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DEFINING RELATIONS
OF CERTAIN INFINITE DIMENSIONAL GROUPS

BY

V.G. KAC and D.H. PETERSON

In our papers [8], [4], [5], we began a systematic study of the “smallest” group $G(A)$ associated to a Kac-Moody algebra and of its “unitary form” $K(A)$. The groups $G(A)$ and $K(A)$ are connected simply-connected topological groups, in general infinite-dimensional. A complex semisimple (resp. compact) connected simply-connected Lie group $G$ (resp. $K$), and a certain central extension by $\mathbb{C}^\times$ (resp. $S^1$) of the group of polynomial maps of $\mathbb{C}^\times$ into $G$ (resp. $S^1$ into $K$), provide the simplest examples of such groups $G(A)$ (resp. $K(A)$).

In the present paper, we define the groups $G(A)$ axiomatically, without reference to the corresponding Kac-Moody algebras. We then give a detailed exposition of the structure theory of the group $G(A)$ sketched in [8]. For that, we develop a theory of “refined Tits systems” (§ 3), which are groups satisfying certain axioms which describe the groups $G(A)$ more adequately than the axioms of usual Tits systems. In a similar, axiomatic fashion, we study the groups $K(A)$.

The second objective of the paper is to establish presentation theorems for the groups $G(A)$ and $K(A)$. In fact, both are special cases of a general presentation theorem for certain subgroups of a group with the structure of a Tits system (THEOREM A). The presentation theorem for $G(A)$ states that this group is an amalgamated product of its “standard parabolic subgroups of rank ≤ 2” (this follows also from a theorem of Tits [9]). On the other hand, one can reduce the problem of explicit presentation of $G(A)$ to that of the “Borel subgroup” of $G(A)$ in terms of its generating 1-parameter subgroups. We solve the latter problem in the rank 2 case (PROPOSITIONS 3.5 and 4.3) and state a conjecture in the general case. As an application
(Corollary 3.5), we generalize a theorem of Nagao [9].

The presentation of $K(A)$ is especially simple and elegant (Theorem B). It is achieved by decomposing $K(A)$ into a disjoint union of "cells", which also provides a solution to the word problem. Loosely speaking, our presentation is a "real-analytic" continuation of a presentation of an extension of a certain Coxeter group $W(A)$ by a power of $\mathbb{Z}/2\mathbb{Z}$. More precisely, we show that $K(A)$ is an amalgamated product of compact groups of semisimple rank one and two, and moreover, write the relations among the subgroups of rank one explicitly.

The "cellular decomposition" of $K(A)$ mentioned above may be regarded as an algebraic fact underlying the cellular decomposition of the associated flag variety. This decomposition plays a key role in our forthcoming work on the topological structure of the groups $K(A)$ [7].

A weaker form of the presentation theorem for compact groups was obtained in [2] by making use of a topological argument, which does not generalize to the infinite-dimensional situation. Theorem B shows that the definition of $K(A)$ given in [2] coincides with ours.

Theorem B was presented at the conference "Combinatorics and algebraic groups" in Oberwolfach in June 1983 and in a lecture course by the first author at the University of Paris in the fall of 1983. After writing this paper, we learned about the paper [13], where a presentation theorem for compact Lie groups is proved by a similar method.

It is a pleasure to acknowledge the two main sources of inspiration during our work on this paper: the book of Steinberg [10] and the lectures [12] by and discussions with Tits.

1. Coxeter systems

Let $S$ be a finite set, and let $(m_{s,t})_{s,t \in S}$ be a Coxeter matrix on $S$, i.e., a symmetric matrix of non-negative integers such that $m_{s,t} = 1$ if and only if $s = t$. Let $W$ be the associated Coxeter group, i.e., $W$ is the group on generators $S$ with defining relations

$$(st)^{m_{s,t}} = 1 \text{ for } s, t \in S.$$ (Note that for $s = t$, this relation gives $s^2 = 1$.) The pair $(W, S)$ is called a Coxeter system. If $J$ is a subset of $S$, then $W_J$ denotes the subgroup of $W$ generated by $J$.

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* A description of some of the results of this work is contained in the paper of the first author Constructing groups associated to infinite-dimensional Lie algebras, MSRI publications # 4, Springer-Verlag, 1985.
Given $w \in W$, an expression $w = s_1 \cdots s_k$, where $s_1, \ldots, s_k \in S$, is called reduced if $k$ is minimal possible, and one writes $l(w) = k$.

The following two operations on words on $S$ are called elementary:

- (E1) delete a consecutive subword $ss$;
- (E2) replace a consecutive subword $sts \cdots (m_{s,t}$ factors) by $tst \cdots (m_{s,t}$ factors).

Now we can state the first crucial lemma of the paper.

**Lemma 1.1.** Any two words on $S$ representing the same element of $W$ can be transformed to a common word by elementary operations.

**Proof.** This follows from [1, Ch. IV, § 1, n° 1.5, Proposition 4 and Lemma 4].

**Corollary 1.1.** If $R$ and $R'$ are reduced expressions of an element of a Coxeter group $W$, then $R'$ can be obtained from $R$ by elementary operations of the form (E2).

Let $A = (a_{s,t})_{s,t \in S}$ be a generalized Cartan matrix, i.e. $a_{s,s} = 2$, $a_{s,t}$ is a non-positive integer for $s \neq t$, and $a_{s,t} = 0$ implies $a_{t,s} = 0$. Put $m_{s,s}^A = 1$ and, for distinct $s, t \in S$, put $m_{s,t}^A = 2, 3, 4, 6$ or 0 according as $a_{s,t}a_{t,s} = 0, 1, 2, 3$ or $\geq 4$. Let $(W(A), S)$ be the Coxeter system associated to the Coxeter matrix $(m_{s,t}^A)$.

Let $Q$ and $Q^v$ be free abelian groups on symbols $\alpha_s$ and $\alpha_s^v$, $s \in S$, respectively. Define a bilinear pairing $Q \times Q^v \to \mathbb{Z}$ by $\langle \alpha_s, \alpha_s^v \rangle = a_{s,t}$.

**Lemma 1.2.** The formulas

\begin{equation}
(1.1) 
    s \cdot \alpha_t = \alpha_t - a_{s,t} \alpha_s; \quad s \cdot \alpha_t^v = \alpha_t^v - a_{t,s} \alpha_s^v
\end{equation}

define faithful actions of the group $W(A)$ by automorphisms of $Q$ and $Q^v$ respecting the pairing $\langle \ , \ \rangle$.

**Proof.** See e.g. [3, Proposition 3.13].

**Remark.** If every off-diagonal entry of a Coxeter matrix is $2, 3, 4, 6$ or 0, then the associated Coxeter group is called crystallographic since then, by Lemma 1.2, it has a faithful reflection representation by integral matrices (the converse is also true). These are precisely the Coxeter groups appearing in the sequel as the Weyl groups of certain infinite-dimensional groups $G(A)$; the lattices $Q$ and $Q^v$ will appear as the root and coroot lattices of the group $G(A)$. The Coxeter system $(W(A), S)$ and its action on $Q$ (or $Q^v$) determines the group $G(A)$ uniquely.
2. The group $G(A)$

Let $A = (a_{s,s'})_{s,s' \in S}$ be a generalized Cartan matrix. We associate to $A$ a group $G(A)$ as follows.

For $t \in \mathbb{C}^\times$ and $u \in \mathbb{C}$, introduce the following elements of $SL_2(\mathbb{C})$:

$$h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Let $\epsilon$ denote the compact involution of $SL_2(\mathbb{C})$, i.e. $\epsilon(a) = t^{-a^{-1}}$, so that the fixed point set of $\epsilon$ is $SU_2$.

The following axioms (G1), (G2) and (G3) determine, up to a unique isomorphism, a group $G(A)$ and homomorphisms $\varphi_s : SL_2(\mathbb{C}) \to G(A)$ for $s \in S$. Here and further on, $\varphi_s(h(t)), \varphi_s(x(u))$ and $\varphi_s(y(u))$ are denoted by $h_s(t), x_s(u)$ and $y_s(u)$, for short.

(G1) There exists a faithful $G(A)$-module $(V, \pi)$ over $\mathbb{C}$ such that each $SL_2(\mathbb{C})$-module $(V, \pi \circ \varphi_s)$ is a direct sum of rational finite-dimensional submodules.

(G2) a) $h_s(t)x_{s'}(u)h_s(t)^{-1} = x_{s'}(t^{a_{s,s'}}u)$ and $h_s(t)y_{s'}(u)h_s(t)^{-1} = y_{s'}(t^{-a_{s,s'}}u)$ for all $s, s' \in S, t \in \mathbb{C}^\times$ and $u \in \mathbb{C}$;

b) $x_s(u)y_{s'}(v) = y_{s'}(v)x_s(u)$ for all distinct $s, s' \in S$ and all $u, v \in \mathbb{C}$.

(G3) If a group $G$ and homomorphisms $\varphi_s' : SL_2(\mathbb{C}) \to G(s \in S)$ satisfy (G1) and (G2), then there exists a unique homomorphism $\psi : G(A) \to G$ such that $\varphi_s' = \psi \circ \varphi_s$ for all $s \in S$.

Put $G_s = \varphi_s(SL_2(\mathbb{C}), s \in S$. It follows from the axioms that the subgroups $G_s, s \in S$, generate the group $G(A)$. Put $H_s = \{h_s(t) | t \in \mathbb{C}^\times\}$, and let $H$ be the subgroup of $G(A)$ generated by the subgroups $H_s$. Since the $x(u)$ and $y(u)$ generate $SL_2(\mathbb{C})$, (G2a) implies

$$(2.1) \quad h_s(t)\varphi_{s'}(\begin{pmatrix} a & b \\ c & d \end{pmatrix})h_s(t)^{-1} = \varphi_{s'}\left(\begin{pmatrix} a & t^{a_{s,s'}}b \\ -a_{s,s'}c & d \end{pmatrix}\right).$$

In particular, $H$ is abelian.

In order to proceed, we need a digression on Kac-Moody algebras.

Recall that the Kac-Moody algebra $\mathfrak{g}'(A)$ associated to a generalized Cartan matrix $A$ is the Lie algebra on generators $e_s, f_s, \alpha_s^v, s \in S$, with the following defining relations:

- (g1) $[\alpha_s^v, e_t] = a_{s,t}e_t, \quad [\alpha_s^v, f_t] = -a_{s,t}f_t; \quad [e_s, f_t] = 0$ if $s \neq t$;
- (g2) $[e_s, f_s] = \alpha_s^v; \quad [\alpha_s^v, \alpha_t^v] = 0$;
(g3) $(\text{ad } e_s)^{1-a_s,t} e_t = 0, \quad (\text{ad } f_s)^{1-a_s,t} f_t = 0$ if $s \neq t$.

Then the $\alpha_s^\vee$ are linearly independent [3, Chapter 1] and the group $W(A)$ acting on the coroot lattice $Q^\vee = \sum_{s \in S} Z \alpha_s^\vee$ by (1.1) is called the Weyl group of $g'(A)$. For brevity, we write, $W_J$ for $W(A)_J$ if $J \subset S$.

The Lie algebra $g'(A)$ admits a gradation $g'(A) = \bigoplus_{\alpha \in Q} g_\alpha$ by the free abelian group $Q$ on symbols $\alpha_s$, $s \in S$, which is called the root lattice, such that $g_0 = \bigoplus_{s \in S} C \alpha_s^\vee$, $g_{\alpha_s} = C e_s$ and $g_{-\alpha_s} = C f_s$ [3, Chapter 1]. The height of $\sum_k k_s \alpha_s \in Q$ is $\sum_s k_s$.

Let $\Delta = \{ \alpha \in Q \mid g_\alpha \neq 0, \alpha \neq 0 \}$ be the set of roots of $g'(A)$; it is $W(A)$-invariant [3, Chapter 3]. Put $Z_+ = \{0,1,2,\ldots\}$ and $Q_+ = \sum_s Z_+ \alpha_s \subset Q$. Elements of $\Delta_+ := Q_+ \cap \Delta$ are called positive roots. One knows that $\Delta = \Delta_+ \sqcup -\Delta_+$ ($\sqcup$ denotes a disjoint union). Elements of $\Delta^\text{re} := \{w \cdot \alpha_s \mid w \in W(A), s \in S\}$ are called real roots. Put $\Delta^\text{re}_+ = \Delta^\text{re} \cap \Delta_+$; then $\Delta^\text{re}_+ = \Delta^\text{re}_+ \sqcup -\Delta^\text{re}_+$ (see [3, Chapters 1 and 5] for details).

In § 4, we will need

**Lemma 2.1.**

(a) If $w \in W(A)$ and $w \neq 1$, then there exists $s \in S$ such that $w \cdot \alpha_s \in -\Delta^\text{re}_+$

(b) If $J$ is a subset of $S$, then

$$\bigcap_{w \in W_J} w \cdot \Delta^\text{re}_+ = \Delta^\text{re}_+ \setminus \sum_{s \in J} Z \alpha_s.$$

(c) If $s \in S$, then the set $\Phi_s := \{ \beta \in \Delta^\text{re}_+ \setminus Z \alpha_s \mid \langle \alpha_s^\vee, \beta \rangle \geq 0 \}$ satisfies the following two properties.

(i) $\Delta^\text{re}_+ = \Phi_s \cup (s \cdot \Phi_s) \cup \{\alpha_s\}$

(ii) if $\beta \in \Phi_s$, then $\Delta_+ \cap (\beta + Z_+ \beta + Z_+ \alpha_s) = \Phi_s \cap \{\beta, \beta + \alpha_s\}$.

**Proof.** — (a) is proved e.g. in [3, Lemma 3.11]. Since $\langle \alpha_s, w \cdot \alpha_s^\vee \rangle > 0 \Leftrightarrow \langle w \cdot \alpha_t, \alpha_s^\vee \rangle > 0$ for all $s,t \in S$ and $w \in W(A)$ by [6, p. 139], the argument proving [8, Lemma 1] proves (c). (These arguments are reproduced also in [3, 2nd ed., Exercise 5.19].)

To prove (b), first note that, for any $\beta \in Q$, $\beta + \sum_{s \in J} Z \alpha_s$ is $W_J$-invariant. Hence if $\beta \in Q$ and $W_J \cdot \beta$ intersects $Q_+$ and $-Q_+$, then $\beta \in \sum_{s \in J} Z \alpha_s$. This shows that $\Delta^\text{re}_+ \setminus \sum_{s \in J} Z \alpha_s$ is $W_J$-invariant, so that $\Delta^\text{re}_+ \setminus \sum_{s \in J} Z \alpha_s \subset \bigcap_{w \in W_J} w \cdot \Delta^\text{re}_+$. Conversely, if $\beta \in \bigcap_{w \in W_J} w \cdot \Delta^\text{re}_+$, choose $\gamma \in W_J \cdot \beta$ of minimal height. Then $\gamma \in \Delta^\text{re}_+$ and $\langle \gamma, \alpha_s^\vee \rangle \leq 0$ for all $s \in J$ since $s \cdot \gamma = \gamma - \langle \gamma, \alpha_s^\vee \rangle \alpha_s$. If also $\gamma \in \sum_{s \in J} Z \alpha_s$, then $\gamma \in \sum_{s \in J} Z_+ \alpha_s$ forces $\langle \gamma, \alpha_s^\vee \rangle \leq 0$ for all $s \in S \setminus J$, since $\langle \alpha_t, \alpha_s^\vee \rangle \leq 0$ for all distinct $s,t \in S$, so that $\langle \gamma, \alpha_s^\vee \rangle \leq 0$ for all $s \in S$, which by [3, Proposition 5.1e] contradicts $\gamma \in \Delta^\text{re}_+$. This proves (b). \[\square\]
A complex \(G(A)\)-module \((V, \pi)\) is called differentiable if the \(SL_2(\mathbb{C})\)-modules \((V, \pi \circ \varphi_s)\) are direct sums of rational finite-dimensional submodules. Given such a module, we have a module \((V, d\pi)\) over \(g'(A)\) defined by:

\[
\begin{align*}
    d\pi(e_s) &= \frac{d}{du}\pi(x_s(u)) \bigg|_{u=0}, \\
    d\pi(f_s) &= \frac{d}{du}\pi(y_s(u)) \bigg|_{u=0}, \\
    d\pi(\alpha_s^\nu) &= \frac{d}{dt}\pi(h_s(t)) \bigg|_{t=1}.
\end{align*}
\]

To check this, we have to show that the relations (g1)-(g3) are annihilated by \(\pi\). Indeed, (g1) follows from (G2); the first part of (g2) is standard and the second part is clear from (2.1); (g3) follows from (g1) and (g2) by [4, Lemma 1.1]. Moreover, the \(g'(A)\)-module \((V, d\pi)\) is integrable (in the terminology of [8]), i.e. all \(d\pi(e_s)\) and \(d\pi(f_s)\) are locally nilpotent. Conversely, an integrable \(g'(A)\)-module \((V, d\pi)\) gives rise to a unique differentiable \(G(A)\)-module \((V, \pi)\) satisfying \(\pi(x_s(u)) = \exp d\pi(ue_s), \pi(y_s(u)) = \exp d\pi(uf_s), u \in \mathbb{C}\). It follows that the definition of the group \(G(A)\) by axioms (G1)-(G3) coincides with that of [8].

If \(s, t \in S\) and \(a_{s,t} = a_{t,s} = 0\), then (g1) and (g3) show that \(e_s\) and \(f_s\) commute with \(e_t\) and \(f_t\), and therefore \(G_s\) and \(G_t\) commute.

The adjoint \(g'(A)\)-module \((g'(A), \text{ad})\) gives rise to the adjoint \(G(A)\)-module \((g'(A), \text{Ad})\), which is related to a differentiable \(G(A)\)-module \((V, \pi)\) by

\[
\text{(2.2)} \\
    d\pi(\text{Ad}(g)x) = \pi(g)d\pi(x)(g)^{-1} \text{ for } g \in G(A), \ x \in g'(A).
\]

This follows from the well-known formula \((\exp d\pi(a))d\pi(x)(\exp -d\pi(a)) = d\pi((\exp \text{ad } a)x)\), for any elements \(x\) and \(a\) of a Lie algebra and any of its modules \(d\pi\) such that \(\text{ad } a\) and \(d\pi(a)\) are locally nilpotent (see e.g. [3, (3.8.1)]).

It is convenient to introduce an exponential map \(\exp\) from certain subset of \(g'(A)\) into \(G(A)\), as follows. Let \(x \in g'(A)\) be such that \(d\pi(x)\) is locally-finite for every integrable \(g'(A)\)-module \((V, d\pi)\). If there exists \(g \in G(A)\) such that \(\pi(g) = \exp d\pi(x)\) for every integrable \(g'(A)\)-module \((V, d\pi)\), we write : \(g = \exp x\). It is shown in [8] that \(\exp\) is defined on the set of all \(\text{ad}\)-locally-finite elements of \(g'(A)\) (but we will not use this fact). Note that \(x_s(u) = \exp(ue_s), y_s(u) = \exp(uf_s)\) and \(h_s(e^u) = \exp u\alpha_s^\nu\) for all \(s \in S\) and \(u \in \mathbb{C}\). It follows from (2.2) that

\[
\text{(2.3)} \\
    g(\exp x)g^{-1} = \exp(\text{Ad}(g)x), \ g \in G(A).
\]

Using integrable highest weight \(g'(A)\)-modules, one easily deduces as in [8] the following.
LEMMA 2.2.

(a) The homomorphism \((C^\times)^S \rightarrow G(A)\) defined by \((t_s)_{s \in S} \mapsto \prod_s h_s(t_s)\) is an isomorphism onto \(H\).

(b) The homomorphisms \(\varphi_s\) are injective

(c) \(G_s \cap G_{s'} = \{1\}\) for \(s \neq s'\).

Put \(H_+ = \{\prod_s h_s(t_s) \mid (t_s)_{s \in S} \in R_+^S\}\), where \(R_+\) denotes the multiplicative group of positive real numbers, and put \(T = \{\prod_s h_s(t_s) \mid (t_s)_{s \in S} \in (S^1)^S\}\), where \(S^1\) denotes the unit circle. The homomorphism of LEMMA 2.2(a) induces isomorphisms: \(R_+^S \xrightarrow{\sim} H_+, (S^1)^S \xrightarrow{\sim} T\). Note that \(H = H_+ \times T\).

Put \(\tilde{s} = \varphi_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, s \in S; \) we have

\[
\tilde{s}^2 = h_s(-1). 
\]

Recall formula (1.1). One knows that [3, LEMMA 3.8]:

\[
\text{Ad}(\tilde{s})g_\alpha = g_{s', \alpha}; \quad \text{Ad}(h)g_\alpha = g_\alpha \quad \text{for} \quad h \in H.
\]

Using (2.1), we have

\[
\tilde{s}'h_s(t)\tilde{s}'^{-1} = h_s(t)h_{s'}(t^{-a_s}t') \quad \text{for} \quad t \in C^\times.
\]

Another useful relation, obtained by calculating in \(SL_2(C)\), is

\[
y_s(t) = x_s(t^{-1})\tilde{s}h_s(-t)x_s(t^{-1}), \quad \text{for} \quad t \in C^\times.
\]

LEMMA 2.3. — If \(s \neq s'\), then

\[
\tilde{s}s'\tilde{s} \cdots = \tilde{s}' \tilde{s} \tilde{s}' \cdots \quad (m_{s, s'}^A \text{ factors on each side}).
\]

Proof ([11]). — We may assume that \(m_{s, s'}^A \neq 0\). Let \(g\) and \(g'\) denote the left- and right-hand sides of (2.8). Then, putting \(t = s\) or \(s'\) according as \(m_{s, s'}^A\) is odd or even, we obtain, using (2.3) and (2.5):

\[
gG_tg^{-1} = G_{s'}.
\]

(We also use the fact that \(SL_2(C)\) is generated by the \(x(u)\) and \(y(u)\).) Therefore we have:

\[
g'g^{-1} = \tilde{s}'g\tilde{t}^{-1}g^{-1} \in \tilde{s}'G_tg^{-1} = \tilde{s}'G_{s'} = G_{s'}.
\]
Interchanging \( s \) and \( s' \), we get \( g'g^{-1} \in G_S \). By \textsc{Lemma} 2.2(c), it follows that \( g'g^{-1} = 1 \).

\textit{Remark.} — If we take

\[
\tilde{s} = \varphi_s \begin{pmatrix} 0 & t_s \\ -t_s^{-1} & 0 \end{pmatrix},
\]

where the \( t_s \in \mathbb{C}^\times \) are arbitrary, \textsc{Lemma} 2.3 and its proof remain valid.

Let \( N \) be the subgroup of \( G(A) \) generated by \( H \) and all the \( \tilde{s}, s \in S \). Then \( H \) is a normal subgroup of \( N \) by (2.6). The group \( W = N/H \) is called the Weyl group of \( G(A) \).

\textsc{Proposition} 2.1. — There exists a unique isomorphism of \( W \) onto \( W(A) \) taking \( \tilde{s}H \) to \( s \) for all \( s \in S \).

\textit{Proof.} — (2.5) and \textsc{Lemma} 1.2 show that there exists a unique homomorphism from \( W \) to \( W(A) \) taking \( \tilde{s}H \) to \( s \) for all \( s \in S \). Formulas (2.4) and (2.8) show that there exists a unique homomorphism from \( W(A) \) to \( W \) taking \( s \) to \( \tilde{s}H \) for all \( s \in S \).

Using \textsc{Proposition} 2.1, we identify \( S \) with a subset of \( W \) by identifying \( s \) with the coset \( \tilde{s}H \in N/H = W \). In the same way, we sometimes also identify \( W(A) \) and \( W \).

\textsc{Corollary} 2.1.

(a) \((W, S)\) is a Coxeter system with Coxeter matrix \((m^A_{s, s'})_{s, s' \in S}\).

(b) \( N \) is the group on generators \( \tilde{s}(s \in S) \) and \( h_s(t) \) \((s \in S \text{ and } t \in \mathbb{C}^\times)\) with defining relations :

\begin{align*}
(N1) \quad & h_s(t)h_s(t') = h_s(tt') ; \\
(N2) \quad & h_s(t)h_{s'}(t') = h_{s'}(t')h_s(t) ; \\
(N3) \quad & \tilde{s}'h_s(t)\tilde{s}'^{-1} = h_s(t)h_{s'}(t^{-a_{s, s'}}) ; \\
(N4) \quad & \tilde{s}^2 = h_s(-1) ; \\
(N5) \quad & \tilde{s}\tilde{s}'\cdots = \tilde{s}'\tilde{s}\tilde{s}'\cdots(m^A_{s, s'} \text{ factors on each side}).
\end{align*}

\textit{Proof.} — (a) is immediate from \textsc{Proposition} 2.1. Let \( N_0 \) be the group with the generators and relations in (b), and let \( H_0 \) be the abelian normal subgroup of \( N_0 \) generated by the \( h_s(t), s \in S \) and \( t \in \mathbb{C}^\times \). Since the relations \((N1 - N5)\) hold in \( N \) by formulas (2.1), (2.4), (2.6) and (2.8), there exists a homomorphism \( \mu \) of \( N_0 \) onto \( N \) mapping the generators to the corresponding elements of \( N \). By \((N1), (N2)\) and \textsc{Lemma} 2.2(a), there exists a homomorphism \( \varphi \) of \( H \) onto \( H_0 \) such that \( \mu \circ \varphi = \text{id}_H \). Hence, \( H_0 \cap \ker \mu = \{1\} \). But \( H_0 = \mu^{-1}(H) \) by (a), so that \( \ker \mu \subset H_0 \). Hence, \( \ker \mu = \{1\} \), proving (b).
Corollary 2.2. — The centralizer of $H$ in $N$ is $H$.

Proof. — $H$ is clearly an abelian normal subgroup of $N$. Since $C$ is an infinite field, the corollary now follows from Proposition 2.1, Lemma 1.2 and formula (2.6).

Let $\widetilde{W}$ be the subgroup of $N$ generated by the $\tilde{s}$, $s \in S$, and let $H_{(2)}$ be the subgroup of $H$ generated by the $\tilde{s}^2 = h_s(-1)$, $s \in S$. (Note that $\widetilde{W}$ is the fixed point set in $N$ of the involution of $G(A)$ defined by $x_s(u), y_s(-u)$.)

Corollary 2.3.

(a) $H_{(2)} = \{h \in H | h^2 = 1\}$, and the inclusion $\widetilde{W} \subset N$ induces an isomorphism from $\widetilde{W}/H_{(2)}$ onto $W = N/H$.

(b) There exists a unique map $w \mapsto \tilde{w}$ from $W$ into $\widetilde{W}$ satisfying

- $\tilde{1} = 1$;
- $\tilde{s} = \varphi_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for all $s \in S$;
- $\tilde{ww'} = \tilde{w}\tilde{w'}$ if $w, w' \in W$ and $l(ww') = l(w) + l(w')$.

If $\psi : \widetilde{W} \rightarrow W$ is the canonical map, then $w \mapsto \tilde{w}$ is a well-defined section of the map $\psi$.

Proof. — $H_{(2)} = \{h \in H | h^2 = 1\}$ by Lemma 2.2(a). By Proposition 2.1 and Lemma 2.3, $N = \widetilde{W}H$ and $\widetilde{W} \cap H$ is generated by the $\tilde{s}^2$, $s \in S$. (a) follows. (b) follows from Lemma 2.3 and Corollary 1.1.

Corollary 2.4. — $\widetilde{W}$ is the group on generators $\tilde{s}$, $s \in S$, with defining relations:

- (n1) $\tilde{t}\tilde{s}^2\tilde{t}^{-1} = \tilde{s}^2\tilde{t}^{-2a_{st}}$;
- (n2) $\tilde{s}\tilde{t}\tilde{s}\ldots = \tilde{s}\tilde{t}\ldots$ ($m_{st}$ factors on each side).

Proof. — For $s \in S$, put $h_s = \tilde{s}^2$. Then (n1) and (n2) imply:

- (m1) $h_s^2 = 1$;
- (m2) $h_s h_t = h_t h_s$;
- (m3) $\tilde{t} h_s \tilde{t}^{-1} = h_s h_t^{-as,t}$;
- (m4) $\tilde{s}^2 = h_s$;
- (m5) $\tilde{s}\tilde{t}\tilde{s}\ldots = \tilde{s}\tilde{t}\ldots$ ($m_{st}$ factors on each side).

Indeed, (m3), (m4) and (m5) are clear, and (m1) follows from (n1) with $t = s$. To check (m2), write $h_t h_s h_t^{-1} = \tilde{t}(\tilde{t} h_s \tilde{t}^{-1})\tilde{t}^{-1} = \tilde{t}(h_s h_t^{-as,t})\tilde{t}^{-1} = (\tilde{t} h_s \tilde{t}^{-1})(\tilde{t} h_t^{-as,t} \tilde{t}^{-1}) = (h_s h_t^{-as,t})(h_t^{-as,t} h_t^{2as,t}) = h_s$ by (m3) and (m4). This verifies (m2).
The rest of the proof is essentially the same as that of Corollary 2.1(b).
(One uses \(-1 \neq 1\) in \(C^x\) to construct the analogue of \(\varphi\).)  

Introduce the 1-parameter subgroups \(U_{\alpha s} = \{x_s(u) \mid u \in C\}, s \in S\), of \(G(A)\). For a real root \(\alpha = w \cdot \alpha_s\), take \(n \in N\) such that \(w = nH\) and put \(U_{\alpha} = nU_{\alpha s}n^{-1}\). We have \(U_{\alpha} = n(\exp g_{\alpha s})u^{-1} = \exp(\text{Ad}(n)g_{\alpha s}) = \exp g_{w \cdot \alpha s} = \exp g_{\alpha}\); hence, the 1-parameter group \(U_{\alpha}\) depends only on \(\alpha\).

Note that \(U_{-\alpha s} = \{y_s(u) \mid u \in C\}\). We have:

\[ nU_{\alpha}n^{-1} = U_{w \cdot \alpha} \quad \text{for} \quad n \in N, \quad w \in nH, \quad \alpha \in \Delta^{re}. \]

Recall that \(\Delta^{re} = \Delta^{re}_+ \sqcup -\Delta^{re}_+\). Let \(U_+\) (resp. \(U_-\)) be the subgroup of \(G\) generated by the subgroups \(U_\alpha\) (resp. \(U_-\alpha\), \(\alpha \in \Delta^{re}_+\). (This definition is due to Tits [12]). These subgroups are analogues of maximal unipotent subgroups of reductive algebraic groups; they play an important role in the structure theory of the groups \(G(A)\), which we will discuss in §§ 3 and 4.

Finally, it is clear from the axioms (G1)–(G3) that there exists a unique involution \(\omega\) of \(G(A)\) such that \(\varphi_s \circ \epsilon = \omega \circ \varphi_s\) for all \(s \in S\) (recall that \(\epsilon\) is the compact involution of \(SL_2(C)\)). We call \(\omega\) the compact involution of \(G(A)\). It is clear that the subgroups \(G_s\) and \(H\) are stable under \(\omega\) and that \(\widehat{W}\) is pointwise fixed by \(\omega\). Furthermore, \(\omega(U_{\alpha}) = U_{-\alpha}\) for all \(\alpha \in \Delta^{re}\), and therefore \(\omega(U_+) = U_-\).

Remark. — \(g'(A)\) can be characterized by axioms similar to (G1)–(G3). Also, the category of all integrable \(g'(A)\)-modules and all \(g'(A)\)-module homomorphisms is isomorphic in the obvious way to the category of all differentiable \(G(A)\)-modules over \(C\) and all \(G(A)\)-module homomorphisms, and this isomorphism is compatible with tensor products, etc.


We call a 6-tuple \((G, N, U_+, U_-, H, S)\) a refined Tits system if the following axioms hold*:

\begin{itemize}
  \item[(RT1)] \(G\) is a group, and \(N, U_+\) and \(U_-\) are subgroups of \(G\); \(G\) is generated by \(N\) and \(U_+\); \(H\) is a normal subgroup of \(N\); \(H\) normalizes \(U_+\) and \(U_-\); \(S\) is a subset of \(W := N/H\); \(S\) generates \(W\); \(s^2 = 1\) for all \(s \in S\).
  
  For a subgroup \(M\) of \(G\) and \(w = nH \in W\), we write \(wM\) for \(nM\) and \(Mw\) for \(Mn\) if \(M \supset H\), and \(M^w\) for \(n^{-1}Mn\) if \(H\) normalizes \(M\).
  
  \item[(RT2)] For \(s \in S\), put \(U_s = U_+ \cap U_-^s\). If \(s \in S\) and \(w \in W\), then :
\end{itemize}

* The reader may compare this definition with that of a split BN-pair, extensively used in finite group theory.
(a) \( U_s^w \setminus \{1\} \subset U_sHsU_s; U_s^w \neq \{1\}. \)
(b) \( U_s^w \subset U_+ \) or \( U_s^w \subset U_. \)
(c) \( U_+ = U_s(U_+ \cap U_s^w). \)

(RT3) If \( u_- \in U_-, n \in N, u_+ \in U_+ \) and \( u_-nu_+ = 1 \), then \( u_- = n = u_+ = 1. \)

Throughout this section, we assume only that \((G, N, U_+, U_-, H, S)\) is a refined Tits system. We will show in §4 that \((G(A), \ldots)\) is a refined Tits system.

Let \( B \) be the subgroup of \( G \) generated by \( H \) and \( U_+ \), so that \( B = H \cdot U_+ \) by (RT1,3).

Remark. — If \((G, N, U_+, U_-, H, S)\) is a refined Tits system, and if \( M \) is a subgroup of \( G \) such that \( U_s \cup U_s^w \subset M \) for all \( s \in S \), and \( M \) is generated by \( N \cap M \) and \( U_+ \cap M \), then \((M, N \cap M, U_+ \cap M, U_- \cap M, H \cap M, S_M)\) is a refined Tits system, where \( S_M \) corresponds to \( S \) under the isomorphism \((N \cap M)/(H \cap M) \cong N/H \) induced by the inclusion \( N \cap M \subset N \). In particular, the subgroup of \( G \) generated by the \( U_s \) and \( U_s^w \), and the subgroup of \( G \) generated by \( N \) and the \( U_s \), satisfy these conditions.

**LEMMA 3.1.**
(a) \( B \cap N = H. \)
(b) If \( s \in S \), then \( sBs \neq B. \)
(c) Let \( s \in S \) and \( w \in W. \) Then:
   (i) Exactly one of the following holds:
       \( U_s^w \subset U_+ \) and \( U_s^w \subset U_-; \)
       \( U_s^w \subset U_+ \) and \( U_s^w \subset U_. \)
   (ii) \( sBw \subset BswU_s^w \) and \( sBw \subset Bsw \cup BwU_s^w. \)

**Proof.** — (a) follows from (RT3).
To prove (b), note that \( U_s \cap sBs = (U_s^w \cap B)^s \subset (U_- \cap B)^s = \{1\} \neq U_s = U_s \cap B. \)

To prove c(i), note that \( U_s^w \) is contained in exactly one of \( U_+ \) and \( U_- \), and \( U_s^w \) is contained in exactly one of \( U_+ \) and \( U_- \). But by (RT2a), \( U_s^wU_s^wU_s^w \cap N \neq \{1\}. \) Since \( U_- \cap N = \{1\} \neq U_+ \cap N \) by (RT3), \( U_s^w \) and \( U_s^w \) cannot both be contained in \( U_- \) or in \( U_+. \) This proves c(i).

To prove c(ii), we write \( sBw = s[(U_+ \cap U_+)HU_s]w = (U_+ \cap U_+)HswU_s^w \subset BswU_s^w \) and \( sBw = (U_+ \cap U_+)U_s^wHsw \subset (U_+ \cap U_+)\{(1) \cup U_sHsU_s\}Hsw \subset Bsw \cup BwU_s^w. \)

**LEMMA 3.1** shows that \((G, B, N, S)\) is a Tits system (see [1] for the definition). The following are some well-known properties of Tits systems [1]:

1. \( G = \coprod_{w \in W} BwB \) (Bruhat decomposition);
2. \((W, S)\) is a Coxeter system;
(3.3) \( l(sw) > l(w) \iff sBw \subset BswB \) for \( s \in S \) and \( w \in W \).

(3.4) \( P_J := BW_jB \) is a subgroup of \( G \) for any \( J \subset S \), and any subgroup of \( G \) containing \( B \) is of this form.

Since \( (W, S) \) is a Coxeter system by (3.2), we have its Coxeter matrix
\[
(m_{s,t})_{s,t \in S} = (m_{s,t} \mathbb{Z} = \{n \in \mathbb{Z} \mid (st)^n = 1\}).
\]

The groups \( P_J \) of (3.4) are called standard parabolic subgroups of \( G \). We sometimes write \( P_s \) for \( P_{\{s\}} \), \( s \in S \); these are called minimal standard parabolics. Note that for any \( J \subset S \), \((P_J, W_J, H, U_+, U_- \cap P_J, H, J)\) is a refined Tits system.

**Corollary 3.1.** — The normalizer of \( U_+ \) in \( G \) is \( B \).

**Proof.** — The normalizer, say \( P \), of \( U_+ \) in \( G \) clearly contains \( B \). If \( P \neq B \), then \( sH \subset P \) for some \( s \in S \) by (3.4), so that \( sH \) also normalizes \( B = HU_+ \). This contradicts \( sBs \neq B \) from Lemma 3.1.1.

Let \( I \) be a set, and let \( (M_i)_{i \in I} \) be an indexed set of groups. For \( i, j \in I \), let \( M_{\{i,j\}} \) be a group and let \( \varphi_{ij} = M_{\{i,j\}} \to M_i \) be a homomorphism. (Note that \( M_{\{i,j\}} = M_{\{j,i\}} \).) The amalgamated product of the \( \varphi_{ij} \) is a pair \((M, (\varphi_i)_{i \in I})\), unique up to a unique isomorphism, satisfying :

**AP1** \( M \) is a group, and the \( \varphi_i : M_i \to M \) are homomorphisms satisfying \( \varphi_i \circ \varphi_{ij} = \varphi_j \circ \varphi_{ji} \) for all \( i, j \in I \).

**AP2** If \( L \) is a group and if \( \psi_i : M_i \to L, i \in I \), are homomorphisms satisfying \( \psi_i \circ \varphi_{ij} = \psi_j \circ \varphi_{ji} \) for all \( i, j \in I \), then there exists a unique homomorphism \( \psi : M \to L \) satisfying \( \psi_i = \psi \circ \varphi_i \) for all \( i \in I \).

If the \( M_i \) are subgroups of a group \( F \) and \( \varphi_{ij} \) is the inclusion \( M_i \cap M_j \subset M_i \) for all \( i, j \in I \), then we say that the group \( M \) defined above is the amalgamated product of the \( M_i \). If, moreover, the canonical homomorphism \( \psi : M \to F \) defined by (AP2) is bijective, then we say that \( F \) is the amalgamated product of its subgroups \( M_i \).

We say that a subgroup \( M \) of \( G \) is \( W \)-graded if, putting \( M_w = M \cap BwB \), we have for all \( w, w' \in W \):

\[
(3.5) \quad M_{ww'} = M_w M_{w'} \quad \text{if} \quad l(ww') = l(w) + l(w').
\]

The next two results hold for arbitrary Tits systems.

**Theorem A.**

(a) Any \( W \)-graded subgroup \( M \) of \( G \) is the amalgamated product of its intersections with the \( P_J, |J| \leq 2 \).

(b) \( G \) and \( N \) are \( W \)-graded subgroups of \( G \). If \( L \) is a \( W \)-graded subgroup of \( G \), and if \( M \) is a subgroup of \( G \) satisfying \( M(L \cap B) = L \), then \( M \) is a \( W \)-graded subgroup of \( G \).
(c) Let $L$ be a $W$-graded subgroup of $G$, and let $Z_s$, $s \in S$, be subsets of $G$ such that $L \cap BsB = Z_s(L \cap B)$ for all $s \in S$. Let $M$ be a subgroup of $L$ containing the $Z_s$. Then $M$ is a $W$-graded subgroup of $G$, and $M \cap BsB = Z_s(M \cap B)$ for all $s \in S$. For $s, t \in S$ and $z_1 \in Z_s$, $z_2 \in Z_t$, $z_3 \in Z_s, \ldots$, choose $z'_1 \in Z_t$, $z'_2 \in Z_s$, $z'_3 \in Z_t, \ldots$ and $b \in M \cap B$ such that

$$z_1z_2z_3 \cdots = (z'_1z'_2z'_3 \cdots)b \quad (m_{s,t} \text{ factors } z \text{ on each side}).$$

Then $M$ is the amalgamated product of $M \cap B$ and the $M \cap P_s$, $s \in S$, modulo the relations (3.6).

Proof. — Let $L$ be a $W$-graded subgroup of $G$, put $B_L = L \cap B$, and let the $Z_s$, $s \in S$, be subsets of $G$ satisfying $L \cap BsB = Z_sB_L$. Note that $L \cap P_s = Z_sB_L \cup B_L \supset B_L Z_s$. Since $L$ is $W$-graded, we have $L \cap Bs_1 \cdots s_kB = (L \cap Bs_1B) \cdots (L \cap Bs_kB) = (Z_{s_1}B_L) \cdots (Z_{s_k}B_L) = Z_{s_1} \cdots Z_{s_k}B_L$ for every reduced expression $s_1 \cdots s_k$. In particular, $B_L$ and the $Z_s$ generate $L$. Choose relations (3.6) as in (c) (with $M = L$), and let $\tilde{L}$ be the amalgamated product of $B_L$ and the $L \cap P_s$, $s \in S$, modulo the chosen relations. We may regard $B_L$ and the $Z_s$ as subsets of $\tilde{L}$. We clearly have:

(i) $B_L$ is a subgroup of $\tilde{L}$.

(ii) $Z_s$, $s \in S$, is a subset of $\tilde{L}$.

(iii) $B_L$ and the $Z_s$ generate $\tilde{L}$.

(iv) For all $s \in S$, $B_L \cup Z_sB_L (= L \cap P_s)$ is a subgroup of $\tilde{L}$.

(v) For all $s, t \in S$, $Z_sZ_tZ_s \cdots B_L = Z_tZ_sZ_t \cdots B_L$

$$(m_{s,t} \text{ factors } Z \text{ on each side}).$$

Using Lemma 1.1, we deduce that for every $g \in \tilde{L}$, there exists a reduced expression $s_1 \cdots s_k$, where $s_1, \ldots, s_k \in S$, such that $g \in Z_{s_1} \cdots Z_{s_k}B_L$. Now let $\psi : \tilde{L} \to L$ be the canonical surjective homomorphism defined by (AP2). If $\psi(g) = 1$, then by (3.1), $\psi(g) \in Bs_1 \cdots s_kB$ forces $k = 0$ and hence $g \in B_L$. Since $\psi$ is the identity on $B_L$, we deduce that $g = 1$. Hence, $\psi$ is bijective. This verifies (a) and also the case $M = L$ of (c).

We now prove (b). By (3.2) and (3.3), $G$ and $N$ are $W$-graded subgroups of $G$. Now let $L$ be a $W$-graded subgroup of $G$ and let $M$ be a subgroup of $G$ satisfying $M(L \cap B) = L$. For $w \in W$, put $L_w = L \cap BwB$ and $M_w = M \cap BwB$. Then, if $w, w' \in W$ and $l(ww') = l(w) + l(w')$, we have

$$M_{ww'}(L \cap B) = L_{ww'} = L_wL_{w'} = M_w(L \cap B)L_{w'}$$

$$= M_wL_{w'} = M_wM_{w'}(L \cap B),$$

and hence

$$M_{ww'} = M_{ww'}(M \cap B) = M_{ww'}(L \cap B) \cap M$$

$$= M_wM_{w'}(L \cap B) \cap M = M_wM_{w'}(M \cap B) = M_wM_{w'}.$$
This verifies (b). (c) follows from (b) and the special case \( M = L \) of (c).

Remark. — For \( M = G \), TITS (see [9]) has proved a stronger version of (a): \( G \) is the amalgamated product of \( N, B \) and the \( P_s \). Actually, Tits defined the groups associated to \( g'(A) \) in this way [12]. Our results imply that our group \( G(A) \) is isomorphic to his “minimal” group. In [12] one can find also a discussion of the relationship of these groups to that considered by other authors.

If \( X \) and \( Y_1, \ldots, Y_k \) are subsets of \( G \), we write \( X = Y_1 \cdots Y_k \) [unique] if \((g_1, \ldots, g_k) \mapsto g_1 \cdots g_k\) defines a bijection from \( Y_1 \times \cdots \times Y_k \) onto \( X \).

The following crucial statement is a generalization of a theorem of STEINBERG [10, THEOREM 15].

PROPOSITION 3.1. — If \( w, w' \in W \) satisfy \( l(ww') = l(w) + l(w') \), and if \( X, Y \) are subsets of \( G \) satisfying \( BwB = XB \) [unique] and \( Bw'B = YB \) [unique], then \( Bww'B = X'YB \) [unique].

Proof. — Fix subsets \( X_s \) of \( G \), \( s \in S \), such that \( BsB = X_sB \) [unique]. First, consider the case \( w = s \in S \). Then by (3.3), we have \( Bsw'B = (BsB)(Bw'B) = X_sBw'B = X_sYB \). To prove uniqueness, suppose \( xyb = x'y'b' \), where \( x, x' \in X_s, y, y' \in Y, b, b' \in B \). If \((x')^{-1}x \in BsB\), then, by (3.3), \( y'b' = (x')^{-1}xyb \in Bsw'B \), which is impossible since \( y'b' \in Bw'B \) (the decomposition (3.1) is disjoint). Hence, by (3.4), the only possibility is that \( x'^{-1}x \in B \). It follows that \( x \in x'B \) and hence \( x = x' \). But then \( yb = y'b' \) and hence \( y = y' \), \( b = b' \). (This argument is due to STEINBERG [10].)

Now, fix \( w \in W \). Taking a reduced expression \( w = s_1 \cdots s_k \), we deduce by induction on \( k \) from what has already been proved:

\[ Bs_{s_1} \cdots s_kw'B = (Bs_1B)(Bs_2 \cdots s_kw'B) = X_{s_1}X_{s_2} \cdots X_{s_k}YB \text{[unique]} \]

Put \( X' = X_{s_1} \cdots X_{s_k} \) for short; we have proved \( Bww'B = X'YB \) [unique] for any choice of \( Y \). We have: \( Bww'B = X'YB = X'(BYB) = (X'B)YB = (XB)YB = X(BYB) = XYB \). To prove uniqueness for any choice of \( X \), we show:

\[ z, z' \in BwB \quad \text{and} \quad zBw'B \cap z'Bw'B \neq \emptyset \Rightarrow zB = z'B. \]

Indeed, write \( z = xb, z' = x'b' \), where \( x, x' \in X' \) and \( b, b' \in B \). Then \( xBw'B \cap x'Bw'B \neq \emptyset \), hence \( xYB \cap x'YB \neq \emptyset \), hence \( x = x' \) and (3.7) is proved.

If now \( x, x' \in X \) but \( xYB \cap x'YB \neq \emptyset \), then \( xB = x'B \) from (3.7), so \( x = x' \), which implies the uniqueness in question.

LEMMA 3.2. — The following three assertions on \( s \in S \) and \( w \in W \) are equivalent:

(3.7) \( z, z' \in BwB \quad \text{and} \quad zBw'B \cap z'Bw'B \neq \emptyset \Rightarrow zB = z'B. \)
(i) \( U_s^w \subset U_+; \)
(ii) \( U_s^{sw} \subset U_-; \)
(iii) \( l(sw) > l(w). \)

Proof. — By Lemma 3.1(c) and (3.3) we have: \( U_s^w \subset U_+ \Rightarrow sBw \subset BswB \Rightarrow l(sw) > l(w) \Rightarrow sBsw \notin BwB \Rightarrow U_s^{sw} \notin U_+ \Rightarrow U_s^{sw} \subset U_- \Rightarrow U_s^{w} \subset U_+. \)

For \( s \in S, \) let \( G_s \) be the subgroup of \( G \) generated by \( U_s \) and \( U_s^w. \)

Corollary 3.2. — If \( s, t \in S \) and \( w \in W, \) then:
(i) \( U_s^w = U_t \Leftrightarrow wt = sw \) and \( l(sw) > l(w); \)
(ii) \( U_s^{sw} = U_t^t \Leftrightarrow wt = sw \) and \( l(sw) < l(w); \)
(iii) \( \{U_s^w, U_s^{sw}\} = \{U_t, U_t^t\} \Leftrightarrow wt = sw; \)
(iv) \( G_s^w = G_t^t \Leftrightarrow wt = sw. \)

Proof. — If \( wt = sw \) and \( l(sw) > l(w), \) then \( U_s^w \subset U_+ \) and \( U_s^{wt} = U_s^{sw} \subset U_- \) by Lemma 3.2, so that \( U_s^w \subset (U_+ \cap U_-) = U_t; \) since \( U_t^{w-1} \subset U_s \) by symmetry, we get \( U_s^w = U_t. \) Now suppose that \( U_s^w = U_t. \) Then \( U_s^w \subset U_+ \) and \( U_s^{wt} \subset U_- \), so that \( l(sw) > l(w) \) and \( l(sw) < l(wt) \) by Lemma 3.2. By [1], we deduce that \( wt = sw. \)

This proves (i); (ii) follows from (i), and (iii) follows from (i) and (ii). (iv) follows from (iii) since \( \{U_s^w, U_s^{sw}\} = \{G_s^w \cap U+, G_s^w \cap U_-\} \) and \( \{U_t, U_t^t\} = \{G_t \cap U+, G_t \cap U_-\}. \)

We now prove analogues of several of the results of § 2 for arbitrary refined Tits systems.

Corollary 3.3.
(a) Let \( s, t \in S, \) and assume that \( G_s \cap G_t = \{1\}. \) Choose \( \tilde{s} \in G_s \cap sH \) and \( \tilde{t} \in G_t \cap tH. \) Then

\[
\tilde{s}\tilde{t}\tilde{s} \cdots = \tilde{t}\tilde{s}\tilde{t} \cdots (m_{s,t} \text{ factors on each side}).
\]

(b) Assume that \( G_s \cap G_t = 1 \) whenever \( s, t \in S \) and \( m_{s,t} \geq 2, \) and choose elements \( \tilde{s}, \tilde{t} \) of \( G_s \cap sH, \) \( s \in S. \) Let \( \tilde{W} \) be a subgroup of \( N \) containing the \( \tilde{s}, \) \( s \in S. \) Then:

(i) There exists a function \( w \rightarrow \tilde{w} \) from \( W \) into \( \tilde{W} \) satisfying: \( \tilde{1} = 1; \) \( \tilde{s}, \) \( s \in S, \) is as selected; \( \tilde{ww'} = \tilde{w}\tilde{w'} \) if \( w, w' \in W \) and \( l(ww') = l(w) + l(w'); \) \( \tilde{w}H = \tilde{w} \) for all \( w \in W. \)

(ii) \( \tilde{W} \) is the amalgamated product of its subgroups \( \tilde{W} \cap B = \tilde{W} \cap H \) and \( \tilde{W} \cap P_s = \tilde{W} \cap (H \cup sH), s \in S, \) modulo the relations (3.8).

Proof. — To prove (a), let \( g \) and \( g' \) be the left-hand and right-hand sides of (3.8), respectively, and put \( w = sts \cdots (m_{s,t} \text{ factors}) \) and \( r = w^{-1}tw. \)
Using \( s^2 = t^2 = (st)^{m_{s,t}} = 1 \), we have: \( r = s \) or \( r = t \), so that \( r \in S \), and \( \tilde{t} = g' \tilde{r} \). Using Corollary 3.2, we have \( gg' \mathbf{t}^{-1} = g \tilde{r} g^{-1} \tilde{t}^{-1} \in gG_t g^{-1} G_t = G^r g^{-1} G_t = G_t G_t = G_t \) and, similarly, \( g' g^{-1} \in G_s \). Hence, \( g' \mathbf{t}^{-1} \in G_s \cap G_t = \{ 1 \} \), so that \( g = g' \), proving (a). \( b(i) \) follows from (a) and Corollary 1.1, \( b(i) \) follows from (a) and Theorem A.

**Proposition 3.2.**

(a) \( G = \bigsqcup_{n \in N} U_+ n U_+ \) (Bruhat decomposition).

(b) If \( w \in W \), then \( U_+ w B = U_+ (w H) (U_+ \cap U_+) \) [unique].

(c) \( G = U_+ U_- \).

(d) If \( w, w' \in W \) satisfy \( l(w w') = l(w) + l(w') \), then:

\[ \begin{align*}
\text{(i)} \quad U_- \cap U_+ w' = (U_- \cap U_+) w' (U_- \cap U_+) \quad \text{[unique]}; \\
\text{(ii)} \quad U_+ \cap U_- w' = (U_+ \cap U_-) w' (U_+ \cap U_-) \quad \text{[unique]}; \\
\text{(iii)} \quad U_+ \cap U_+ w' = (U_+ \cap U_-) w' (U_+ \cap U_+ w') \quad \text{[unique]}. 
\end{align*} \]

**Proof.** By the axioms, we have \( B^s B = U^s B \) [unique] for each \( s \in S \). By repeated use of Proposition 3.1, we deduce that if \( l(w) = k \) and \( w = s_1 \cdots s_k \), where \( s_1, \ldots, s_k \in S \), then \( B^w B = U^{s_1 \cdots s_k} U^{s_2 \cdots s_k} \cdots U^{s_k} B \) [unique]. But \( U^{s_1 \cdots s_k} U^{s_2 \cdots s_k} \cdots U^{s_k} \subset U_- \cap U_+ \) by Lemma 3.2, and \( (U_- \cap U_+) B \subset B^w B \). Since \( U_- \cap B = \{ 1 \} \), we deduce that

\[ U_- \cap U_+ w = U^{s_1 \cdots s_k} U^{s_2 \cdots s_k} \cdots U^{s_k} \quad \text{[unique]} \]

and

\[ (3.8.1.) \quad B^w B = (U_- \cap U_+) B \quad \text{[unique]}. \]

The first equality applied to \( w w' \) implies d(i), and (3.8.1) applied to \( w^{-1} \) implies (b) by taking inverses. By applying d (i) to \( w^{-1} w^{-1} \), taking inverses and conjugating by \( w w' \), we obtain d(ii).

By induction on \( l(w) \), we next prove

\[ (3.8.2) \quad U^w_+ = (U^w_+ \cap U_-)(U^w_+ \cap U_+) \quad \text{[unique]}. \]

We may assume \( w \neq 1 \). Choose \( s \in S \) such that \( l(sw) < l(w) \). Then \( U_+ \subset U_s U_+ \) by (RT2), so that \( U^w_+ \subset U^w_s U^s_+ \). Since \( U^w_s \subset U_- \) by Lemma 3.2, the induction hypothesis gives \( U^w_+ \subset U_- U_+ \). Therefore,

\[ U^w_+ \cap B = (U^w_+ \cap U_- U_+) \cap B = U^w_+ \cap (U_- U_+ \cap B) = U^w_+ \cap U_+, \]

the last equality by (RT3). Since \( U^w_+ \subset B^w B \), (3.8.2) now follows from (3.8.1).
We now prove (iii). Using (3.8.2) applied to \(w'\) and \(w\), we obtain
\[
U_+ = (U_+ \cap U_+^w)(U_+ \cap U_+^w) \quad \text{and} \quad U_+^w = (U_+^w \cap U_+^w)(U_+^w \cap U_+^w).
\]
Since \(U_+ \cap U_+^w = \{1\}\), we deduce
\[
U_+ = (U_+ \cap U_+^w \cap U_+^w)(U_+ \cap U_+^w \cap U_+^w).
\]

But \(U_+^w \cap U_+^w \subset U_-\) by d(i), so that the first factor \(U_+ \cap U_+^w \cap U_+^w\) is \(\{1\}\); therefore, \(U_+ \cap U_+^w = U_+ \cap U_+^w \cap U_+^w\), i.e., \(U_+ \cap U_+^w \subset U_+^w\). By (3.8.2) applied to \((w')^{-1}\) and d(ii), we have
\[
U_+ = (U_+ \cap U_-^w)(U_+ \cap U_-^w) \quad \text{[unique]}
\]
\[
= (U_+ \cap U_-^w)(U_+ \cap U_-^w)(U_+ \cap U_-^w) \quad \text{[unique]}
\]

Since the first and third factors are contained in \(U_+^w\), and the second factor intersects \(U_+^w\) in \(\{1\}\), we obtain d(iii).

By (3.8.2) applied to \(w = nH\), we have \(U_+ n U_+ \subset nU_+ U_+\) for all \(n \in N\). If \(n, n' \in N\) and \(U_+ n U_+ \cap U_+ n' U_+ \neq \emptyset\), then \(n' \in U_+ n U_+ \subset nU_+ U_+\) and so \(n' = n\) by (RT3). Using (3.1), we deduce (a) and \(G = NU_+ U_+\). (c) follows by taking inverses.

**Corollary 3.4.** \(\bigcap_{w \in W} U_+ w = \{1\}\).

**Proof.** Suppose \(u \in \bigcap_{w \in W} U_+ w\). By (3.1) and Proposition 3.2 (b) write \(u = u_+ u_- n\), where \(u_+ \in U_+\), \(n \in N\) and \(u_- \in U_- \cap nU_+ n^{-1}\). Then \([U_-(nn^{-1})^{-1}]nu_+ = 1\), and \(nu_- n^{-1} \in U_-\) by assumption, so that by (RT3), \(u_- = nun^{-1}\) and \(n = 1\). Since \(u_- \in U_- \cap nU_+ n^{-1} = U_- \cap U_+ = \{1\}\), we have \(u = 1\).

**Proposition 3.3.**

(a) \(G = \bigsqcup_{n \in N} U_- n U_+\) (Birkhoff decomposition).

(b) If \(w \in W\), then \(U_- w B = U_- (wH)(U_+ \cap U_+^w)\) [unique].

(c) \(G = U_- U_+ N\).

**Proof.** If \(s \in S\) and \(w \in W\), then \(sBw \subset BswU_- \cup BwU_-\) by Lemma 3.1 (c). We conclude that \(U_+ N U_-\) is stable under left multiplication by \(N\) and \(U_+\) and hence equals \(G\). Hence, \(G = G^{-1} = U_- U_+\). By (3.8.2) applied to \(w = n^{-1}H\), we have \(U_- n U_+ \subset U_- U_+ n\) for all \(n \in N\). If \(n, n' \in N\) and \(U_- n U_+ \cap U_- n' U_+ \neq \emptyset\), then \(n' \in U_- n U_+ \subset U_- U_+ n\) and so \(n' = n\) by (RT3). Using (3.1), we deduce (a) and (c). (b) follows from (3.8.2) applied to \(w^{-1}\) and (RT3).

**Proposition 3.4.** \(U_-\) is generated by its subgroups \(U_s^w\), where \(s \in S\) and \(w \in W\) are such that \(l(sw) < l(w)\).
Proof. — Let $U'$ be the subgroup of $U_-$ generated by these $U_s^w$. Then $G = U'NU_+$ by the argument proving Proposition 3.3(a). (We also use Lemma 3.2 here.)

Hence, $U' \subset U_- \subset U'NU_+$, which implies $U_- = U'$ by (RT3).

We now determine the structure of $U_-$ in certain cases.

Proposition 3.5.

(a) If $s \in S$, $w \in W$ and $l(w^{-1}sw) = 2l(w) + 1$, then

$$U_- \cap U_s^{sw} \subset Bw^{-1}swB \cup (U_- \cap U_s^{w}).$$

(b) If $|S| = 2$ and $s \in S$, then

$$U_-^{(s)} := U_- \cap \left( \bigcup_{w \in W, l(w) > l(ws)} U_s^{w} \right)$$

is a subgroup of $U_-$. 

(c) If $S = \{s,t\}$ and $m_{s,t} = 0$, so that $W$ is an infinite dihedral group, then $U_-$ is the free product of its subgroups $U_-^{(s)}$ and $U_-^{(t)}$ defined in (b).

Proof. — In the situation of (a), write $w = s_1 \cdots s_k$, where $k = l(w)$. Then we have, by Proposition 3.2 d(i) applied to $sw$ and by (RT2):

$$U_w^{-1} \cap (U_s^w \setminus U_s) = (U_w^{-1} \cap U_s)(U_s^w \setminus \{1\}) \subset B(U_sHsU_s) \subset BsB,$$

and hence, by (3.3),

$$U_- \cap (U_s^{sw} \setminus U_s^{w}) \subset w^{-1}BsBw \subset Bw^{-1}swB.$$

This proves (a).

We now prove (b). Let $S = \{s,t\}$. If $m_{s,t} \neq 0$, we put $w_0 = sts \cdots (m_{s,t} \text{ factors})$. Using Proposition 3.4, we then deduce that $U_+^{w_0} \supset U_-$ and hence that $U_-^{(s)} = U_-^{(t)} = U_-$. If $m_{s,t} = 0$, then it is easy to check that for $n = 1,2,3, \ldots$, there exists a unique $w_n \in W$ satisfying $l(w_n) = n > l(w_ns)$, and by using Proposition 3.2 d(i) that $U_- \cap U_+^{wn} \subset U_- \cap U_+^{wn+1}$, so that $U_-^{(s)}$ is an increasing union of subgroups of $U_-$ and hence is a subgroup of $U_-$. This proves (b).

To prove (c), note that, by using Proposition 3.4, $U_-^{(s)}$ and $U_-^{(t)}$ generate $U_-$. For $r \in S$, put $W^{(r)} = \{w \in W \mid l(rw) = l(wr) < l(w)\}$; then $U_-^{(r)} \setminus \{1\} \subset \bigcup_{w \in W^{(r)}} BwB$ by using (a). Moreover, it is easy to check that if
\[ w_1 \in W^{(s)}, w_2 \in W^{(t)}, w_3 \in W^{(s)}, \ldots, \text{then } l(w_1 \cdots w_n) = l(w_1) + \cdots + l(w_n) \]

for \( n = 1, 2, 3, \ldots \) Hence, by (3.1) and (3.3), if \( u_1 \in U^{(s)}, u_2 \in U^{(t)}, u_3 \in U^{(s)}, \ldots \) and \( u_1 u_2 u_3 \ldots \neq 1 \), then \( u_1 u_2 \cdots u_n \neq 1 \) for \( n = 1, 2, \ldots \). Similarly, \( u_2 u_3 \cdots u_{n+1} \neq 1 \) for \( n = 1, 2, \ldots \). This proves (c).

**Conjecture.** — \( U^- \) is the amalgamated product of its subgroups \( U^- \cap U^w \), \( w \in W \). (PROPOSITION 3.5(c) confirms this when \( W \) is an infinite dihedral group; the conjecture is trivial when \( W \) is finite.)

We can now prove a generalization of a theorem of NAGAO [9]:

**COROLLARY 3.5.** — Assume that \( S = \{s, t\} \) and \( m_{s, t} = 0 \), (so that \( W \) is an infinite dihedral group), and that \( U^- = U^s \triangleleft (U^- \cap U^s) \). Then the “opposite minimal parabolic” \( P^-_s := HG_s \triangleleft (U^- \cap U^s) \) is the amalgamated product of its subgroups \( HG_s \) and \( HU_-^{(s)} \) (defined in PROPOSITION 3.5 (b)).

**Proof.** — Put \( U_1 = U_-^{(s)} \cap U^s \). Clearly, \( H \) normalizes \( U_1 \) and \( U_1^s \cap U_1 = \{1\} \). LEMMA 3.2 and the assumption \( U_- = U^s \triangleleft (U_- \cap U^s) \) imply that \( U^s \) normalizes \( U_1 \). PROPOSITION 3.2(d) shows that \( U_-^{(s)} = U^s_1 U_1 \) and that \( U_1^s = U_-^{(t)} \). We therefore obtain :

\[ U_-^{(s)} = U^s \triangleleft U_1, \text{ and } H \text{ normalizes } U_1. \]

\[ U_-^{(t)} = U^s_1. \]

By using (RT2a), we obtain :

\[ HG_s = HU^s_1 \cup U^s_1 U^s U^s. \]

Now, let \( \tilde{P}^-_s \) be the amalgamated product of the subgroups \( HG_s \) and \( HU_-^{(s)} \) of \( P^-_s \), and let \( \Psi : \tilde{P}^-_s \rightarrow P^-_s \) be the canonical map. Identifying \( HG_s \) and \( HU_-^{(s)} \) with subgroups of \( \tilde{P}^-_s \), let \( F \) be the subgroup of \( \tilde{P}^-_s \) generated by \( \bigcup_{g \in HG_s} gU_1 g^{-1} \). Fixing \( n \in sH \), (3.9) and (3.11) imply that \( F \) is generated by \( U_1 \) and \( \bigcup_{u \in U^s_1} uU_1(\text{un}^{-1}) \). Let \( \tilde{U}_- \) be the subgroup of \( \tilde{P}^-_s \) generated by \( U^s_1 \) and \( F \). Using (3.9), we see that \( \tilde{U}_- \) is generated by \( U_-^{(s)} \) and \( nU_1 n^{-1} \). Clearly, \( \Psi = \text{id} \) on \( U_-^{(s)} \), and \( \Psi \) maps \( nU_1 n^{-1} \) isomorphically onto \( U_-^{(t)} \) by (3.10). Hence, by PROPOSITION 3.5 (c), \( \Psi \) maps \( \tilde{U}_- \) isomorphically onto \( U_- \). Since also \( \Psi = \text{id} \) on \( HG_s \), we see that \( \Psi \) is surjective. By using (3.11) and \( \tilde{P}^-_s = HG_s F \), we have

\[ \tilde{P}^-_s = H \tilde{U}_- \cup U^s_1 nH \tilde{U}_- \]

\[ = H \tilde{U}_- \cup nH U^s_1 \tilde{U}_- \subset (HG_s \cap NU_+) \tilde{U}_-. \]
If \( g \in \tilde{P}_s^- \) and \( \Psi(g) = 1 \), write \( g = g'u \), where \( g' \in HG_s \cap NU_+ \) and \( u \in \tilde{U}_- \). Since \( \Psi = \text{id} \) on \( HG_s \) and \( \Psi(\tilde{U}_-) \subseteq U_- \), we have \( 1 = \Psi(g) = \Psi(g') \Psi(u) = g' \Psi(u) \) and hence \( g' = \Psi(u) = 1 \) by (RT3). But \( \Psi \) is injective on \( \tilde{U}_- \). Therefore, \( u = 1 \) and so \( g = 1 \). This shows that \( \Psi \) is injective. 

Let \( k \) be a field, Nagao’s theorem states that \( SL_2(k[t^{-1}]) \) is the amalgamated product of its subgroups

\[
SL_2(k) \quad \text{and} \quad \left\{ g \in SL_2(k[t^{-1}]) \mid g = \begin{pmatrix} \ast & 0 \\ \ast & \ast \end{pmatrix} \right\}.
\]

We deduce this result from Corollary 3.5, as follows. Put

\[
G = SL_2(k((t))), \quad H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^x \right\},
\]

\[
U_+ = \left\{ g \in SL_2(k(t)) \mid g(t = 0) = \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \right\},
\]

\[
U_- = \left\{ g \in SL_2(k[t^{-1}]) \mid g(t = \infty) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}.
\]

Let \( N \) be the subgroup of \( G \) generated by \( H \) and \( n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix}. \)

Put \( S = \{n_1H, n_2H\} \subseteq N/H = W \) and \( s = n_1H \in S \). It is easy to check that \( (G, N, U_+, U_-, H, S) \) is a refined Tits system. (To check (RT3), one notes that \( n \in N \) and \( U_- \cap nU_+n^{-1} = \{1\} \) imply \( n \in H \).) Since \( U_- = U_+^s \alpha (U_- \cap U^s_-) \), and since \( W \) is an infinite dihedral group, Corollary 3.5 applies. The conclusion is Nagao’s theorem.

Remark. — In the example above, it is easy to check that \( G \) is generated by \( N \) and \( U_+ \) by using the fact that \( k((t)) \) is a field. The corresponding fact for \( k[t, t^{-1}] \) may be proved by using the density of \( k[t, t^{-1}] \) in \( k((t)) \) and the fact that \( U_+ \) is an open subgroup. Furthermore, using the involution \( t \rightarrow t^{-1} \) of \( k[t, t^{-1}] \), we deduce by using Proposition 3.4 the well-known fact that \( SL_2(k[t, t^{-1}]) \) is generated by its subgroups

\[
SL_2(k) \quad \text{and} \quad \left\{ \begin{pmatrix} a & bt^{-1} \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k) \right\}.
\]

Define a map \( \theta : U_- H U_+ \rightarrow H \) by \( \theta(u_- hu_+) = h \).

Proposition 3.6. — If \( w, w' \in W \) and if \( l(ww') = l(w) + l(w') \), then

\[
\theta(n'^{-1}gn'g') = n'^{-1} \theta(g)n' \theta(g').
\]
for all $g \in B^w B$, $g' \in B^{w'} B$ and $n' \in w'H$.

Proof. — First, we prove (3.12) for $g' = 1$. By (3.8.1), write $g = u_- hu_+$, where $u_- \in U_- \cap U_+$, $h \in H$ and $u_+ \in U_+$. By (3.8.2), write $n'^{-1}u_+n' = u'_-u'_+$, where $u'_- \in U_-$ and $u'_+ \in U_+$. By PROPOSITION 3.2 d (i), $(U_- \cap U_+^w)^w \subset U_-$, so $n'^{-1}u_- n' \in U_-$. It follows that

$$n'^{-1}gn' = ((n'^{-1}u_-n')((n'^{-1}hn')u'_-(n'^{-1}hn')^{-1}))(n'^{-1}hn')u'_+ \in U_-(n'^{-1}hn')U_+,$$

and hence $\theta(n'^{-1}gn') = n'^{-1}hn' = n'^{-1}\theta(g)n'$.

The proof of (3.12) for arbitrary $g' \in B^{w'} B$ follows by a straightforward calculation. Write $g' = n'^{-1}bn'b'$, where $b, b' \in B$. Then

$$\theta(n'^{-1}gn'g') = \theta(n'^{-1}(gb)n'b') = \theta(n'^{-1}(gb)n')\theta(b')$$

$$= n'^{-1}\theta(gb)n'\theta(b') = n'^{-1}\theta(g)\theta(b)n'\theta(b')$$

$$= n'^{-1}\theta(g)n'(n'^{-1}\theta(b)n')\theta(b')$$

$$= n'^{-1}\theta(g)n'\theta(n'^{-1}bn')\theta(b')$$

$$= n'^{-1}\theta(g)n'\theta(n'^{-1}bn'b')$$

$$= n'^{-1}\theta(g)n'\theta(g').$$

PROPOSITION 3.7. — Let $K$ be a subgroup of $G$ satisfying $K \cap U_+ = \{1\}$, and put $T = \theta(K \cap B)$. Let $H_+$ be a normal subgroup of $N$, and assume that $H = H_+ T[\text{unique}]$. Assume that $U_s \subset KBs$ for all $s \in S$. Assume that $w \mapsto \tilde{w}$ is a map from $W$ to $N$ satisfying : $s = \tilde{s}H$ for all $s \in S$; $\tilde{1} = 1$; $\tilde{ww'} = \tilde{w}\tilde{w'}$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$. For $w \in W$, put

$$(3.13) \quad Z_w = \{k \in K \cap BwB \mid \theta(\tilde{w}^{-1}k) \in H_+\}.$$  

Then :

(a)  
(i)  
$G = KH_+U_+$ [unique];

(ii)  
for all $w \in W$, $BwB = Z_w B$ [unique];

(iii)  
for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$, $Z_{ww'} = Z_w Z_{w'}$ [unique].

(b) For $s, t \in S$ and $m_s, t$ elements $z_1 \in Z_s$, $z_2 \in Z_t$, $z_3 \in Z_s, \ldots$, there exists a unique sequence of $m_s, t$ elements $z'_1 \in Z_t$, $z'_2 \in Z_s$, $z'_3 \in Z_t, \ldots$ satisfying

$$(3.14) \quad z_1z_2z_3 \cdots = z'_1z'_2z'_3 \cdots (m_s, t \text{ factors on each side }).$$
Furthermore, $K$ is the amalgamated product of its subgroups $K \cap B$ and $K \cap P_s$, $s \in S$, modulo the relations (3.14).

**Proof.** — For $s \in S$, we have $BsB = U_s sB \subset KBsB = KB$, and hence $G = KB$ by (3.1) and Proposition 3.1. But $B = TH_+U_+ = TU_+H_+ = (K \cap B)U_+H_+ = (K \cap B)H_+U_+$. Hence, $G = KH_+U_+$. If $k, k' \in K$, $h, h' \in H_+$, $u, u' \in U_+$ and $khu = k'h'u'$, then $k^{-1}k' \in K \cap B$ and $\theta(k^{-1}k') = hh^{-1} \in H_+$. Since $\theta(K \cap B) \cap H_+ = \{1\}$, we conclude that $h = h'$ and hence $k^{-1}k' = h'u'hu^{-1}h'^{-1} \in K \cap U_+ = \{1\}$, so that $k = k'$ and $u = u'$. This proves a(i).

To prove a (ii), fix $w \in W$. If $k \in K \cap BwB$, choose $t \in K \cap B$ such that $\theta(\tilde{w}^{-1}k) \in H_+\theta(t)$. Then $k = (kt^{-1})t \in ZWB$. Using a (i), we deduce that $BwB = KB \cap BwB = ZWB$. Now suppose that $z, z' \in Zw, b, b' \in B$ and $zb = z'b'$. Put $g = z^{-1}z' = bb'^{-1} \in K \cap B$, so that $\theta(\tilde{w}^{-1}z') = \theta(\tilde{w}^{-1}zg) = \theta(\tilde{w}^{-1}z\theta(g))$. Hence, $\theta(g) = \theta(\tilde{w}^{-1}z)^{-1}\theta(\tilde{w}^{-1}z') \in T \cap H_+ = \{1\}$, so that $g \in U_+ \cap K = \{1\}$. This shows that $z = z'$ and $b = b'$, verifying a (ii).

To prove a (iii), fix $w, w' \in W$ such that $l(ww') = l(w) + l(w')$. We claim that $ZwZw' \subset Z_{ww'}$. To verify this, let $k \in Zw$ and $k' \in Z_{w'}$. Then

$$kk' \in ZwZ_{w'} \subset (K \cap BwB)(K \cap Bw'B) \subset K \cap (BwB)(Bw'B) = K \cap Bww'B,$$

and also

$$\theta(\tilde{ww'}^{-1}kk') = \theta((\tilde{w}\tilde{w'})^{-1}kk') = \theta(\tilde{w'}^{-1}(\tilde{w}^{-1}k)\tilde{w}'(\tilde{w'}^{-1}k'))$$

$$= \tilde{w'}^{-1}\theta(\tilde{w}^{-1}k)\tilde{w}'\theta(\tilde{w'}^{-1}k')$$

$$\in \tilde{w'}^{-1}H_+\tilde{w}'H_+ = H_+,$$

the third equality by Proposition 3.6. This proves the claim. We have : $ZwZ_{w'} \subset Z_{ww'}$; $BwB = ZwB$ [unique] and $Bw'B = Zw'B$ [unique]; $Bww'B = Z_{ww'}B$[unique]. Using Proposition 3.1, we deduce a(iii).

(b) follows from (a) and Theorem A.
4. \(G(A)\) is a refined Tits system.

Fix a generalized Cartan matrix \(A\). Let \(G(A)\) be the corresponding group, defined in §2. Recall the subgroups \(N, U_+, U_-\) and \(H\) of \(G(A)\), the Weyl group \(W = N/H\) and the subset \(S\) of \(W\), introduced in §2.

For \(s \in S\), put \(U(s) = U_{\alpha_s}(= \exp g_{\alpha_s})\) for short. We keep the “exponential” notation \(M^w\) of §3. We shall see that \(U(s) = U_+ \cap U_-^s\).

**Proposition 4.1.**

(a) \(G(A)\) is generated by \(N\) and \(U_+\). The group \(H\) is a normal subgroup of \(N\); it normalizes \(U_+\) and \(U_-\). The set \(S\) generates \(W\), and \(s^2 = 1\) for all \(s \in S\).

(b) If \(s \in S\) and \(w \in W\), then:

(i) \(U(s)\) is a subgroup of \(U_+ \cap U_-^s\), and \(H\) normalizes \(U(s)\).

(ii) \(U(s) \neq \{1\}\).

(iii) \(U(s) \setminus \{1\} \subset U(s)HsU(s)\).

(iv) \(U^w(s) \subset U_+\) or \(U^w(s) \subset U_-\).

(v) \(U_+ \subset U(s)U_-^s\).

(c) (i) If \(w \in W\) and \(w \neq 1\), then \(U^w(s) \subset U_-\) for some \(s \in S\).

(ii) If \(u_- \in U_-\), \(h \in H\), \(u_+ \in U_+\) and \(u_- h u_+ = 1\), then \(u_- = h = u_+ = 1\).

Before proving Proposition 4.1 we use it to deduce:

**Proposition 4.2.** — \((G(A), N, U_+, U_-, H, S)\) is a refined Tits system, and \(U(s) = U_+ \cap U_-^s\) for all \(s \in S\).

**Proof.** — (RT1) follows from Proposition 4.1 (a). By Proposition 4.1 (cii), \(U_- \cap U_+ = \{1\}\). Hence, by Proposition 4.1 (b, iv), \(U(s) = U_+ \cap U_-^s\), which is \(U_s\) from §3. (RT2) now follows from Proposition 4.1 (b). To prove (RT3), suppose that \(u_- \in U_-\), \(n \in N\), \(u_+ \in U_+\) and \(u_- n u_+ = 1\). Then, since \(U_- \cap U_+ = \{1\}\) by Proposition 4.1 (cii), we have

\[
\{1\} = U_- \cap (u_- n u_+) U_+ (u_- n u_+)^{-1} = u_- (U_- \cap n U_+ n^{-1}) u_-^{-1},
\]

so that \(U_- \cap n U_+ n^{-1} = \{1\}\). By Proposition 4.1 (b, ii) and (c, i), this forces \(n \in H\). Now \(u_- = n = u_+ = 1\) follows from Proposition 4.1 (cii), proving (RT3).  

Parts (a) and (b, iv) of Proposition 4.1 are clear. Part (b, ii) is clear since \(\text{Ad}(x_1) f_s = f_s + \alpha^w_s - e_s \neq f_s\). Part (b, iii) follows from formula (2.7), and part (c, i) follows from Lemma 2.1 (a), Proposition 2.1 and formula (2.9). To prove parts (b, v) and (c, ii), we need some constructions.
Henceforth, $U(g)$ denotes the universal enveloping algebra of a Lie algebra $g$. The use of $U(n_+)$ to investigate $U_+$, exploited below, was one of the ingredients of Tits [12].

Recall that the Kac-Moody algebra $g'(A)$ has a triangular decomposition $g'(A) = n_+ + g_0 + n_+$, where

$$n_\pm = \bigoplus_{\alpha \in \Delta_\pm} g_{\pm \alpha} = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} g_{\pm \alpha}.$$

We complete the universal enveloping algebra $U(n_+)$ with respect to its induced algebra gradation, obtaining an algebra $\tilde{U}(n_+)$ consisting of all formal sums $\sum_{\alpha \in Q_+} u_\alpha$, where $u_\alpha \in U(n_+)_\alpha$. Let $U(n_+)$ be the subalgebra of $\tilde{U}(n_+)$ consisting of all such formal sums $\sum_{\alpha \in Q_+} u_\alpha$ satisfying the following condition: If $(V, d\pi)$ is an integrable $g'(A)$-module and $v \in V$, then $d\pi(u_\alpha)v = 0$ for all but a finite number of $\alpha \in Q_+$. Such a $(V, d\pi)$ then becomes a $\tilde{U}(n_+)$-module $(V, \tilde{\pi})$ by: $\tilde{\pi}(\sum u_\alpha)v = \sum d\pi(u_\alpha)v$.

For $\alpha \in \Delta_+^e$, define a map $\widetilde{\exp} : g_\alpha \to U(n_+)$ by:

$$\widetilde{\exp} x = \sum_{n=0}^{\infty} (n!)^{-1} x^n.$$

Let $\tilde{U}_+$ be the subset of $\tilde{U}(n_+)$ generated by the $\widetilde{\exp} g_\alpha$, $\alpha \in \Delta_+^e$, under multiplication, so that $\tilde{U}_+$ is a group under multiplication with identity $1$.

**Lemma 4.1.** There exists a unique surjective homomorphism $\Psi : \tilde{U}_+ \to U_+$ such that $\tilde{\pi} = \pi \circ \Psi$ for every integrable $g'(A)$-module $(V, d\pi)$. We have $\exp = \Psi \circ \widetilde{\exp}$ on $g_\alpha$ for every $\alpha \in \Delta_+^e$.

**Proof.** Let $(V, d\pi)$ be an integrable $g'(A)$-module such that the associated $G(A)$-module $(V, \pi)$ is faithful. Clearly, we have $\tilde{\pi}(\widetilde{\exp} x) = \pi(\exp x)$ for all $x \in g_\alpha$, $\alpha \in \Delta_+^e$. Hence, $\tilde{\pi}(\tilde{U}_+) = \pi(U_+)$. Since $\pi$ is injective on $U_+$, we conclude that there exists a unique map $\Psi : \tilde{U}_+ \to U_+$ such that $\tilde{\pi} = \pi \circ \Psi$; clearly, $\Psi$ is a surjective homomorphism, and $\exp = \Psi \circ \widetilde{\exp}$ on every $g_\alpha$. If $(V', d\pi')$ is another integrable $g'(A)$-module, then the same reasoning applied to $(V \oplus V', d\pi \oplus d\pi')$ yields a homomorphism $\Psi_0 : \tilde{U}_+ \to U_+$ satisfying $\tilde{\pi} \oplus \tilde{\pi}' = (\pi \oplus \pi') \circ \Psi_0$, i.e., $\tilde{\pi} = \pi \circ \Psi_0$ and $\tilde{\pi}' = \pi' \circ \Psi_0$. Then $\Psi_0 = \Psi$ by the first equality and the uniqueness of $\Psi$, so that $\tilde{\pi}' = \pi' \circ \Psi$ by the second one. \qed

For $s \in S$, put

$$Y_s^\pm = \bigcup_{\alpha} U_\alpha,$$
where $\alpha$ runs over $\Delta^e_\infty \setminus \{\alpha_s\}$ with $\pm \langle \alpha, \alpha_s^v \rangle \geq 0$.

**Lemma 4.2.** — Let $s \in S$. Then:

(a) $Y_s^\pm = nY_s^\pm n^{-1}$ for all $n \in sH$.

(b) $U_+$ is generated by $U(s)$, $Y_s^+$ and $Y_s^-$.

(c) $uzu^{-1}z^{-1} \in Y_s^+$ for all $u \in U(s)$ and $z \in Y_s^+$.

**Proof.** — (a) and (b) are clear. If $u = \exp a \in U(s)$ and $z = \exp b \in Y_s^+$, then $(ad a)^2 b = 0 = (ad b)^2 a$ by Lemma 2.1 (c). We have:

$$uzu^{-1} = \Psi(\exp a)\Psi(\exp b)\Psi(\exp a)$$

$$= \Psi(\exp a)(\exp b)(\exp a)$$

$$= \Psi\left(\sum_{n=0}^{\infty} (n!)^{-1} x^n\right),$$

where $x = (\exp ad a)b = b + [a, b]$. Since $x$ and $b$ commute, we get

$$uzu^{-1}z^{-1} = \Psi\left(\sum_{n=0}^{\infty} (n!)^{-1} x^n\right)\Psi\left(\sum_{m=0}^{\infty} (m!)^{-1}(-b)^m\right)$$

$$= \Psi\left(\sum_{n=0}^{\infty} (n!)^{-1} x^n \sum_{m=0}^{\infty} (m!)^{-1}(-b)^m\right)$$

$$= \Psi\left(\sum_{n=0}^{\infty} (n!)^{-1}(x - b)^n\right)$$

$$= \Psi\left(\sum_{n=0}^{\infty} (n!)^{-1}[a, b]^n\right).$$

Since $\exp[a, b] \in Y_s^+$ by Lemma 2.1 (c), we get $uzu^{-1}z^{-1} = \Psi(\exp[a, b]) = \exp[a, b] \in Y_s^+$. This proves (c).

**Corollary 4.1.** — Let $s \in S$, and let $U(s)$ be the subgroup of $U_+$ generated by $\{uzu^{-1} | u \in U(s), z \in Y_s^+ \cup Y_s^-\}$. Then $U_+ = U(s)U(s), U(s)$ normalizes $U(s)$, and $H \cup sH$ normalizes $U(s)$.

**Proof.** — By Lemma 4.2 (a,b), $U_+ = U(s)U(s)$, and $U(s)$ and $H$ normalize $U(s)$. Thus, it suffices to show that if $u \in U(s)$ and $z \in Y_s^+ \cup Y_s^-$, then there exists $n \in sH$ such that $nuzu^{-1}n^{-1} \in U(s)$. If $z \in Y_s^+$, then $uzu^{-1} \in Y_s^+Y_s^+$ by Lemma 4.2 (c), and hence $nuzu^{-1}n^{-1} \in Y_s^-Y_s^- \subset U(s)$ for all $n \in sH$ by Lemma 4.2 (a). If $u = 1$ and $z \in Y_s^-$, then $nuzu^{-1}n^{-1} = nzn^{-1} \in Y_s^+ \subset U(s)$ for all $n \in sH$. Finally, suppose $u \neq 1$ and $z \in Y_s^-$. By
using PROPOSITION 4.1 b(iii), choose $n \in sH$ and $u_1, u_2 \in U(s)$ such that $nu = u_1nu_2n^{-1}$. Then
\[
nuzu^{-1}n^{-1} = u_1nu_2n^{-1}znu_2^{-1}n^{-1}u_1^{-1} \in u_1nu_2Y_s + u_2^{-1}n^{-1}u_1^{-1}
\subset u_1ny_sY_s^+ + u_2^{-1}n^{-1}u_1^{-1}
\subset u_1Y_s - Y_s^- u_1^{-1} \subset U(s).\]

A $g'(A)$-module $(V, d\pi)$ is called $Q$-graded if there is a vector space decomposition $V = \bigoplus_{\beta \in Q} V_\beta$ satisfying $d\pi(\alpha_\lambda) V_\beta \subset V_{\alpha + \beta}$.

**LEMMA 4.3.** — There exists a $Q$-graded integrable $g'(A)$-module $V$ which is a faithful $U(n_+)$-module.

**Proof.** — One can take for $V$ the direct sum of all integrable lowest weight $g'(A)$-modules. In more detail, given $A = (\lambda_s)_{s \in S} \in \mathbb{Z}^S_+$, define a 1-dimensional $U(g_0 + n_-)$-module $C_{V_A}$ by $\alpha^s(\lambda_A) = -\lambda_s v_A, n - (v_A) = 0$. Let
\[
M^*(A) = U(g'(A)) \otimes_{U(g_0 + n_-)} C_{V_A},
\]
regarded as a $Q$-graded $g'(A)$-module, where the action is defined by left multiplication and the $Q$-gradation is induced from that of $U(g'(A))$ by putting $v_A = 0$. Then it is easy to see that the $Q$-graded $g'(A)$-module $L^*(A) = M^*(A) / \sum_s U(n_+) e_s^{\lambda_s+1}(v_A)$ is integrable (cf. [3, LEMMA 3.4]). We put
\[
V = \bigoplus_{\Lambda \in \mathbb{Z}^S_+} L^*(\Lambda).
\]
If $u \in U(n_+)_\beta$, $u \neq 0$ and $u(v_A) = 0$ in $L^*(A)$, then $\beta - (\lambda_s + 1)\alpha_s \in \mathbb{Q}_+$ for some $s \in S$. It follows that $V$ is a faithful $U(n_+)$-module.

We say that a subgroup $F$ of $G(A)$ is graded if $u_- \in U_-, h \in H, u_+ \in U_+$ and $u_- hu_+ \in F$ imply $u_-, h, u_+ \in F$.

**LEMMA 4.4.** — Let $(V, \pi)$ be a $Q$-graded integrable $g'(A)$-module. Then:
(a) $\ker \pi$ is a graded subgroup of $G(A)$.
(b) If $V$ is a faithful $U(n_+)$-module, then $V$ is a faithful $U(\widetilde{n}_+)$-module.

**Proof.** — If $u \in U_+$ and $v \in V_\beta$, then
\[
\pi(u)v - v \in \sum_{\alpha \in \mathbb{Q}_+ \setminus \{0\}} V_{\beta + \alpha},
\]
so that $U_+$ is “upper triangular” on $V$. Similarly, $H = \exp g_0$ is “diagonal” on $V$ and $U_-$ is “lower triangular” on $V$. If now $u_- \in U_-, h \in H, u_+ \in U_+$,
and \( u_- h u_+ \in \ker \pi \), then, for all \( v \in V_\beta, \beta \in Q \), we have
\[
\pi(u_+)v - v = \pi(h^{-1}u_-)v - v \in \left( \sum_{\alpha \in Q_+ \setminus \{0\}} V_{\beta+\alpha} \right) \cap \left( \sum_{\alpha \in -Q_+} V_{\beta+\alpha} \right) = (0).
\]
Hence, \( \pi(u_+) = 1 \), so that \( u_+ \in \ker \pi \) and, similarly, \( u_- \in \ker \pi \) and so finally \( h \in \ker \pi \). (a) follows. (b) is clear.

**Corollary 4.2.**

(a) The homomorphism \( \Psi \) of Lemma 4.1 is an isomorphism from \( \tilde{U}_+ \) onto \( U_+ \).

(b) If \( u_- \in U_-, h \in H, u_+ \in U_+ \) and \( u_- h u_+ = 1 \), then \( u_- = h = u_+ = 1 \).

(c) If \( s \in S \), then \( U(s) \neq \{1\} \) and \( U_+ = U(s) \ltimes (U_+ \cap U_+^s) \).

**Proof.** — (a) is clear from Lemmas 4.1, 4.3 and 4.4(b). Suppose \( u_- \in U_-, h \in H, u_+ \in U_+ \) and \( u_- h u_+ = 1 \). By Lemma 4.4(a), \( u_+ \in \ker \pi \) for every \( Q \)-graded integrable \( g'(A) \)-module \( (V,d\pi) \); by Lemmas 4.1, 4.3 and 4.4(b), this forces \( u_+ = 1 \). Similarly, by using the involution \( \omega \) of \( G(A) \), we conclude that \( u_- = 1 \). Hence, \( h = 1 \) also, proving (b). The first part of (c) follows from (a). Fix \( s \in S \). Then
\[
U(s) \cap U_+^s \subset U_+^s \cap U_+^s = (U_\cap U_+)^s = \{1\}
\]
by using (b). By Corollary 4.1, \( U_+ = U(s)U(s), U(s) \subset U_+ \cap U_+^s \), and \( U(s) \) normalizes \( U(s) \). Hence, \( U_+ = U(s) \ltimes U(s) \) and \( U(s) = U_+ \cap U_+^s \). This proves (c).

Proof of the remainder of Proposition 4.1 is immediate from Corollary 4.2.

**Proposition 4.3.** — Let \( A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix} \) be a \( 2 \times 2 \) matrix with \( m, n \in \mathbb{Z}_+ \) and \( mn \geq 4 \). Let \( (W(A),S) \) be the associated Coxeter system, so that \( S = \{s,t\} \) and \( m_{s,t} = 0 \). Put
\[
\Delta_+^s = \{(st)^k \cdot \alpha_s \mid k \in \mathbb{Z}_+\} \cup \{(st)^k \cdot \alpha_t \mid k \in \mathbb{Z}_+\}
\]
and
\[
\Delta_+^t = \{(ts)^k \cdot \alpha_t \mid k \in \mathbb{Z}_+\} \cup \{(ts)^k \cdot \alpha_s \mid k \in \mathbb{Z}_+\},
\]
so that \( \Delta_+^{sr} = \Delta_+^s \cup \Delta_+^t \). For \( r \in S \), let \( U_+^{(r)} \) be the subgroup of \( U_+ \subset G(A) \) generated by the \( U_\alpha, \alpha \in \Delta_+^r \). Then \( U_+ \) is the free product of its subgroups \( U_+^{(s)} \) and \( U_+^{(t)} \).
Proof. — Using the involution \( \omega \), this is clear from Proposition 3.5(c).

Remarks. — (1) For \( m = n = 2 \), i.e. for the case \( A_1^{(1)} \), Proposition 4.3 was stated in \([8, \text{Example}]\).

(2) For \( m, n \geq 2 \), each group \( U_+^{(r)} \) from Proposition 4.3 is the direct sum of its one-parameter subgroups \( U_\alpha, \alpha \in \Delta^r_+ \); otherwise, each \( U_+^{(r)} \) is a two-step nilpotent group.

(3) We conjecture that, in general, \( U_+ \) is the amalgamated product of its subgroups \( U_+ \cap U_-^w, w \in W \). (This is a special case of the conjecture of § 3.)

We now explore some features of \( G(A) \), which are related to the \( Q \)-gradation of \( g'(A) \).

\( S \) is called **indecomposable** if, whenever \( J \) is a subset of \( S \) such that \( J \neq \emptyset \) and \( J \neq S \), there exist \( s \in J \) and \( t \in S \setminus J \) such that \( st \neq ts \). (This corresponds to the indecomposability of \( A \).) The following are general properties of Tits systems [1]:

(4.1) If \( S \) is indecomposable and \( F \) is a normal subgroup of \( G(A) \), then \( FB = B \) or \( FB = G(A) \).

(4.2) The center of \( G(A) \) is contained in \( B \).

We will also use the following special properties of \( G(A) \).

(4.3) \( G(A) \) is generated by the \( U_s \) and \( U_s^* \), \( s \in S \).

(4.4) \( \bigcap_{w \in W} U_+^w = \{1\} \).

Indeed, (4.3) is clear, and (4.4) follows from Corollary 3.4 by using the involution \( \omega \).

We call a subgroup \( F \) of \( G(A) \) **weakly graded** if \( F \cap U_s^* B = (F \cap U_s^*)(F \cap B) \) for all \( s \in S \). Note that every graded subgroup of \( G(A) \) is weakly graded. Let \( C \) be the center of \( G(A) \).

**Proposition 4.4.**

(a) \( C \subset H \).

(b) Let \( F \) be a weakly graded normal subgroup of \( G(A) \), and suppose that \( S \) is indecomposable. Then \( F = G(A) \) or \( F \subset C \).

Proof. — \( C \subset H \) follows from (4.2) and (4.4). Now let \( F \) be a weakly graded normal subgroup of \( G(A) \), and assume that \( S \) is indecomposable. Suppose that \( FB = B \). Then \( F \subset B \) and hence, using (4.4), \( F \subset \bigcap_{w \in W} B^w = H \). If \( h \in F \) and \( u \in U_+ \), then \( huh^{-1} u^{-1} \in F \cap U_+ = \{1\} \). Hence, \( h \) centralizes \( U_+ \); similarly, \( h \) centralizes \( U_- \). (4.3) now shows that \( F \subset C \). Now suppose that \( FB \neq B \). Then \( FB = G(A) \) by (4.1). Hence, for all \( s \in S \),

\[
U_s^* B = U_s^* B \cap FB = (U_s^* B \cap F) B = (U_s^* \cap F) (B \cap F) B = (U_s^* \cap F) B.
\]
Since $U_s^s \cap B = \{1\}$, we conclude that $U_s^s \subseteq F$ and therefore $U_s \subseteq F$ for all $s \in S$. Hence, by (4.3), $F = G(A)$. 

We sometimes write $H(A)$ for $H$, $U_+(A)$ for $U_+$, etc., to emphasize the dependence on $A$.

**Corollary 4.3.**

(a) Let $A'$ be an indecomposable generalized Cartan matrix, and let $\Psi : G(A') \to G(A)$ be a homomorphism such that $\Psi(U_{\pm}(A')), \subseteq U_{\pm}$ and $\Psi(H(A')) \subseteq H$. Then either $\ker \Psi = G(A')$ or else

$$\ker \Psi \subseteq \text{Center}(G(A')) \subseteq H(A').$$

(b) If $J$ is a subset of $S$ and $A_J = (a_{s,t})_{s,t \in J}$ is the corresponding principal submatrix of $A$, then the obvious homomorphism $G(A_J) \to G(A)$ is injective.

**Proof.** — (a) follows from Proposition 4.4, since $\ker \Psi$ is graded and hence weakly graded. Since the homomorphism of (b) is injective on $H$, (b) follows from (a).

**Corollary 4.4.**

(a) If $(V, d\pi)$ is a $Q$-graded integrable $\mathfrak{g}'(A)$-module and if $A$ is indecomposable, then $\ker \pi = G(A)$ or $\ker \pi \subseteq C \subseteq H$ for the corresponding $G(A)$-module.

(b) The direct sum of all irreducible highest weight modules with fundamental highest weights (see [3, Chapter 10] for the definition) is a faithful differentiable $G(A)$-module.

**Proof.** — (a) follows from Proposition 4.4, since $\ker \pi$ is graded and hence weakly graded. Since the module of (b) is a faithful $H$-module, (b) follows from (a).

**Corollary 4.5.** — Assume that the generalized Cartan matrix $A$ is indecomposable and not of affine type, and let $(V, d\pi)$ be an integrable $\mathfrak{g}'(A)$-module. Then $\ker \pi = G(A)$ or $\ker \pi \subseteq C \subseteq H$ for the corresponding $G(A)$-module.

**Sketch of proof.** — Since $A$ is not of affine type, there exist integers $k_s$, $s \in S$, such that $\alpha_{s'}(\sum_{s \in S} k_s \alpha_s) > 0$ for all $s' \in S$ [3, Theorem 4.3]. For $t \in \mathbb{C}^X$, put $h(t) = \prod_{s \in S} h_s(t)^{k_s}$. Define a $\mathbb{Z}$-gradation $\mathfrak{g}'(A) = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ by

$$\mathfrak{g}_n = \{x \in \mathfrak{g}'(A) \mid \text{Ad}(h(t))x = t^n x \text{ for all } t \in \mathbb{C}^X\}.$$ 

Now let $(V, d\pi)$ be an integrable $\mathfrak{g}'(A)$-module. Define a $\mathbb{Z}$-gradation $V = \oplus_{n \in \mathbb{Z}} V_n$ by

$$V_n = \{v \in V \mid \pi(h(t))v = t^n v \text{ for all } t \in \mathbb{C}^X\}.$$
These gradations are compatible, and by imitating the arguments proving Lemma 4.4(a), one shows that \( \ker \pi \) is a graded subgroup of \( G(A) \). Corollary 4.5 now follows from Proposition 4.4.

**Corollary 4.6.** — \( \text{Ad} \) is faithful on \( U_+ \). Moreover, \( \ker \text{Ad} = C \subset H \).

*Proof.* — This is clear from Proposition 4.4.

**Remark.** — One may also prove the first part of Corollary 4.6 by defining a map \( \log \) from \( U_+ \) to \( \hat{\mathfrak{h}}_+ \subset U(\mathfrak{n}_+) \) and noting that the center of \( g'(A) \) is contained in \( g_0 \). However, this procedure is not valid over a field of positive characteristic, and also involves the Campbell-Hausdorff formula. For these reasons, we omit this approach here.

The following statement is clear from \((G2a)\) (we use that \( t^2 \neq 1 \) for some \( t \in C^X \)):

\[(4.5) \text{If } s \in S, \text{ then the centralizer of } H \text{ in } U_s \text{ is } \{1\}.
\]

**Proposition 4.5.**

(a) Let \( F \) be a graded subgroup of \( G(A) \) containing \( N \) such that \( F \cap U_s = \{1\} \) for all \( s \in S \). Then \( F = N \).

(b) The normalizer of \( H \) in \( G(A) \) is \( N \).

*Proof.* — We first deduce (b) from (a). Let \( \tilde{N} \) be the normalizer of \( H \) in \( G(A) \). Then \( \tilde{N} \) contains \( N \). Suppose \( u_- \in U_- \), \( h \in H \), \( u_+ \in U_+ \) and \( u_- hu_+ \in \tilde{N} \). Put \( n = u_- hu_+ \). If \( h' \in H \), then

\[ u_+ h' u_+^{-1} h'^{-1} = (u_- h)^{-1}(n h' n^{-1})(u_- h) h'^{-1} \in U_+ \cap H U_- = \{1\}, \]

so that \( u_+ \) centralizes \( H \) and, similarly, \( u_- \) centralizes \( H \). Along with \((4.5)\), this verifies the hypotheses of (a) with \( F = \tilde{N} \). Hence, by (a), \( \tilde{N} = N \), proving (b).

We now prove (a). We first show that \( N \) normalizes \( F \cap U_+ \). Indeed, suppose that \( s \in S \), \( n \in s H \) and \( u \in F \cap U_+ \). By \((3.8.2)\), write \( nun^{-1} = u_1 u_2 \), where \( u_1 \in U_- \cap n U_+ n^{-1} \) and \( u_2 \in U_+ \). Since \( n, u \in F \) and \( F \) is graded, we obtain \( u_1, u_2 \in F \). But then \( n^{-1} u_1 n \in F \cap U_s = \{1\} \), so that \( u_1 = 1 \) and hence \( nun^{-1} = u_2 \in F \cap U_+ \). This shows that \( N \) normalizes \( F \cap U_+ \). Hence, \( F \cap U_+ \subset \bigcap_{w \in W} U_+^w = \{1\} \) by \((4.4)\). Now let \( g \in F \). By Proposition 3.2(a,b) write \( g = u_+ u_- n \), where \( n \in N \), \( u_- \in U_- \cap n U_+ n^{-1} \) and \( u_+ \in U_+ \). Since \( g, n \in F \) and \( F \) is graded, we have \( u_-, u_+ \in F \). Hence, \( u_+, n^{-1} u_- n \in F \cap U_+ = \{1\} \), so that \( g = n \in N \). This proves (a).

**Corollary 4.7.** — The centralizer of \( H \) in \( G(A) \) is \( H \).

*Proof.* — This follows from Proposition 4.5(b) and Corollary 2.2.
We now discuss Levi decompositions of parabolics.

**Proposition 4.6.** — Let \( J \) be a subset of \( S \), and put \( M_J = P_J \cap \omega(P_J) \), \( U_J = M_J \cap U_+ \) and \( U^J = \bigcap_{w \in W_J} U_+^w \). Then \( P_J = M_J \rtimes U^J \) and, moreover:

(a) \( M_J \) is generated by \( H \) and the \( G_s, s \in J \).

(b) \( U_J \) is generated by the \( U_\alpha, \alpha \in \Delta^+ \cap \sum_{s \in J} Z\alpha_s \).

(c) \( U^J \) is the smallest normal subgroup of \( U_+ \) containing the \( U_\alpha, \alpha \in \Delta^+ \cap \sum_{s \in J} Z\alpha_s \).

**Proof.** — Let \( \tilde{M}_J, \tilde{U}_J \) be the subgroups asserted in (a), (b) and (c) to be \( M_J, U_J \) and \( U^J \). Clearly, we have:

\[
(4.6) \quad U_+ = \tilde{U}_J \tilde{U}^J.
\]

\[
(4.7) \quad HW_J \subseteq \tilde{M}_J \subseteq M_J.
\]

We shall prove the following assertions:

\[
(4.8) \quad \tilde{U}_J \subseteq \tilde{M}_J.
\]

\[
(4.9) \quad \tilde{M}_J \text{ normalizes } \tilde{U}_J.
\]

\[
(4.10) \quad M_J \cap U^J = \{1\}.
\]

We first show that these assertions suffice to validate the proposition.

Since \( HW_J \subseteq \tilde{M}_J \) by (4.7) and \( \tilde{U}_J \subseteq U_+ \) by (4.6), (4.9) gives \( \tilde{U}_J \subseteq U^J \).

By (4.6,7,8), \( \tilde{U}_J \subseteq U_J \); by (4.6), \( U_J \tilde{U}_J \subseteq \tilde{U}_J \tilde{U}^J \); by (4.10), \( U_J \cap U^J = \{1\} \). These yield:

\[
(4.11) \quad \tilde{U}_J = U_J \text{ and } \tilde{U}^J = U^J.
\]

By (4.6,7,8), \( \tilde{M}_J \) and \( \tilde{U}^J \) generate \( P_J \); by (4.9), \( \tilde{M}_J \) normalizes \( \tilde{U}_J \); by (4.7), \( M_J \subseteq \tilde{M}_J \subseteq P_J \); by (4.10, 11), \( M_J \cap \tilde{U}^J = \{1\} \). These yield:

\[
(4.12) \quad P_J = \tilde{M}_J \rtimes \tilde{U}^J \text{ and } \tilde{M}_J = M_J.
\]

The proposition follows from (4.11) and (4.12).

It remains to verify (4.8), (4.9) and (4.10). (4.8) follows from Lemma 2.1(b). Corollary 3.6 applied to the refined Tits system \( (P_J, HW_J, U_+, P_J \cap U_-, H, J) \) implies \( \bigcap_{w \in W_J} (P_J \cap U_-)^w = \{1\} \); applying \( \omega \), we deduce (4.10). Finally, we verify (4.9). Suppose \( s \in J \), and put

\[
X^\pm = \bigcup_{\alpha} U_\alpha,
\]

where \( \alpha \) runs over \( (\Delta^+ \cap \sum_{t \in J} Z\alpha_t) \setminus \{\alpha_s\} \) with \( \pm \langle \alpha, \alpha_s^w \rangle \geq 0 \) and

\[
Y^\pm = \bigcup_{\alpha} U_\alpha,
\]
where \( \alpha \) runs over \( \Delta^\text{re}_+ \setminus \sum_{\tau \in J} \mathbb{Z} \alpha_\tau \) with \( \pm \langle \alpha, \alpha_\tau^\vee \rangle \geq 0 \).

Let \( U_1 \) be the subgroup of \( U_+ \) generated by \( \{ uxu^{-1} \mid u \in U_x, x \in X^+ \cup X^- \} \) and let \( U_2 \) be the subgroup of \( U_+ \) generated by \( \{ uyu^{-1} \mid u \in U_y, y \in Y^+ \cup Y^- \} \). Using \textsc{Lemma} 2.1 (c), the argument proving \textsc{Corollary} 4.1 shows that \( HG_s \) normalizes \( U_1 \) and \( U_2 \). Let \( U_3 \) be the subgroup of \( U_+ \) generated by \( \{ u_1 u_2 u_1^{-1} \mid u_1 \in U_1, u_2 \in U_2 \} \); since \( U_s \), \( U_1 \) and \( U_2 \) generate \( U_+ \), and since \( U_s \) normalizes \( U_1 \) and \( U_2 \), we deduce that \( U_3 \) is the smallest normal subgroup of \( U_+ \) containing \( U_2 \). Hence, \( U_3 = U_J \), so that \( HG_s \) normalizes \( U_J \). Varying \( s \in J \), we obtain (4.9).

\textbf{Remark.} — It is easy to show that, for all \( j \in J \), \( P_j \) is the normalizer of \( U_J \) in \( G(A) \) and \( M_j \) is the normalizer of \( M_j \) in \( P_j \).

We conclude this section with some technical results about “finite-dimensional” subgroups of \( U_+ \).

\textbf{Proposition 4.7.} — Let \( \alpha, \beta \in \Delta^\text{re}_+ \). Then the following assertions are equivalent:

\begin{enumerate}
    \item \( |(Z_+ \alpha + Z_+ \beta) \cap \Delta^\text{re}_+| < \infty \).
    \item For some \( w \in W \), one has \( w \cdot \alpha, w \cdot \beta \in -\Delta^\text{re}_+ \).
    \item \( (U_\alpha, U_\beta) \) is contained in the subgroup of \( U_+ \) generated by the \( U_\gamma \) where \( \gamma \in (Z_+ \alpha + Z_+ \beta) \cap \Delta^\text{re}_+ \) and \( \gamma \neq \alpha, \beta \).
\end{enumerate}

\textit{Sketch of proof} — (We use here some notions defined e.g. in [3, Chapter 5]. First, suppose \( \langle \alpha, \beta^v \rangle > 0 \) and \( \langle \beta, \alpha^v \rangle > 0 \). Then (a) and (c) hold by \textsc{Lemma} 2.1(c)(ii) and the argument proving \textsc{Lemma} 4.2(c). We have \( (1 - \langle \beta, \alpha^v \rangle \langle \alpha, \beta^v \rangle) \beta = \langle \beta, \alpha^v \rangle r_\beta \cdot \alpha + r_\alpha \cdot \beta \), hence \( r_\alpha \cdot \beta < 0 \) or \( r_\beta \cdot \alpha < 0 \). If \( r_\alpha \cdot \beta < 0 \) (resp. \( r_\beta \cdot \alpha < 0 \)), then \( w = r_\alpha \) (resp. \( = r_\beta \)) satisfies (b).

Now, suppose \( \langle \alpha, \beta^v \rangle = 0 = \langle \beta, \alpha^v \rangle \). Then (a) and (c) hold by \textsc{Lemma} 2.1(c)(ii) and the argument proving \textsc{Lemma} 4.2(c), and \( w = r_\alpha r_\beta \) satisfies (b).

By [6, p. 139] or [3, 2nd ed., Exercise 5.19], the only remaining case is \( \langle \alpha, \beta^v \rangle < 0 \) and \( \langle \beta, \alpha^v \rangle < 0 \). By using \( W \), we may assume that \( \beta = \alpha_s \) for some \( s \in S \). If \( \alpha - \alpha_s \in \Delta \), put \( \gamma = \alpha - \alpha_s \); otherwise, put \( \gamma = \alpha \). Then:

\[ \beta, \gamma \in \Delta^\text{re}_+ \; ; \; \langle \beta, \gamma^v \rangle < 0 \; \text{and} \; \langle \gamma, \beta^v \rangle < 0 \; ; \; \gamma - \beta \notin \Delta. \]

Put \( T = \{ 1, 2 \} \), and define a generalized Cartan matrix \( B = (b_{t,u})_{t,u \in T} \) by \( b_{11} = b_{22} = 2, b_{12} = \langle \gamma, \beta^v \rangle, b_{21} = \langle \beta, \gamma^v \rangle \). Let \( \alpha_1, \alpha_2 \) be the corresponding generators of the root lattice of the Kac-Moody algebra \( g'(B) \). One can show that there exists a homomorphism \( \Psi : g'(B) \rightarrow g'(A) \) such that, if \( k, l \in \mathbb{Z} \) and \( \delta = k\beta + l\gamma \), \( \tilde{\delta} = k\alpha_1 + l\alpha_2 \), then \( \delta \in \Delta^\text{re}_+ \iff \tilde{\delta} \in \Delta^\text{re}_+(B) \), and \( \Psi(g'(B)_{\tilde{\delta}}) = g'(A)_{\delta} \) if \( \delta \in \Delta^\text{re}_+ \). Since the induced homomorphism from
$G(B)$ to $G(A)$ is injective on $U_+(B)$ by Corollary 4.3(a), and since the implication $(b) \Rightarrow (a)$ always holds by Lemma 4.5(e) below, this reduces us to the following case:

A is a generalized Cartan matrix $(\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix})$ where $m, n > 0; \beta = \alpha_1; \alpha = \alpha_2$ or $\alpha_2 + \alpha_1; (\alpha, \beta^w) < 0$.

First, suppose $mn \geq 4$. Then the $(r_{\alpha_1}r_{\alpha_2})^k \cdot \alpha_1$, $k = 0, 1, 2, \ldots$, are distinct elements of $(\mathbb{Z}_+ \alpha + \mathbb{Z}_+ \beta) \cap \Delta_+^{re}$, so that (a) is false and hence (b) is false. Moreover, in this case (c) is false by Proposition 4.3.

Finally, suppose $mn \leq 3$. Then $W(A)$ is a finite dihedral group and $w_0(\Delta_+(A)) = -\Delta_+(A)$ for the longest element $w_0$ of $W(A)$. Therefore (b) holds, and hence (a) holds. One can show that (c) holds by using the theory of algebraic groups over $\mathbb{C}$, but we will give a self-contained argument instead. Put $w = r_{\alpha_2}$ if $\alpha = \alpha_2$ and $w = r_{\alpha_1}r_{\alpha_2} = r_{\alpha_2}r_{\alpha}$ if $\alpha = \alpha_1 + \alpha_2$. Using Corollary 4.2(c), one shows that $U_{\alpha}$ normalizes $U_{\alpha_2} \cap U_{\alpha_2}^{-1}$, and $U_{\alpha_2} \subset U_{\alpha_2} \cap U_{\alpha_2}^{-1} \subset U_+^{w}$ so that $(U_{\alpha}, U_{\beta}) \subset U_+^{w}$, $U_{\beta}$ normalizes $U_+ \cap U_{\alpha_2}^{w}$, and $U_{\alpha} \subset U_+ \cap U_{\alpha_2}^{w}$, so that $(U_{\alpha}, U_{\beta}) \subset U_+^{w}$. Therefore, $(U_{\alpha}, U_{\beta}) \subset U_+^{w} \cap U_+^{w}$. But by using Proposition 3.3(d), one sees that $U_+^{w} \cap U_+^{w}$ is the subgroup defined in (c). Hence, (c) holds.

This verifies that in all cases, (a), (b) and (c) are true or false simultaneously.

For $w \in W$, put $\Phi(w) = \Delta_+^{re} \cap -w \cdot \Delta_+^{re}$.

**Lemma 4.5.** Let $w, w' \in W$ satisfy $l(ww') = l(w) + l(w')$. Then:

(a) $\Phi(w) = \Delta_+^{re} \cap \sum_{\alpha \in \Phi(w)} \mathbb{Z}_+ \alpha$.
(b) For $\alpha \in \Delta_+^{re}, \alpha \in \Phi(w)$ if and only if $U_+ \subset U_+ \cap U_+^{-1}$.
(c) $\Phi(1) = \emptyset$. For $s \in S$, $\Phi(s) = \{\alpha_s\}$.
(d) $\Phi(ww') = \Phi(w) \cup w \cdot \Phi(w')$.
(e) $|\Phi(w)| = l(w)$.

**Proof.** Since $\Delta_+^{re}$ is $W$-invariant, $Q_+$ is a semigroup and $\Delta_+^{re} = \Delta_+^{re} \cap Q_+$, (a) is clear. We have $U_{\alpha_2}^{w} = U_{\alpha_2}^{-1}$, $U_+ \cap U_+^{-1} = \{1\}$, $\Delta_+^{re} = \Delta_+^{re} \cap -\Delta_+^{re}$, and $U_+ \subset U_+ \Leftrightarrow \alpha \in \pm \Delta_+^{re}$ for $\alpha \in \Delta_+^{re}$, so that (b) is clear. (c) is clear, and (e) follows from (c) and (d).

It is easy to deduce (d) from Proposition 3.2(d).

**Lemma 4.6** [10]. Let $F$ be a group, and let $F_1, \ldots, F_k$ be subgroups of $F$ satisfying: for $i = 1, \ldots, k$, $F_iF_{i+1} \cdots F_k$ is a normal subgroup of $F$; $F = F_1F_2 \cdots F_k$ [unique]. Then we have, for any permutation $\sigma$ of $\{1, \ldots, k\}$, $F = F_{\sigma(1)}F_{\sigma(2)} \cdots F_{\sigma(k)}$ [unique].

**Proposition 4.8.** Let $\Phi$ be a finite subset of $\Delta_+^{re}$ satisfying $\Phi =
\[ \Delta^+ \cap \sum_{\beta \in \Phi} Z_+ \beta, \text{ and let } \beta_1, \ldots, \beta_n \text{ be an enumeration of } \Phi. \text{ Then } U = U_{\beta_1} \cdots U_{\beta_n} \text{ [unique], where } U \text{ is the subgroup of } U_+ \text{ generated by the } U_{\beta_k}. \]

**Proof.** — We may assume by using \( W \) that \( \alpha_s \in \Phi \) for some \( s \in S \). Let \( \gamma_1 = \alpha_s, \gamma_2, \ldots, \gamma_n \) be an enumeration of \( \Phi \) such that the height of \( \gamma_{i-1} \) is at most that of \( \gamma_i \), \( 2 \leq i \leq n \). By **Proposition 4.7**, \( U = U_{\gamma_1} \cdots U_{\gamma_n} \), and \( U_{\gamma_k} \cdots U_{\gamma_n} \) is a normal subgroup of \( U \) for \( k = 1, \ldots, n \).

Put \( U' = U_{\gamma_2} \cdots U_{\gamma_n} \). Since \( U_{\gamma_1} \cap U' \subset U_+ \cap U_+^s = \{1\} \) and \( U = U_{\gamma_1} U' \), we obtain \( U = U_{\gamma_1} U' \) [unique]. By induction on \( n \),

\[ U' = U_{\gamma_2} \cdots U_{\gamma_n} \text{ [unique].} \]

Hence,

\[ U = U_{\gamma_1} U_{\gamma_2} \cdots U_{\gamma_n} \text{ [unique].} \]

Now we apply **Lemma 4.6**.

**Corollary 4.8.** — If \( w \in W \), then

\[ U_+ \cap U_w^{-1} = U_{\beta_1} \cdots U_{\beta_n} \text{ [unique]} \]

for any enumeration \( \beta_1, \ldots, \beta_n \) of \( \Phi(w) \).

**Proof.** — We proceed by induction on \( l(w) \), the cases \( l(w) \leq 1 \) being trivial. Choose \( s \in S \) such that \( l(sw) < l(w) \). Then \( U_+ \cap U_w^{-1} = (U_+ \cap U_s^w)(U_+ \cap U_s^{(sw)^{-1}})^s \) by **Proposition 3.2(d)**. By the induction hypothesis, \( U_+ \cap U_s^{(sw)^{-1}} \) is generated by the \( U_{\beta} \), \( \beta \in \Phi(sw) \), and hence \( (U_+ \cap U_s^{(sw)^{-1}})^s \) is generated by the \( U_{\beta} \), \( \beta \in s \cdot \Phi(sw) \). Since \( U_+ \cap U_s^w = U_{\alpha_s} \), we conclude that \( U_+ \cap U_w^{-1} \) is generated by \( \{\alpha_s\} \cup s \cdot \Phi(sw) \), which equals \( \Phi(w) \) by **Lemma 4.5**. **Lemma 4.5(a)** and **Proposition 4.8** complete the proof.

**5. The group \( K(A) \)**

Recall the involution \( \omega \) of \( G(A) \) from § 2, and let \( K(A) \) be the fixed-point set of \( \omega \). We shall give explicit generators and relations for \( K(A) \).

Let \( D = \{ u \in \mathbb{C} \mid |u| \leq 1 \} \) be the closed unit disc, let \( S^1 = \{ t \in \mathbb{C} \mid |t| = 1 \} \) be the unit circle and let \( \hat{D} = D \setminus S^1 \). For \( u \in D \), put

\[ z(u) = \begin{pmatrix} u & (1 - |u|^2)^{1/2} \\ -\overline{(1 - |u|^2)^{1/2}} & \overline{u} \end{pmatrix} \in SU_2. \]

Note that \( z(t) = h(t) \) if \( t \in S^1 \) (cf. § 2).
For \( s \in S, u \in D \) and \( t \in S' \), put \( z_s(u) = \varphi_s(z(u)) \) and \( h_s(t) = \varphi_s(h(t)) \). For \( s \in S \), put \( K_s = K \cap G_s \). Note that \( z_s(u) \in K_s = \varphi_s(SU_2) \) and \( z_s(0) = \tilde{s} \) (cf. § 2). Recall the subgroups \( H_+ \) of \( G(A) \) and \( T \) of \( K(A) \) introduced in § 2.

**Proposition 5.1.**

(a) \( G(A) = K(A)H_+U_+ \) [unique] (Iwasawa decomposition).

(b) \( K(A) \) is generated by the \( K_s, s \in S \).

(c) If \( w = s_1 \cdots s_k \) is a reduced expression and \( g \in K(A) \cap BwB \), then there exist unique \( u_1, \ldots, u_k \in \hat{D} \) and \( t \in T \) such that

\[
g = z_{s_1}(u_1) \cdots z_{s_k}(u_k)t.
\]

(d) For all \( s, t \in S \), there exists a unique map \( \Gamma_{s,t} : \hat{D}^{m_s,t} \to (\hat{D})^{m_s,t} \) such that if \( u = (u_1, u_2, \ldots) \in (\hat{D})^{m_s,t} \) and \( \Gamma_{s,t}(u) = v = (v_1, v_2, \ldots) \in (\hat{D})^{m_s,t} \), then

\[
z_s(u_1)z_t(u_2)z_s(u_3) \cdots = z_t(v_1)z_s(v_2)z_t(v_3) \cdots
\]

(e) \( K \) is the amalgamated product of its subgroups \( K \cap P_s, s \in S \), modulo the relations in (d).

**Proof**. — We use Proposition 3.7. If \( h \in H, u_+ \in U_+ \) and \( hu_+ \in K(A) \), then \( \omega(hu_+) = hu_+ \) and hence \( \omega(u_+)^{-1}(\omega(h)^{-1}h)u_+ = 1 \). Since \( \omega(u_+) \in U_-, \omega(h)^{-1}h \in H \) and \( u_+ \in U_+ \), we deduce that \( \omega(h)^{-1}h = 1 \) and \( u_+ = 1 \). Hence, \( hu_+ = h \in H \cap K(A) \). Using Lemma 2.2(a), it is easy to check that \( H \cap K(A) = T \). Hence, \( K(A) \cap U_+ = \{1\} \) and \( T = \theta(K \cap B) \). Clearly, \( H_+ \) is a normal subgroup of \( N \) and \( H = H_+T \) [unique]. If \( u \in C \), then \( z(-(1+|u|^2)^{-1/2}u)^{-1}x(u)z(0) \) is of the form \( \binom{0}{*} \). This shows that \( U_s \subseteq KBs \) for all \( s \in S \). By Corollary 2.3(b), there exists a unique map \( w \to \tilde{w} \) from \( W \) to \( N \) satisfying : \( \tilde{1} = 1 ; \tilde{s} = z_s(0) \) for all \( s \in S \); \( \tilde{ww'} = \tilde{w}\tilde{w'} \) if \( w, w' \in W \) and \( l(ww') = l(w) + l(w') \).

This verifies the hypotheses of Proposition 3.7 and shows that \( T = K \cap B \), and \( U_s \subseteq z_s(\hat{D})Bs \) for all \( s \in S \). Recall \( Z_w \) defined by (3.13). If \( s \in S \), then \( BsB = U_sBs \subseteq (z_s(\hat{D})Bs)sB = z_s(\hat{D})B \); \( z_s \) defines an injection from \( \hat{D} \) into \( Z_s \) by an easy calculation; \( BsB = Z_sB \) [unique] by Proposition 3.7. Hence, \( z_s \) defines a bijection from \( \hat{D} \) onto \( Z_s \) for all \( s \in S \). Proposition 3.7 now shows that \( (a), (c), (d) \) and \( (e) \) hold, and that \( K(A) \) is generated by \( T \) and the \( Z_s \). Since, \( Z_s \subseteq K_s \), and since \( T \) is generated by the \( K_s \cap T \), (b) follows.

Note the following corollary of Proposition 5.1(c).

* The proof of the Iwasawa decomposition is a straightforward generalization of that of Steinberg [10]. In the affine case this has been done in [14].
COROLLARY 5.1. — For $J \subset S$, denote by $K_J$ the subgroup of $K(A)$ generated by the $K_s$ with $s \in J$. Then $K(A) \cap P_J = K_J T$.

We wish to determine the maps $\Gamma_{s,t}$ of PROPOSITION 5.1(d). Using COROLLARY 4.3(b), we see that $\Gamma_{s,t}$ depends only on $a_{s,t}$ and $a_{t,s}$. Clearly, $\Gamma_{s,t} \circ \Gamma_{t,s} = \text{id}$, and $\Gamma_{s,s} = \text{id}$. If $a_{s,t} = a_{t,s} = 0$, then $G_s$ and $G_t$ commute and so $\Gamma_{s,t}(\alpha, \beta) = (\beta, \alpha)$. If $m_{s,t} = 0$, then $\Gamma_{s,t}$ is trivial. If $a_{s,t} = -1$ and $a_{t,s} = -k$, $k = 1, 2$ or 3, we write $\Gamma_k$ for $\Gamma_{s,t}$. We must calculate $\Gamma_1, \Gamma_2$ and $\Gamma_3$.

LEMMA 5.1.
(a) If $S = \{1, 2\}$ and $A$ is the generalized Cartan matrix $(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array})$, then $C^3$ is a faithful $G(A)$-module by:

$$\varphi_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)(x, y, z) = (ax + by, cx + dy, z)$$

and

$$\varphi_2 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)(x, y, z) = (x, ay + bz, cy + dz).$$

(b) If $S = \{1, 2\}$ and $A$ is the generalized Cartan matrix $(\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array})$, then $C^4$ is a faithful $G(A)$-module by:

$$\varphi_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)(x, y, z, w) = (x, ay + bw, z, cy + dw)$$

and

$$\varphi_2 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)(x, y, z, w) = (ax + by, cx + dy, dz - cw, -bz + aw).$$

Moreover,

$$\varphi_1 \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right) \rightarrow \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha + \beta j \end{array} \right)$$

and

$$\varphi_2 \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right) \rightarrow \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right)$$

defines a faithful representation of $K(A)$ by quaternionic matrices.

Proof. — Using COROLLARY 4.4 and LEMMA 2.2(a), we see that the modules defined in the lemma are faithful. Let $H$ be the associative $\mathbb{R}$-algebra of quaternions, with standard $\mathbb{R}$-basis $1, i, j, k$, with $ij = k, jk = i, ki = j,$ and $i^2 = j^2 = k^2 = -1$. $C^4$ becomes a right $H$-module
under \((x, y, z, w)i = (xi, yi, zi, wi)\) and \((x, y, z, w)j = (\bar{z}, \bar{w}, -\bar{x}, -\bar{y})\), which is free on generators \(v_1 = (1, 0, 0, 0)\) and \(v_2 = (0, 1, 0, 0)\). It is easy to check that \(\varphi_1(SU_2)\) and \(\varphi_2(SU_2)\) give \(H\)-module endomorphisms of \(C^4\) under the action defined in (b). But

\[
\sigma \to \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},
\]

where \(\sigma(v_i) = v_1q_{1i} + v_2q_{2i}\), defines an isomorphism from \(\text{End}_H(C^4)\) onto the ring of 2-by-2 matrices over \(H\). The lemma now follows from a calculation.

**COROLLARY 5.2.** — If \(\alpha_i \in \mathcal{D}\) and \(u_i = (1 - |\alpha_i|^2)^{1/2}\), \(1 ≤ i ≤ 4\), then :

(a) \((\beta_1, \beta_2, \beta_3) = \Gamma_1(\alpha_1, \alpha_2, \alpha_3)\) \text{ if and only if :}

\[
(1 - |\beta_1|^2)^{-1/2} \beta_1 = (u_2u_3)^{-1}(u_1\alpha_3 + \bar{\alpha}_1\alpha_2u_3),
\]

\[
\beta_2 = \alpha_1\alpha_3 - u_1\alpha_2u_3,
\]

\[
(1 - |\beta_3|^2)^{-1/2} \beta_3 = (u_1u_2)^{-1}(\alpha_1u_3 + u_1\alpha_2\bar{\alpha}_3).
\]

(b) Define \(A, B, C, D, E, F, G, H \in C\) by :

\[
\begin{pmatrix} 1 & 0 \\ 0 & \alpha_1 + u_1j \end{pmatrix} \begin{pmatrix} \alpha_2 & u_2 \\ -u_2 & \bar{\alpha}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_3 + u_3j \end{pmatrix} \begin{pmatrix} \alpha_4 & u_4 \\ -u_4 & \bar{\alpha}_4 \end{pmatrix} = \begin{pmatrix} A + Bj & C + Dj \\ E + Fj & G + Hj \end{pmatrix}
\]

in \(M_2(H)\).

Then \((\beta_1, \beta_2, \beta_3, \beta_4) = \Gamma_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) \text{ if and only if :}

\[
(1 - |\beta_1|^2)^{-1/2} \beta_1 = B^{-1}F,
\]

\[
(1 - |\beta_2|^2)^{-1/2} \beta_2 = (|B|^2 + |F|^2)^{-1}(AB + EF),
\]

\[
(1 - |\beta_1|^2)^{-1/2} \beta_3 = B^{-1}(AF - BE),
\]

\[
(1 - |\beta_2|^2)^{-1/2} \beta_4 = (|B|^2 + |F|^2)^{-1}(BG - CF).
\]

**Proof.** — **Lemma 5.1** and a calculation show that the given formulas hold if \((\beta_1, \cdots) = \Gamma_k(\alpha_1, \cdots)\). Since \((1 - |\beta|^2)^{-1/2} \beta\) determines \(\beta\) for \(|\beta| < 1\), the corollary follows.

Unfortunately, a similar calculation of \(\Gamma_3\), i.e. a matrix calculation for the exceptional 14-dimensional group \(G_2\), seems difficult. As an alternative,
we shall utilize the embedding of $G_2$ in $D_4$. For that, we apply to $D_4$ the following lemma:\* 

**Lemma 5.2.** Let $\mathcal{A}$ be a group of permutations $\sigma$ of $S$ satisfying $a_{\sigma(s),\sigma(s')} = a_{s,s'}$, for all $s, s' \in S$. For $\sigma \in \mathcal{A}$ define an automorphism $\tilde{\sigma}$ of $G(A)$ by $\tilde{\sigma} \circ \varphi_s = \varphi_{\sigma(s)}$ for all $s \in S$ and an automorphism $\tilde{\tau}$ of $W(A)$ by $\tilde{\tau}(s) = \tau(s)$. Let $G(A)^{\mathcal{A}}$ and $W(A)^{\mathcal{A}}$ be the corresponding fixed-point subgroups. Let $S/\mathcal{A}$ be the set of all orbits of $\mathcal{A}$ on $S$. Assume that if $t \in S/\mathcal{A}$, and $s$ and $s'$ are distinct elements of $t$, then $a_{s,s'} = 0$, so that $G_s$ and $G_{s'}$ commute. For $t, u \in S/\mathcal{A}$, fix $s \in u$ and put $b_{t,u} = \sum_{r \in t} a_{r,s}$. Then $B = (b_{t,u})_{t,u \in S/\mathcal{A}}$ is a generalized Cartan matrix.

Define homomorphisms $g \mapsto \overline{g}$ from $G(B)$ into $G(A)$ and $w \mapsto \overline{w}$ from $W(B)$ into $W(A)$ by:

$$\overline{\varphi_t}(x) = \prod_{s \in t} \varphi_s(x) \text{ for all } t \in S/\mathcal{A} \text{ and } x \in SL_2(\mathbb{C});$$

$$\overline{t} = \prod_{s \in t} s \text{ for all } t \in S/\mathcal{A}.$$ 

Then:

(a) $g \mapsto \overline{g}$ is an isomorphism from $G(B)$ onto $G(A)^{\mathcal{A}}$.

(b) $w \mapsto \overline{w}$ is an isomorphism from $W(B)$ onto $W(A)^{\mathcal{A}}$. For any reduced expression for $w \in W(B)$, the corresponding expression for $\overline{w} \in W(A)^{\mathcal{A}}$ is reduced.

**Proof.** It is easy to check that $B$ is a generalized Cartan matrix. We denote the homomorphisms $g \mapsto \overline{g}$ and $w \mapsto \overline{w}$ by $\Psi$. For any subset $F$ of $G(A)$, we put $F^{\mathcal{A}} = F \cap G(A)^{\mathcal{A}}$. It is easy to check that $\Psi$ is well-defined and that $\Psi(G(B)) \subset G(A)^{\mathcal{A}}$, $\Psi(U_\pm(B)) \subset U_\pm(A)^{\mathcal{A}}$. It is easy to check $\Psi(H(B)) \subset H(A)^{\mathcal{A}}$, $\Psi(N(B)) \subset N(A)^{\mathcal{A}}$, and that $\Psi$ on $G(B)$ induces $\Psi$ on $W(B)$. Using Lemma 2.2(a), it is easy to see that $\Psi(H(B)) = H(A)^{\mathcal{A}}$, and using Corollary 4.3(a), it is easy to check that $\Psi$ is injective on $G(B)$. Hence, $\Psi$ is injective on $W(B)$.

If $w \in W(A)^{\mathcal{A}}$ and $w \neq 1$, choose $t \in S/\mathcal{A}$ such that $l(sw) < l(w)$ for some $s \in t$. Since $w \in W(A)^{\mathcal{A}}$, we deduce that $l(sw) < l(w)$ for all $s \in t$, so that $l(sw) = l(w) - l(t) = l(w) - |t|$ (here $|t|$ means Card $(t)$) by a standard fact about Coxeter groups [1].

By induction on $l(w)$, we deduce:

(5.1) If $w \in W(A)^{\mathcal{A}}$, then there exist $t_1, \ldots, t_n \in S/\mathcal{A}$ such that $w = \overline{t_1} \cdots \overline{t_n}$ is a reduced expression.

\* We use some arguments of [15] in the proof of this lemma.
We next prove:

(5.2) If \( w \in W(A)^A \), then \( (U_+(A) \cap U_-(A)^w)^A \subset \Psi(G(B)) \).

If \( w = 1 \), (5.2) is clear. Suppose \( w = \bar{t} \) for some \( t \in S/A \). Let \( s_1, \ldots, s_m \) be an enumeration of \( t \). If \( g \in (U_+(A) \cap U_-(A)^w)^A \), write

\[
g = x_{s_1}(u_1) \cdots x_{s_m}(u_m)
\]

by Proposition 3.2(d), where \( u_1, \ldots, u_m \in C \) are determined by \( g \). If \( \sigma \in A \), let \( \tau \) be the permutation of \( \{1, \ldots, m\} \) defined by \( \sigma(s_i) = s_{\tau(i)} \). Then

\[
\tilde{\sigma}(g) = \tilde{\sigma}(x_{s_1}(u_1)) \cdots \tilde{\sigma}(x_{s_m}(u_m))
\]

\[
= x_{\sigma(s_1)}(u_1) \cdots x_{\sigma(s_m)}(u_m)
\]

\[
= x_{s_1}(u_{\tau^{-1}(1)}) \cdots x_{s_m}(u_{\tau^{-1}(m)})
\]

since \( G(A)s_1, \ldots, G(A)s_m \) commute. Since \( g \) determines the \( u_i \), we must have \( u_1 = u_{\tau^{-1}(1)} \). Varying \( \sigma \), we conclude that \( u_1 = \cdots = u_m \), so that \( g = \Psi(x_t(u_1)) \), verifying (5.2).

Now suppose \( w \in W(A)^A \), \( w \neq 1 \). By (5.1), choose \( t \in S/A \) such that \( l(\bar{t}w) = l(w) - l(\bar{t}) \). If \( g \in (U_+(A) \cap U_-(A)^w)^A \), use Proposition 3.2(d) to write \( g = g_1g_2 \), where \( g_1 \in (U_+(A) \cap U_-(A)^w)^A \) and \( g_2 \in U_+(A) \cap U_-(A)^w \). Using (5.1), choose \( n \in N(B) \) such that \( \Psi(n) \in \bar{t}wH(A) \), and put \( g' = \Psi(n)g \Psi(n)^{-1} \), \( g'_1 = \Psi(n)g_1 \Psi(n)^{-1} \) and \( g'_2 = \Psi(n)g_2 \Psi(n)^{-1} \). Then \( g' \in G(A)^A \), \( g'_1 = g'_1g'_2 \), \( g'_2 \in U_+(A) \) and \( g'_2 \in U_-(A) \). If \( \sigma \in A \), then \( g' = \tilde{\sigma}(g') = \tilde{\sigma}(g'_1)\tilde{\sigma}(g'_2) \), where \( \tilde{\sigma}(g'_1) \in U_+(A) \) and \( \tilde{\sigma}(g'_2) \in U_-(A) \).

Since \( U_+(A) \cap U_-(A) = \{1\} \), we deduce that \( \tilde{\sigma}(g'_1) = g'_1 \) and \( \tilde{\sigma}(g'_2) = g'_2 \). Hence, \( g'_1 \in (U_+(A) \cap U_-(A)^w)^A \subset \Psi(G(B)) \). Similarly, by induction on \( l(w) \), \( g'_2 \in \Psi(G(B)) \) and hence \( g \in \Psi(G(B)) \). This proves (5.2).

We next prove:

(5.3) \( G(A)^A \subset \Psi(G(B))U_+(A)^A \).

To avoid confusion, let \( B_+ \) denote the subgroup \( H(A)U_+(A) \) of \( G(A) \).
Suppose \( w \in W(A) \) and \( G(A)^A \cap B_+wB_+ \neq \emptyset \). Since \( \tilde{\sigma}(B_+wB_+) = \tilde{\sigma}(B_+)\tilde{\sigma}(w)\tilde{\sigma}(B_+) = B_+\tilde{\sigma}(w)B_+ \) for all \( \sigma \in A \), (3.1) forces \( w \in W(A)^A \).

Using (5.1), choose \( n \in N(B) \) such that \( \Psi(n) \in wH(A) \). If \( g \in G(A)^A \cap B_+wB_+ \), write \( g = g_1\Psi(n)hg_2 \), where \( g_1 \in U_+(A) \cap U_-(A)^wH(A), h \in H(A) \), \( g_2 \in U_+(A) \). As before, we deduce that \( g_1 \in (U_+(A) \cap U_-(A)^w)^A \), \( h \in H(A)^A \) and \( g_2 \in U_+(A)^A \). By (5.2), we have \( g_1 \in \Psi(G(B)) \), and also \( h \in H(A)^A = \Psi(H(B)) \). Hence, \( g \in \Psi(G(B))g_2 \subset \Psi(G(B))U_+(A)^A \). By (3.1), this proves (5.3).
To prove (a), it remains to show that \( U_+(A)A \subseteq \Psi(G(B)) \). Let \( g \in U_+(A)A \). Then \( \omega(g) \in U_-(A)A \). By (5.3), choose \( g' \in G(B) \) and \( g'' \in U_+(A) \) such that \( \omega(g) = \Psi(g')g'' \). Write \( g' = g_1 g_2 \), where \( g_1 \in U_-(B) \), \( n \in N(B) \) and \( g_2 \in U_+(B) \). Then

\[
(\omega(g)^{-1}\Psi(g_1))\Psi(n)(\Psi(g_2)g'') = 1
\]

and hence, by (RT3), \( \omega(g)^{-1}\Psi(g_1) = 1 \). We conclude that

\[
g = \omega^2(g) = \omega(\Psi(g_1)) = \Psi(\omega(g_1)).
\]

This proves (a).

It remains to prove the assertion of (b) about reduced expressions. We need:

(5.4) There exists a function \( l \) on \( W(B) \) such that \( \tilde{l}(t_1 \cdots t_n) = |t_1| + \cdots + |t_n| \) if \( t_1 \cdots t_n \) is a reduced expression.

Indeed, by Lemma 1.1, we need only to show that if \( t, u, v \in S/A \), then

\[
|t| + |u| + |v| + \cdots = |u| + |t| + |u| + \cdots (m_{t,u}^B \text{ summands on each side}.)
\]

If \( m_{t,u}^B \) is even, this is clear. Suppose \( t \neq u \) and \( m_{t,u}^B \) is odd. Then since \( B \) is a generalized Cartan matrix, \( b_{t,u} = -1 = b_{u,t} \). Hence, \( a_{r,s} = 0 \) or \(-1 \) for all \( r \in t \) and \( s \in u \), since otherwise \( b_{t,u} = \sum_{r \in t} a_{r,s} \) would be less than \(-1 \). Similarly, \( a_{s,r} = 0 \) or \(-1 \) for all \( r \in t \) and \( s \in u \). Since \( A \) is a generalized Cartan matrix, we deduce that \( a_{r,s} = a_{s,r} \) for all \( r \in t \) and \( s \in u \), and hence

\[
|u| = |u| b_{t,u} = \sum_{r \in t} a_{r,s} = \sum_{s \in u} a_{s,r} = |t| b_{u,t} = -|t|.
\]

Therefore, \( |t| + |u| + |v| + \cdots = |u| + |t| + |u| + \cdots \), proving (5.4).

Now let \( t_1 \cdots t_n \) be a reduced expression. By (5.1), choose \( t'_1, \ldots, t'_m \in S/A \) such that \( \tilde{l}_1 \cdots \tilde{l}_n = \tilde{l}'_1 \cdots \tilde{l}'_m \) and \( \tilde{l}'_1 \cdots \tilde{l}'_m \) is a reduced expression. Since \( \Psi \) is injective on \( W(B) \), we have \( t_1 \cdots t_n = t'_1 \cdots t'_m \). Using Lemma 1.1, we have:

\[
|t'_1| + \cdots + |t'_m| \geq \tilde{l}(t'_1 \cdots t'_m) = \tilde{l}(t_1 \cdots t_n)
\]

\[
= |t_1| + \cdots + |t_n| \geq l(t_1 \cdots t_n) = l(\tilde{l}_1 \cdots \tilde{l}_n) = |t'_1| + \cdots + |t'_m|.
\]

Hence, \( l(\tilde{l}_1 \cdots \tilde{l}_n) = |t_1| + \cdots + |t_n| \), so that \( \tilde{l}_1 \cdots \tilde{l}_n \) is a reduced expression. This proves (b). \( \blacksquare \)

Corollary 5.3. — Let \( k = 2 \) or \( 3 \), let \( S = \{0,1,\ldots,k\} \), and let \( A \) be the generalized Cartan matrix \( (a_{i,j})_{i,j \in S} \) defined by:

\[
a_{i,i} = 2 \text{ for } 0 \leq i \leq k,
\]

\[
a_{0,i} = a_{i,0} = -1 \text{ for } 1 \leq i \leq k,
\]

\[
a_{i,j} = a_{j,i} = 0 \text{ if } 1 \leq i < j \leq k.
\]
Define maps $\tilde{z}_1: \hat{D} \to K(A)$ and $\tilde{z}_2: \hat{D} \to K(A)$ by:

$$\tilde{z}_1(u) = z_0(u), \quad \tilde{z}_2(u) = z_1(u)z_2(u)\cdots z_k(u).$$

Let $u_i, v_i \in \hat{D}$, $1 \leq i \leq 2k$, and put $u = (u_1, \ldots, u_{2k})$, $v = (v_1, \ldots, v_{2k})$. Then $v = \Gamma_k(u)$ if and only if

$$\tilde{z}_1(u_1)\tilde{z}_2(u_2)\tilde{z}_1(u_3)\cdots \tilde{z}_2(u_{2k}) = \tilde{z}_2(v_1)\tilde{z}_1(v_2)\tilde{z}_2(v_3)\cdots \tilde{z}_1(v_{2k}).$$

**Proof.** — Let $A$ be the group of all permutations of $S$ fixing 0, and apply Lemma 5.2(a).

**Corollary 5.4.** — Let $k = 2$ or $3$, and put $N = k(k + 1)$. Define maps $C$, $R$ and $\Gamma$ from $\hat{D}^N$ to $\hat{D}^N$ by:

$$C(x_1, \ldots, x_N) = (x_2, \ldots, x_N, x_1);$$

$$R(x_1, \ldots, x_N) = (x_2, x_1, x_3, \ldots, x_N);$$

$$\Gamma(x_1, \ldots, x_N) = (y_1, y_2, y_3, x_4, \ldots, x_N)$$

if $(y_1, y_2, y_3) = \Gamma_1(x_1, x_2, x_3)$. (We have $\Gamma^2 = \text{id}$.) Define $i: \hat{D}^{2k} \to \hat{D}^N$ and $j: \hat{D}^N \to \hat{D}^{2k}$ by:

$$i(x_1, \ldots, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4)$$

and

$$j(y_1, \ldots, y_6) = (y_2, y_3, y_5, y_6)$$

if $k = 2$;

$$i(x_1, \ldots, x_6) = (x_1, x_2, x_2, x_2, x_3, x_4, x_4, x_4, x_5, x_6, x_6, x_6)$$

and

$$j(y_1, \ldots, y_{12}) = (y_3, y_4, y_7, y_8, y_{11}, y_{12})$$

if $k = 3$.

Define $\tilde{\Gamma}_k$ by:

$$\tilde{\Gamma}_2 = CTC^{-2}TCRC;$$

$$\tilde{\Gamma}_3 = F^{-1}E^{-2}FE^2B^{-1}F^{-1}EBF,$$

where

$$B = C^{-2}\Gamma C^{-2}\Gamma C^4\Gamma C^{-1}, \quad E = RC \quad \text{and} \quad F = C^4.$$

Then

$$\Gamma_k = jC^{-k} \tilde{\Gamma}_k i.$$
Proof. — Let \( S = \{0, \ldots, k\} \) and \( A \) be as in COROLLARY 5.3. If

\[
i_1, \ldots, i_N \in S \quad \text{and} \quad x = (x_1, \ldots, x_N) \in \hat{D}^N,
\]

we put

\[
z_{i_1, \ldots, i_N}(x) = z_{i_1}(x_1) \cdots z_{i_N}(x_N) \in K(A).
\]

Suppose \( k = 2 \). It is easy to check that \( y = C^{-2}z_{2,1,0,1,2,0}(x) \Rightarrow z_{2,1,0,1,2,0}(x) = z_{0,1,2,0,1,2}(x) \). Since 210120 is a reduced expression in \( W(A) \) by LEMMA 5.2(b), \( z_{2,1,0,1,2,0}(y) \) determines \( y \) by PROPOSITION 5.1(c); hence, we obtain

\[
z_{2,1,0,1,2,0}(y) = z_{0,1,2,0,1,2}(x) \Rightarrow y = C^{-2}z_{2,1,0,1,2,0}(x).
\]

Noting that \( z_1(\alpha)z_2(\beta) = z_2(\beta)z_1(\alpha) \) for all \( \alpha, \beta \in \hat{D} \), the case \( k = 2 \) follows from COROLLARY 5.3.

For \( k = 3 \), the argument is similar, using

\[
y = C^{-3}z_3(x) \Leftrightarrow z_{1,2,3,0,1,2,3,0,3,2,1,0}(y) = z_{0,1,2,3,0,3,2,1,0,1,2,3}(x).
\]

We will need:

**LEMMA 5.3.** — \( SU_2 \) is the group on generators \( z(\alpha), \alpha \in D \), with defining relations (we put \( h(t) = z(t) \) for \( t \in S^1 \)):

(a) \( h(t)h(t') = h(tt') \), where \( t, t' \in S^1 \).

(b) \( h(t)z(\alpha) = z(t^2\alpha)h(t^{-1}) \), where \( t \in S^1, \alpha \in D \).

(c) \( z(\text{i}c)h(t)z(\text{i}c)^{-1} = z(c^2t+(1-c^2)i) \), where \( 0 < c \leq 1, t \in S^1, \text{Im} t \geq 0 \).

\[
\text{Proo}
\]

Proof. — Let \( K \) be the group on generators \( z(\alpha), \alpha \in D \), with the given relations. Since these relations hold in \( SU_2 \), and since every element of \( SU_2 \) is uniquely of one of the forms \( h(t), t \in S^1 \), or \( z(\alpha)h(t), \alpha \in \hat{D} \) and \( t \in S^1 \), it suffices to check that every element of \( K \) is of one of these forms. By (a) and (b), we need only do this for \( z(\beta)z(\gamma) \), where \( \beta, \gamma \in \hat{D} \).

Define a homeomorphism \((F, G)\) from \( \hat{D} \) onto \((0,1) \times \{t \in S^1 \mid \text{Im} t > 0\}\) by requiring \( \alpha = F(\alpha)G(\alpha) + (1 - F(\alpha))G(\alpha) \) for all \( \alpha \in \hat{D} \). Define \( H : S^1 \to R \) by \( H(t) = F(t\beta) - F(t\gamma) \). Since \( F(\alpha) + F(-\alpha) = 1 \) for all \( \alpha \in \hat{D} \), we have \( H(1) + H(-1) = 0 \), so that, by the continuity of \( H \), \( H(t_i) = 0 \) for some \( t_i \in S^1 \). Put \( t'_{1,2} = G(t'_{1,2}) \) and \( t_{1,2} = G(t_{1,2}) \). If \( \text{Im} t_{1,2} \geq 0 \), we put \( t = t', t_1 = t'_1, t_2 = t'_2 \); otherwise, we put \( t = it', t_1 = -t'_1, t_2 = -t'_2 \). Put \( c = F(t^2\beta)^{1/2} \). Then we have:

\[
t, t_1, t_2 \in S^1; \quad \text{Im} t_1, \text{Im} t_2, \text{Im} t_1t_2 \geq 0; \quad 0 \leq c \leq 1; \quad \beta = \bar{t}^2(c^2t_1 + (1-c^2)i_1), \quad \gamma = t^2(c^2t_2 + (1-c^2)i_2).
\]
Put $\alpha = c^2t_1t_2 + (1 - c^2)t_1t_2$. Then (a), (b) and (c) imply:

\[
\begin{align*}
&h(t)z(\beta)z(\gamma)h(t) = z(t^2\beta)z(t^2\gamma) \\
&= z(c^2t_1 + (1 - c^2)t_1)z(c^2t_2 + (1 - c^2)t_2) \\
&= [z(ic)h(t_1)z(ic)^{-1}][z(ic)h(t_2)z(ic)^{-1}] \\
&= z(ic)h(t_1t_2)z(ic)^{-1} = z(\alpha).
\end{align*}
\]

Hence,

\[
z(\beta)z(\gamma) = h(t)z(\alpha)h(t) = z(t^2\alpha)h(t^2),
\]

and hence also $z(\beta)z(\gamma) = h(\alpha)$ if $\alpha \in S^1$. This brings $z(\beta)z(\gamma)$ to the required form.

**THEOREM B.** — $K(A)$ is the group on generators $z_s(u), s \in S$ and $u \in D$, with defining relations (we put $h_s(t) = z_s(t)$ if $t \in S^1$):

(K1) $h_s(t)h_s(t') = h_s(tt')$ if $s \in S; t, t' \in S^1$.

(K2) $z_s(ic)h_s(t)z_s(ic)^{-1} = z_s(c^2t + (1 - c^2)i)$ if $s \in S; 0 \leq c \leq 1; t \in S^1,$ Im$t \geq 0$.

(K3) $h_s(t)z_{s'}(u) = z_{s'}(t^as',u)h_{s'}(t^{-as},s')h_s(t)$ if $s, s' \in S; t \in S^1; u \in D$.

(K4) $z_s(u)z_{s'}(v) = z_{s'}(v)z_s(u)$ if $s, s' \in S, m^A_{s,s'} = 2; u, v \in D$.

(K5) $z_s(u_1)z_{s'}(u_2)z_{s'}(u_3)\ldots = z_{s'}(v_1)z_s(v_2)z_{s'}(v_3)\ldots$ (factors on each side) if $s, s' \in S, a_{s,s'} = -1, a_{s',s} = -k; 1 \leq k \leq 3; (v_1, \ldots, v_m^A_{s,s'}) = \Gamma_k(u_1, \ldots, u_m^A_{s,s'})$, and $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are as defined in Corollaries 5.2 and 5.4.

**Proof.** — Let $\widehat{K(A)}$ be the group on the given generators with the given defining relations. We write $\widehat{z_s(u)}$ and $\widehat{h_s(t)}$ for the generators of $\widehat{K(A)}$, to avoid confusion. Relations (K1) and (K2) hold in $K(A)$ due to Lemma 5.3; relations (K3) hold thanks to (2.1); relations (K4) are clear; relations (K5) hold thanks to Corollaries 5.2 and 5.4. Hence, there exists a unique homomorphism $\Psi = \widehat{K(A)} \rightarrow K(A)$ such that $\Psi(\widehat{z_s(u)}) = z_s(u)$ for all $s \in S$ and $u \in D$.

For $s \in S$, Lemma 5.3 and Lemma 2.2(b) show that there exists a unique homomorphism $\tau_s : K_s \rightarrow \widehat{K(A)}$ satisfying $\tau_s(\widehat{z_s(u)}) = z_s(u)$ for all $u \in D$ (here we use (K1), (K2) and (K3)). By Lemma 2.2(a), there exists a unique homomorphism $\tau : T \rightarrow \widehat{K(A)}$ satisfying $\tau(\widehat{h_s(t)}) = h_s(t)$ for all $s \in S$ and $t \in S^1$ (here we use (K1) and (K3) for $u \in S^1$). Clearly, $\tau_s = \tau$ on $K_s \cap T = \{h_s(t) \mid t \in S^1\}$, and $\tau(h)\tau_s(g)\tau(h)^{-1} = \tau_s(hgh^{-1})$ for all $h \in T, s \in S$ and $g \in K_s$ by (K3). Hence, for $s \in S$, there exists a homomorphism $\bar{\tau}_s : TK_s \rightarrow \widehat{K(A)}$ extending $\tau$ and $\tau_s$. 
Let $\tilde{K}(A)$ be the amalgamated product of the $K \cap P_s = TK_s$, $s \in S$. Then there exists a unique homomorphism $\hat{\tau} : \tilde{K}(A) \to \tilde{K}(A)$ such that, for all $s \in S$, $\hat{\tau} \in \hat{\tau}_s$ on $TK_s$. By Proposition 5.1(e) and relations (K4) and (K5), $\hat{\tau}$ induces a homomorphism $\Phi : K(A) \to \tilde{K}(A)$. It is easy to check that $\Phi$ and $\Psi$ are mutually inverse. This proves the theorem.

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