=1}^{s-1} (u + \lambda_j) e^{-\sqrt{\tau}u} du$$

$$= \frac{2C}{\tau} \int_0^\infty \prod_{j=1}^{s-1} (\frac{v}{\tau} + \lambda_j) e^{-\sqrt{v}} dv \leq \frac{C}{\tau} \prod_{j=1}^{s-1} (\frac{1}{\tau} + \lambda_j) ,$$

and together with (7) and (8) the result for $\lambda_1 = 0$ follows. In the general case the lemma follows from

$$\|P\|_\infty \geq a^{\lambda_1} \|x^{\lambda_s - \lambda_1} + \sum_{j=1}^{s-1} c_j x^{\lambda_j - \lambda_1}\|_\infty \cdot \square$$

Using Hölder's inequality in (6), the method above yields for the L^p -norm ($1 < p < \infty$)

$$\|P\|_p \geq C_s \tau a^{\lambda_s} \frac{(q-1)^{1/q}}{(a^{1-q} - b^{1-q})^{1/q}} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)} ,$$

where $p^{-1} + q^{-1} = 1$. For the L^1 -norm we get

$$\|P\|_1 \geq C_s \tau a^{\lambda_s + 1} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)} .$$

THEOREM 2.- If $\gamma \neq 0$ and (δ_n) is any sequence of reals with $\inf \delta_n > -\infty$ and $\inf_{m \neq n} |\delta_m - \delta_n| > 0$, then the sequence (γx^{δ_n}) is c.u.d. mod 1 for almost all $x > 1$.

Proof. We take $\gamma = 1$ to simplify the writing. We indicate the changes that are necessary in the proof of Theorem 1 (see [7]) to obtain Theorem 2. Dropping finitely many initial terms, if necessary, we can assume that all $\delta_n \geq 1$. As in [7] it suffices to show that if the lattice point $\underline{h} = (h_0, \dots, h_{k-1}) \neq \underline{0}$ and $1 < \alpha < \beta$ are fixed, then with

$$I(N) = \int_\alpha^\beta \left| \frac{1}{N} \sum_{n=1}^N e\left(\sum_{j=0}^{k-1} h_j x^{\delta_{n+j}}\right) \right|^2 dx$$

the series $\sum_{N=1}^\infty I(N)/N$ converges. We take $N \geq 3$. The cases (i) and (ii)

in [7] require no changes, if we note that the number of zeros of a Müntz polynomial in $[\alpha, \beta]$ can be bounded by [6, p.49, Hilfssatz].

In case (iii) the function $R_s(x)$ in [7, equ. (2.12)] is now a Müntz polynomial with $\lambda_s = d \leq (\log N)^{\sigma(s)} \leq \log N$, $\lambda_1 = 0$, $2 \leq s \leq k$, and

$$|\lambda_i - \lambda_j| \geq \delta := \inf_{m \neq n} |\delta_m - \delta_n| \text{ for } i \neq j .$$

Instead of the choice $\epsilon = (\log N)^{-2d}$ in [7] we take $\epsilon = (\log N)^{-2(s-1)-d}$.

For a subinterval $[a, b]$ of $[\alpha, \beta]$ with $|R_s(x)| \leq \epsilon$ for all $x \in [a, b]$ we have then by Lemma 4,

$$\epsilon \geq C_s a^{\lambda_s} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j + (1/\tau)} \geq C_s \delta^{s-1} (\lambda_s + \frac{1}{\tau})^{1-s} ,$$

hence

$$\frac{1}{\tau} \geq C'_S \delta (\log N)^2 - \lambda_S \geq C(\underline{h}, \delta) (\log N)^2 \text{ for } N \geq N_0(\underline{h}, \delta).$$

From the definition of τ it follows that

$$\log \frac{b}{a} \leq C_1(\underline{h}, \delta) (\log N)^{-2} \text{ for } N \geq N_0(\underline{h}, \delta),$$

and so

$$b - a \leq [\exp(C_1(\underline{h}, \delta) (\log N)^{-2}) - 1] \beta \\ \leq C(\underline{h}, \delta, \beta) (\log N)^{-2} \text{ for } N \geq N_1(\underline{h}, \delta).$$

Therefore, the Lebesgue measure of $V = \{x \in [\alpha, \beta] : |R_S(x)| \leq \epsilon\}$ is $O((\log N)^{-2})$. If W is the closure of the complement of V in $[\alpha, \beta]$, then for $x \in W$ we have

$$|R_S(x)| \geq (\log N)^{-2(s-1)-d} \geq (\log N)^{-2} (\log N)^{\sigma(s)} \text{ for } N \geq N_2(\underline{h}),$$

and so the argument following (2.13) in [7] can be applied. This yields $I(N) = O((\log N)^{-2})$, and the proof is complete. \square

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