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On the distribution of integers having no large prime factor


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1. Friedlander and Lagarias [2] considered the problem of estimating the number \( \psi(X,Z,Y) \) of integers in the interval \((X-Y,X]\) having no prime factor >\(Z\). Especially they defined \( f(\alpha) \) as the infimum of the values of \( \theta \) for which for all \( \alpha' > \alpha \) one had \( \psi(X,X^{\alpha'},X^{\theta}) > 0 \) for sufficiently large \( X \) and they proved for \( 0 < \alpha \leq \frac{1}{2} \)

\[
(1) \quad f(\alpha) \leq 1 - 2(1 - \frac{1}{\alpha})^\alpha.
\]

The proof was based on a simple combinatorial construction, a special case of it was discovered independently by Balog and Sárközy [1].

Our aim is to develop an alternative method. Instead of \( \psi(X,Z,Y) \) itself we investigate a weighted sum by analytic arguments originated from Heath-Brown and Iwaniec [3]. We have

**THEOREM:** For \( 0 < \alpha \leq 1 \)

\[
(2) \quad f(\alpha) \leq \frac{1}{2}.
\]

The theorem is a simple consequence of our main lemma

**LEMMA:** Let \( k \geq 1 \) be an integer, \( \frac{1}{8k} > \delta > 0 \), \( X > X_0 \) be real numbers, \( |a_m| \leq 1 \) be arbitrary complex coefficients and we define \( M = X^{1/2 - 1/4k} \), \( Y = X^{1/2 + 1/8k + \delta} \), finally

\[
(3) \quad d_n = \sum_{m_1m_2 \mid n} a_{m_1}a_{m_2}.
\]

For any \( A > 0 \) we have
(4) \[ \sum_{X-Y<n \leq X} d_n = Y(\sum_{M \leq m \leq 2M} \frac{a_m}{m})^2 + O\left(\frac{Y}{\log^A X}\right). \]

2. For a given \( \varepsilon > 0 \) and \( 0 < \alpha \leq 1 \) we can choose a \( k > \max\left\{ \frac{1}{2a}, \frac{1}{8\varepsilon} \right\} \) and \( a_m = \begin{cases} 1 & \text{if } m \text{ has no prime factor } > X^\alpha, \\ 0 & \text{otherwise}. \end{cases} \) Our lemma guarantees that the interval \( (X-X^{1/2+\varepsilon}, X] \) contains numbers \( n \) in the form \( n = \lambda m_1 m_2 \) where \( m_1 \) has no prime factor \( > X^\alpha \) and \( E < \frac{X}{M^2} < X^{1/2k} \). This gives the theorem.

3. At first we reduce the proof of (4) to estimating a certain integral. Our basic tool is the Perron integral formula (Lemma 3.12 of [6]). We define

\[ M(s) = \sum_{M \leq m \leq 2M} a_m^{-s}, \quad L(s) = \sum_{L_1 < t \leq L_2} t^{-s}, \]

where \( L_1 = \frac{1}{2}X^{1/2k} \) and \( L_2 = 2X^{1/2k} \). By Perron formula we can express the left hand side of (4) as an integral taken on a vertical line of the complex plane. We have

\[ \sum_{X-Y<n \leq X} d_n = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s)M^2(s)\frac{X^S-(X-Y)^S}{s} \, ds + O\left(\frac{X \log^3 X}{T}\right). \]

We can provide a fairly small error term by choosing \( T = \frac{X^{1+\delta/2}}{Y} = X^{1/2-1/8k-\delta/2} \). The major part of the integral is that around \( \frac{1}{2}+i0 \).

Choosing \( T_0 = X^{1/4k} \) and using the facts that

\[ L(s) = L_2^{1-s}L_1^{1-s} + O(L_2^{-1/2}) \quad \text{for } s = \frac{1}{2}+it, \quad |t| \leq T_0, \]

\[ \frac{X^S-(X-Y)^S}{s} = XY^{s-1} + O(|s-1|Y^2X^{-2}) \quad \text{for } s = \frac{1}{2}+it, \]

we get again from the Perron formula that

\[ \frac{1}{2\pi i} \int_{\frac{1}{2}-iT_0}^{\frac{1}{2}+iT_0} L(s)M^2(s)\frac{X^S-(X-Y)^S}{s} \, ds = \]
for all \( A>0 \). Combining (5) and (6) we arrive at

\[
(7) \quad \sum_{X-Y<n\leq X} d_n = Y\left( \sum_{M<m\leq 2M} \frac{a_m}{m} \right)^2 + O\left( \frac{Y}{\log^A X} + \frac{Y}{X^{1/2}R} \right)
\]

for all \( A>0 \), where

\[
R = \int_{T_0}^{T} |L\left(\frac{1}{2}+it\right)M^2\left(\frac{1}{2}+it\right)| \, dt.
\]

From (7) it is enough to prove that

\[
(8) \quad R \ll \frac{X^{1/2}}{\log^A X} \quad \text{for all} \quad A>0.
\]

4. Next we prove (8). To bound \( R \) we use three important principles, the mean-value theorem of Dirichlet polynomials (Theorem 6.1 of [5]) which states

\[
\int_{-T}^{T} |M\left(\frac{1}{2}+it\right)|^2 \, dt \ll (M+T) \sum \frac{|a_m|^2}{m},
\]

the Halász-Montgomery-Huxley large-value theorem [4] which states

\[
|\{ |t| \leq T : V<|M\left(\frac{1}{2}+it\right)| \leq 2V \} | \ll \left( \frac{M+MT}{V^2 + V^6} \right) \log^2 X,
\]

and the fact that for \( T_0<t\leq T \) and for all \( A>0 \) we have

\[
(9) \quad L\left(\frac{1}{2}+it\right) \ll \frac{L^1_{1/2}}{\log^A X},
\]

for all \( A>0 \).
which follows for example from van der Corput’s bound for trigonometrical sums (Theorem 5.13 of [6]). Note that the mean-value theorem when is applied to $L(\frac{1}{2}+it)^k$ gives

$$ \int_{-T}^{T} |L(\frac{1}{2}+it)|^{2k}dt \ll (L^k T) \sum_{n=1}^{\infty} \frac{\log k}{n} \ll x^{1/2} \log^k x.$$ 

We divide the interval $[T_0, T]$ into parts and denote the integral over the set $\Omega_0$ on which $|M(\frac{1}{2}+it)| \leq M^{1/4}$ by $R_0$ and over the set $\Omega(V)$ on which $V < |M(\frac{1}{2}+it)| \leq 2V$ by $R(V)$. As $|M(\frac{1}{2}+it)| \leq M^{1/2}$ trivially, it is possible to cover the interval $[T_0, T]$ by using $\ll \log X$ sets $\Omega_0(V)$ together with $\Omega_0$. From H"older's inequality and the mean-value theorem

$$ R_0 \ll \left( \int_{\Omega_0} |L(\frac{1}{2}+it)|^{2k}dt \right)^{1/2k} \left( \int_{\Omega_0} |M(\frac{1}{2}+it)|^{\frac{4k}{2k-1}}dt \right)^{1-\frac{1}{2k}} \ll \left( \int_{\Omega} |L|^{2k} dt \right)^{1/2k} \left( \int_{\Omega} |M|^{\frac{4k}{2k-1}} dt \right)^{1-\frac{1}{2k}}$$

$$ \ll \left( x^{1/2+\delta} \right)^{1/2k} (M+T)^{1-1/2k} \log^{1/2} x \ll X^{\frac{1}{4}+\frac{1}{8k}} \left( \log X \right)^{\frac{1}{2k}}$$

$$ \ll X^{\frac{1}{2}+\delta} \log^{1/2} x \ll \frac{x^{1/2}}{\log X}$$

for all $A > 0$. From the large-value theorem for $M^{1/4} < V < T^{1/4}$

$$ \int_{\Omega(V)} 1 dt \ll \frac{MT}{V^6} \log^2 X$$

and

$$ R(V) \ll \left( \int_{\Omega(V)} |L(\frac{1}{2}+it)|^{2k}dt \right)^{1/2k} \left( \int_{\Omega(V)} |M(\frac{1}{2}+it)|^{\frac{4k}{2k-1}}dt \right)^{1-\frac{1}{2k}} \ll \left( \int_{\Omega(V)} |L|^{2k} dt \right)^{1/2k} \left( \int_{\Omega(V)} |M|^{\frac{4k}{2k-1}} dt \right)^{1-\frac{1}{2k}}$$

$$ \ll \left( x^{1/2+\delta} \right)^{1/2k} \left( MTV^{2k-1} \right)^{1-\frac{1}{2k}} \log^3 X \ll X^{1/2+\delta} \log^{3/2} X \ll \frac{x^{1/2}}{\log^3 X}.$$
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\[ \frac{1}{X^{1/4}} \left( 1 + \frac{1}{2} \log \frac{X}{\log A} \right)^{-1/2} \]

for all \( A > 0 \). Finally if \( T^{1/4} < V \) then from the large-value theorem

\[ \int_{\Omega(V)} 1 \, dt \ll \frac{M}{V^2} \log^2 X \]

and from (9)

\[ R(V) \ll \max_{t \in \Omega(V)} \left| L\left( \frac{1}{2} + it \right) \right| V^2 \int_{\Omega(V)} 1 \, dt \ll \frac{1}{X^{1/4} \log^2 X} \frac{X^{1/2}}{\log A} \cdot \]

Now (8) follows from (10), (11) and (12). This completes the proof.

REFERENCES


