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Brownian motion on a small geodesic ball


<http://www.numdam.org/item?id=AST_1985__132__89_0>
Let \( \{X_t, t \geq 0\} \) be the Brownian motion process of a Riemannian manifold \((M, g)\). The exit time from the geodesic ball centered at \( m \in M \) is defined by

\[
T_\varepsilon = \inf\{ t > 0 : d(X_t, m) = \varepsilon \}
\]

where \( d(\cdot, \cdot) \) is the distance function defined by \( g \).

In a previous paper [4] we studied the mean exit time \( E_m(T_\varepsilon) \) and obtained three non-zero terms of the asymptotic expansion when \( \varepsilon \downarrow 0 \). This was used to prove the following stochastic characterization of the Euclidean space \((\mathbb{R}^n, g_0)\): If for each \( m \in M \), \( E_m(T_\varepsilon) = \varepsilon^2/2n + O(\varepsilon^8) \) when \( \varepsilon \downarrow 0 \), then \((M, g)\) is locally isometric to \((\mathbb{R}^n, g_0)\) provided \( n < 6 \). In case \( n = 6 \), we provided an example of a non-flat symmetric Riemannian manifold whose asymptotic expansion is \( \varepsilon^2/2n + O(\varepsilon^{10}) \) when \( \varepsilon \downarrow 0 \).

In this paper we shall extend our analysis to the second moment \( E_m(T_\varepsilon^2) \), \( m \in M \), \( \varepsilon \downarrow 0 \). By combining the previous techniques with the "stochastic Taylor formula" we obtain a three-term asymptotic expansion for the second moment, given at the end of section 4. As a
by-product we have the following characterization of Euclidean space
\((\mathbb{R}^n, g_0)\) valid in any dimension \(n < \infty\): If for each \(m \in M\), \(E_m(T_\varepsilon) = \varepsilon^2 + O(\varepsilon^3)\) and \(E_m(\varepsilon^2) = \text{const.} \varepsilon^4 + O(\varepsilon^5)\) when \(\varepsilon \to 0\), then \((M, g)\) is locally isometric to \((\mathbb{R}^n, g_0)\). Similar characterizations are ob­
tained for any space of constant curvature.

The present work, which could be formulated in non-stochastic
terms, may be viewed as complementary to the general theory of semi­
martingales on manifolds as formulated by Laurent Schwartz [5]. In
particular our stochastic Taylor formula (proposition 2.1 below) is a
consequence of the martingale formulation of diffusion processes.

2. Notations and Definitions

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. We use the
following notations.

- \(\tilde{M}_m\) is the tangent space at \(m \in M\).
- \(B_m(\varepsilon)\) is the ball of radius \(\varepsilon\) in \(M\) with center at \(m \in M\).
- \(\tilde{B}_m(\varepsilon)\) is the ball of radius \(\varepsilon\) in \(\tilde{M}_m\) with center at \(0 \in \tilde{M}_m\).
- \(\exp_m\) is the exponential mapping (which is defined on all of \(\tilde{M}_m\)
in case \(M\) is complete; otherwise it is a mapping) from
  \(\tilde{B}_m(\varepsilon)\) to \(B_m(\varepsilon)\) for sufficiently small \(\varepsilon > 0\).
- \(\phi_\varepsilon\) is the mapping on functions defined by
  \[(\phi_\varepsilon f)(\exp_m \xi) = f(\xi);\]
  \(\phi_\varepsilon\) maps from \(C^\infty(B_m(1))\) to \(C^\infty(B_m(\varepsilon))\) for sufficiently
  small \(\varepsilon > 0\).
- \(\Delta\) is the Laplace-Beltrami operator of the Riemannian
  manifold:
  \[\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right)\]
  where \(g^{ij} = (g^{-1})^{ij}\), \(g = \det(g_{ij})\).

The following result, which will be used repeatedly, was proved in
[4].

**Proposition 2.0:** There exist second order differential operators
\((\Delta_2, \Delta_0, \Delta_1, \ldots)\) on \(C^\infty(\tilde{M}_m)\) such that for each \(N > 0\) and each \(f \in C^\infty(\tilde{M}_m)\)

\[(2.1) \quad \phi_\varepsilon^{-1} \Delta \phi_\varepsilon f = \varepsilon^{-2} \Delta_2 - \varepsilon^j \Delta_j f + O(\varepsilon^{N+1}) \quad (\varepsilon \to 0).\]

\(\Delta_j\) maps polynomials of degree \(k\) to polynomials of degree \(k + j\). In
any normal coordinate chart \((x_1, \ldots, x_n)\) we have
Here $R_{\text{iajb}}$ is the Riemann tensor and $\rho_{ij} = \sum_{a=1}^{n} R_{\text{iaja}}$ is the Ricci tensor at $m \in M$.

Let $(X_t, P_x)$ be the Brownian motion process with infinitesimal generator $\Delta$. For each $m \in M$ let $T_\varepsilon$ be the exit time from the geodesic ball $B_m(\varepsilon)$. To study the moments of $T_\varepsilon$ we invoke the following "stochastic Taylor formula."

Proposition 2.1 [1,2]: Let $(X, P_x)$ be a Feller-Markov process with infinitesimal generator $A$. Let $T$ be a stopping time with $E(T^{N+1})$ finite and let $f$ be a function in the domain of $A^{N+1}$. Then

$$f(x) - E_x f(X_T) = \sum_{k=1}^{N+1} \left( \frac{(-1)^k}{k!} E_x \left\{ T^k A^k f(X_T) \right\} + \frac{(-1)^{N+1}}{N!} E_x \left\{ \int_0^T T^{(N+1)} f(X_u) \, du \right\} \right)$$

(If $N=0$ the sum is empty and we have the Dynkin formula $E_x f(X_T) = f(x)$.)

Corollary 2.2: Let $T_\varepsilon$ be the exit time from the geodesic ball $B_m(\varepsilon)$ and let $u_0 = 1$, $u_k(x) = (1/k!)E_x(T^k f)$ for $k \geq 1$. Then in the interior of $B_m(\varepsilon)$ we have $\Delta u_k = -u_{k-1}$ (for $k=1, 2, \ldots$) and on the boundary we have $u_k = 0$ (for $k=1, 2, \ldots$). In particular $\Delta^k u_N = (-1)^k u_{N-k}$, $0 \leq k \leq N$, $N \geq 1$.

Proof: Let $\tilde{u}_0 = 1$ and let $\tilde{u}_k$ be the classical solution of the elliptic problem $\Delta \tilde{u}_k = -\tilde{u}_{k-1}$ with $\tilde{u}_k = 0$ on the boundary of $B_m(\varepsilon)$. Taking $T = \min(R, T_\varepsilon)$ and $f = \tilde{u}_{N+1}$ in the proposition 2.1 we have

$$\tilde{u}_{N+1}(x) - E_x T^{N+1} f(X_T) = \sum_{k=1}^{N+1} \frac{1}{k!} E_x \left\{ T^k \tilde{u}_{N-k+1}(X_T) \right\} + \frac{1}{(N+1)!} E_x (T^{N+1})$$

Thus

$$\frac{1}{(N+1)!} E_x (T^{N+1}) \leq 2 \| \tilde{u}_{N+1} \|_{\infty} + \sum_{k=1}^{N} \frac{1}{k!} \| \tilde{u}_{N-k+1} \|_{\infty} E_x (T^k)$$

Letting $R \to \infty$ in this inequality and using induction we see that $E_x (T^{N+1})$ is finite. Taking $T = T_\varepsilon$ above yields
This completes the necessary identification.

The exact solution \( u_2 \) is not available for a general Riemannian manifold. Therefore, following [4] we shall construct an approximate solution \( v_2 \) in the form

\[
v_2 = \phi (\varepsilon^4 g_0 + \varepsilon^6 g_2 + \varepsilon^7 g_3 + \varepsilon^8 g_4)
\]

where \( g_0, g_2, g_3, g_4 \) are functions on \( \mathcal{B}_m(1) \) satisfying

\[
\begin{align*}
\Delta_{-2} g_0 &= -f_0 & g_0 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} g_2 + \Delta_0 g_0 &= -f_2 & g_2 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} g_3 + \Delta_1 g_0 &= -f_3 & g_3 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} g_4 + \Delta_0 g_2 + \Delta_2 g_0 &= -f_4 & g_4 |_{\partial \mathcal{B}_m(1)} &= 0
\end{align*}
\]

The functions \( f_0, f_2, f_3, f_4 \) are solutions of the following set of equations:

\[
\begin{align*}
\Delta_{-2} f_0 &= -1 & f_0 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} f_2 + \Delta_0 f_0 &= 0 & f_2 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} f_3 + \Delta_1 f_0 &= 0 & f_3 |_{\partial \mathcal{B}_m(1)} &= 0 \\
\Delta_{-2} f_4 + \Delta_0 f_2 + \Delta_2 f_0 &= 0 & f_4 |_{\partial \mathcal{B}_m(1)} &= 0
\end{align*}
\]

Letting \( v_1 = \phi (\varepsilon^2 f_0 + \varepsilon^4 f_2 + \varepsilon^5 f_3 + \varepsilon^6 f_4) \) we have \( \Delta v_1 = -v_1 + O(\varepsilon^8), \Delta^2 v_2 = 1 + O(\varepsilon^6) \). Applying proposition 2.1 with \( N=1, f=v_2 \) we have \( v_2 \equiv (\beta) \mathcal{E}_p \left( T^2 (1 + O(\varepsilon^6)) \right) = (\beta) \mathcal{E}_p \left( T^2 \right) + O(\varepsilon^{10}). \) To summarize, we have the following:

**Proposition 2.3:** The function \( v_2 \) defined by (2.3) - (2.7) satisfies

\[
\begin{align*}
v_2 |_{\partial \mathcal{B}_m(\varepsilon)} &= 0 = \Delta v_2 |_{\partial \mathcal{B}_m(\varepsilon)}, \quad \Delta v_2 = -v_1 + O(\varepsilon^8), \quad \Delta^2 v_2 = 1 = O(\varepsilon^6), \quad \text{and} \\
v_2 (m) &= \frac{1}{2} E_m (T^2) + O(\varepsilon^{10}) \quad \text{when} \quad \varepsilon \to 0.
\end{align*}
\]
3. Determination of $g_0$, $g_2$

In this section we shall prove

Proposition 3.1. We have

$$g_0 = (1/2n)^2 (1 - r^2) - (1/8n(n+2)) (1 - r^4)$$

$$g_2 = \left( \rho - \frac{\tau r^2}{n} \right) \left[ \frac{n+2}{6n^2 (n+4)} - \frac{n+3}{12n(n+2)(n+4)(n+6)} (1 - r^4) \right]$$

$$+ \tau \left[ \frac{1 - r^2}{24n^3 (n+2)} + \frac{1 - r^4}{24n^3 (n+2)} - \frac{1 - r^6}{24n^2 (n+2)(n+4)} \right]$$

where $\rho = \sum_{i,j=1}^{n} \rho_{ij} x_i x_j$ is the Ricci tensor, $r^2 = \sum_{i=1}^{n} x_i^2$ and $\tau = \sum_{i=1}^{n} \rho_{ii}$ is the scalar curvature.

Proof: Recall from the previous work [4]

$$f_0 = (1/2n)(1 - r^2)$$

$$f_2 = \left( \rho - \frac{\tau r^2}{n} \right) \frac{1 - r^2}{6n(n+4)} + \tau \frac{1 - r^4}{12n^2(n+2)}$$

$$\Delta_{-2}(r^2) = 2n, \quad \Delta_{-2}(r^4) = 4(n+2)r^2, \quad \Delta_{-2}(r^6) = 6(n+4)r^4$$

$$\Delta_0(r^2) = -\frac{2}{3} \rho, \quad \Delta_0(r^4) = -\frac{4}{3} \rho r^2, \quad \Delta_0(r^6) = -2\rho r^4$$

$$\Delta_{-2}(\rho) = 2\tau, \quad \Delta_{-2}(r^2 \rho) = 2\tau r^2 + 2(n+4)\rho, \quad \Delta_{-2}(r^4 \rho) = 2\tau r^4 + 4(n+6)\rho r^2.$$

$$\Delta_0(\rho) = \frac{2}{3} (\rho \nabla R - 2\rho \nabla^2), \quad \Delta_0(r^2 \rho) = \frac{2r^2}{3} (\rho \nabla R - 2\rho \nabla^2) - \frac{2}{3} \rho^2,$$

$$\Delta_0(r^4 \rho) = \frac{2r^4}{3} (\rho \nabla R - 2\rho \nabla^2) - \frac{4}{3} \rho^2 r^2,$$

where in the last two formulas we have used the fact that $\Delta_0(fg) = f\Delta_0 g + g\Delta_0 f$ if $f = f(r)$ is a radial function and $g$ is arbitrary. A lengthy but straightforward computation then shows that $\Delta_{-2} g_0 = -f_0$, $\Delta_{-2} g_2 = -f_2 - \Delta_0 g_0$, as required. Clearly both $g_0, g_2$ satisfy the required boundary conditions.

4. Determination of $g_4(0)$

We introduce the Green's operator:
defined uniquely by the properties that for all \( f \in \mathcal{C}^\infty(\overline{B}_m(1)) \)

\[
\Delta_{-2}(Pf) + f = 0 \quad \text{in } \overline{B}_m(1)
\]

\[
Pf = 0 \quad \text{on } \partial \overline{B}_m(1).
\]

With this notation we have from (2.8) - (2.11)

\[
f_0 = P1
\]

\[
f_2 = P\Delta_0 f_0
\]

\[
f_3 = P\Delta_1 f_0
\]

\[
f_4 = P\Delta_0 f_2 + P\Delta_2 f_0
\]

Similarly equations (2.4) - (2.7) can be written in the form

\[
g_0 = Pf_0
\]

\[
g_2 = Pf_2 + P\Delta_0 g_0
\]

\[
g_3 = Pf_3 + P\Delta_1 g_0
\]

\[
g_4 = Pf_4 + P\Delta_0 g_2 + P\Delta_2 g_0
\]

\[
= P^2\Delta_0 f_2 + P^2\Delta_2 f_0 + P\Delta_0 g_2 + P\Delta_2 g_0.
\]

Therefore to compute \( g_4 \) we must first compute \( \Delta_0 f_2, \Delta_2 f_0, \Delta_0 g_2, \Delta_2 g_0 \).

To handle the terms \( P\Delta_0 g_2 \) and \( P\Delta_2 g_0 \) we may use lemma 6.3 of [4]. To handle the terms \( P^2\Delta_0 f_2 \) and \( P^2\Delta_2 f_0 \) we invoke the following lemma,
Proof. Let \( G(x,y) \) be the Green's function for the biharmonic equation \( \Delta^2 G = \delta \) with the same boundary conditions. Then

\[
j(x) = \int_{B_m(1)} G(x,y) |y|^k g(y/|y|) \, dy.
\]

Let \( \bar{g} = \int_{S^{n-1}} g(\theta) \, d\theta \) be the mean value of \( g \) on the unit sphere. Then

\[
j(0) = \int_{B_m(1)} G(0,y) |y|^k [g(y/|y|) - \bar{g}] + \int_{B_m(1)} G(0,y) |y|^k \, dy.
\]

The first integral is zero, since \( G(0,y) = G(|y|) \), a radial function. The second integral is the solution of the problem \( \Delta^2 j(x) = r^k \bar{g} \), which is directly computed as

\[
j(r) = \frac{-\bar{g}}{(k+2)(n+k)} \left[ \frac{1 - r^2}{2n} - \frac{1 - r^{k+4}}{(k+4)(n+k+2)} \right].
\]

Thus

\[
j(0) = \frac{-\bar{g}}{(k+2)(n+k)} \left[ \frac{1}{2n} - \frac{1}{(k+4)(n+k+2)} \right]
\]

which is of the required form.

For small values of \( k \), we have for example

\[
k = 0: \quad j(0) = \frac{(n+4)}{8n^2(n+2)} \bar{g},
\]

\[
k = 2: \quad j(0) = \frac{(n+6)}{12n(n+2)(n+4)} \bar{g},
\]

\[
k = 4: \quad j(0) = \frac{(n+8)}{16n(n+4)(n+6)} \bar{g}.
\]

We also recall the following integral formulas which were used in [4] where integration is with respect to the normalized uniform surface measure on \( S^{n-1} \).

\[
\int_{S^{n-1}} (\rho - \frac{1}{n})^2 \, d\rho = \frac{2}{n(n+2)} \left( \|\rho\|^2 - \frac{1}{n} \right)
\]

\[
\int_{S^{n-1}} \rho \# R = \frac{\|\rho\|^2}{n}
\]
\[
\int_{S^{n-1}} \rho^* \rho = \frac{\|\rho\|^2}{n}
\]
\[
\int_{S^{n-1}} R \# R = \frac{1}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right)
\]
\[
\int_{S^{n-1}} \nabla^2 \rho = \frac{2}{n(n+2)} \Delta \tau
\]

It is easily checked that this implies that
\[
\int_{S^{n-1}} \Delta_0 \left(\rho - \frac{\tau \rho}{n}\right) = -(2/3n) \left(\|\rho\|^2 - \frac{\tau^2}{n}\right).
\]

**Computation of \( P^2 \Delta_2 f_0 \):** We have
\[
\Delta_2 f_0 = \frac{1}{90} (9V_2 \rho + 2R \# R)
\]

Both of these terms are homogeneous with \( k = 4 \). Applying the above lemmas 4.1 and 4.2 we have
\[
(P^2 \Delta_2 f_0)(0) = \frac{n + 8}{90 \cdot 16n^2 (n+4)(n+6)} \left[ \frac{18}{n(n+2)} \Delta \tau + \frac{2}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right) \right]
\]

**Computation of \( P \Delta_2 g_0 \):** We have
\[
\Delta_2 g_0 = \frac{1}{90} (9V_2 \rho + 2R \# R) \left( \frac{1}{2n^2} - \frac{r^2}{2n(n+2)} \right)
\]

which is a combination of terms with \( k = 4 \) and \( k = 6 \). Applying lemma 6.3 of [4] and lemma 4.2 above, we have
\[
(P \Delta_2 g_0)(0) = \frac{n^2 + 20n + 48}{90 \cdot 48n^2 (n+2)(n+4)(n+6)} \left[ \frac{18}{n(n+2)} \Delta \tau + \frac{2}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right) \right]
\]

**Computation of \( P^2 \Delta_0 f_2 \):** We have
\[
\Delta_0 f_2 = \left(\rho - \frac{\tau \rho}{n}\right) \frac{\rho}{9n(n+4)} + \frac{(1 - r^2)}{6n(n+4)} \left[ \frac{2}{3} (\rho \# R - 2\rho^* \rho) + \frac{2\tau \rho}{3n} \right] + \frac{\tau \rho r^2}{9n^2 (n+2)}
\]

which is a combination of terms with \( k = 2 \) and \( k = 4 \). Applying lemmas 4.1 and 4.2 we have
\[
(P^2 \Delta_0 f_2)(0) = -\frac{n^2 + 12n + 48}{432n^3 (n+2)(n+4)^2(n+6)} \left(\|\rho\|^2 - \frac{\tau^2}{n}\right) + \frac{n + 8}{144n^4 (n+2)(n+4)(n+6)} \tau^2
\]
Computation of $\Delta_0 \gamma_2$: We have

$$
\Delta_0 \gamma_2 = \left( \rho - \frac{\tau r^2}{n} \right) \Delta_0 \left[ \frac{n + 2}{6n^2(n+4)} (1 - r^2) - \frac{n + 3}{12n(n+2)(n+4)(n+6)} (1 - r^4) \right]
$$

$$
+ \left[ \frac{n + 2}{6n^2(n+4)} (1 - r^2) - \frac{n + 3}{12n(n+2)(n+4)(n+6)} (1 - r^4) \right] \Delta_0 \left( \rho - \frac{\tau r^2}{n} \right)
$$

$$
= \left( \rho - \frac{\tau r^2}{n} \right) \left[ \frac{1 - r^2}{24n^3(n+2)} + \frac{1 - r^4}{24n^3(n+2)} - \frac{1 - r^6}{24n^2(n+2)(n+4)} \right]
$$

$$
+ \frac{n + 2}{6n^2(n+4)} (1 - r^2) + \frac{n + 3}{12n(n+2)(n+4)(n+6)} (1 - r^4)
$$

which is a combination of terms with $k = 4$ and $k = 6$. Applying lemma 4.2 above and lemma 6.3 of [4] we have after some lengthy algebra

$$
(P\Delta_0 \gamma_2) (0) = - \frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3(n+2)^2(n+4)(n+6)} \left( \| \rho \|_2^2 - \frac{\tau^2}{n} \right)
$$

$$
+ \frac{5n^2 + 108n + 240}{864n^4(n+2)^2(n+4)(n+6)} \tau^2
$$

These results are recorded in the table in the Appendix. We summarize the result in the following form.

**Theorem 4.3.** For small $\varepsilon > 0$

$$
\frac{1}{\varepsilon} \mathbb{E}_m \left( \tau^2 | \varepsilon \right) = c_0 \varepsilon^4 + c_1 \varepsilon^6 \tau_m + \varepsilon^8 \left[ c_2 \Delta \tau + c_3 \tau^2 + c_4 \| \rho \|_2^2 + c_5 \| R \|_2^2 \right]_m + O(\varepsilon^{10})
$$

where the constants $c_0, c_1, c_2, c_3, c_4, c_5$ depend on the dimension $n$. In fact $c_0 = q_0(0)$ and $c_1 = q_2(0)$ given by proposition 3.1; $c_2, c_3, c_4, c_5$ are given in the appendix. Here $\tau = \sum_{i=1}^{n} \rho_{ii}$ is the scalar curvature.
and $\Delta t = \frac{n}{\ell^2} \Sigma_{i=1}^{n} t_i$ is the Laplacian of the scalar curvature. Also
tensor\text{and} $$\| R \| = \left\{ \sum R_{ijkl} \right\}^{\frac{1}{2}} \text{and} \| \rho \| = \left\{ \sum \rho_{ijkl} \right\}^{\frac{1}{2}}$$ are the lengths of the curvature
tensor\text{and the Ricci curvature.}

5. Converse theorems

Theorem 5.1. Let $(M, g)$ be a Riemannian manifold such that for all $m \in M$ we have 
$E_m^c(T_\varepsilon) = \text{const.} \varepsilon^2 + O(\varepsilon^8)$ and 
$E_m^c(T_\varepsilon^2) = \text{const.} \varepsilon^4 + O(\varepsilon^{10})$
when $\varepsilon \to 0$. Then $(M, g)$ is locally isometric to $(\mathbb{R}^n, g_0)$.

**Proof.** From the first hypothesis and theorem 1.1 of [4] we have that 
for all $m \in M$, $\tau = 0$ and $\| R \| = 0 = \| \rho \|$. From the second hypothesis and 
theorem 4.3 above we have in addition that $c_4 \| R \|^2 + c_5 \| R \|^2 = 0$. This 
is possible for $\| R \| \neq 0$ if and only if $c_4 + c_5 = 0$. From the table of 
values in the Appendix this entails the equality

$$18(n+4)^2(n+6)(2n^2 + 25n + 48) = 33n^5 + 792n^4 + 8292n^3 + 38208n + 69120$$

Multiplying out the left side it is seen that the left side is 
strictly greater than the right side for every $n > 1$. Therefore 
c_4 + c_5 \neq 0 and we must have $\| R \| = 0 = \| \rho \|$ and $(M, g)$ is locally iso-
metric to $(\mathbb{R}^n, g_0)$.

Theorem 5.2. Let $(M, g)$ be a Riemannian manifold such that for all $m \in M$
$E_m^{(M, g)}(T_\varepsilon) - E_m^{(M_\lambda, g_\lambda)}(T_\varepsilon) = O(\varepsilon^8)$ and 
$E_m^{(M, g)}(T_\varepsilon^2) - E_m^{(M_\lambda, g_\lambda)}(T_\varepsilon^2) = O(\varepsilon^{10})$
when $\varepsilon \to 0$ where $(M_\lambda, g_\lambda)$ is a space of constant
sectional curvature $\lambda$. Then $(M, g)$ is locally isometric to $(M_\lambda, g_\lambda)$.

**Proof.** From the first hypothesis and theorem 1.1 of [4] we have that 
for all $m \in M$

$$\tau_m = \tau(\lambda)$$

$$\| R \|^2_m - \| \rho \|^2_m = \| R(\lambda) \|^2 - \| \rho(\lambda) \|^2$$

where $\tau(\lambda), R(\lambda), \rho(\lambda)$ are the values for a space of constant sec-
tional curvature. From the second hypothesis and theorem 4.3 above, 
we have further
The proof of theorem 5.1 above shows that $c_4 + c_5 \neq 0$. Therefore the above equations uniquely determine the values $\|R\|_m^2 = \|R(\lambda)\|_\lambda^2$, $\|p\|_m^2 = \|p(\lambda)\|_\lambda^2$. It is well known that this implies that $(M,g)$ has constant sectional curvature.
6. Appendix. Table of the coefficients of $g_4(0) = c_2 \Delta \tau + c_3 \tau^2 + c_4 \| \rho \|^2 + c_5 \| R \|^2$

<table>
<thead>
<tr>
<th>coefficient of</th>
<th>$\Delta \tau$</th>
<th>$| \rho |^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^2 \Delta_0 g_2$</td>
<td>$0$</td>
<td>$- \frac{n^2 + 12n + 48}{432n^3 (n+2) (n+4)^2 (n+6)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{n + 8}{80n^3 (n+2) (n+4) (n+6)}$</td>
<td>$\frac{n + 8}{720n^3 (n+2) (n+4) (n+6)}$</td>
</tr>
<tr>
<td>$P \Delta_2 g_0$</td>
<td>$\frac{n^2 + 20n + 48}{240n^3 (n+2)^2 (n+4) (n+6)}$</td>
<td>$\frac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4) (n+6)}$</td>
</tr>
<tr>
<td>$\frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3 (n+2)^2 (n+4)^3 (n+6)^2}$</td>
<td>$\frac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4) (n+6)}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{n^2 + 20n + 48}{120n^3 (n+2)^2 (n+4) (n+6)}$</td>
<td>$\frac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4) (n+6)}$</td>
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<tr>
<td>$\frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3 (n+2)^2 (n+4)^3 (n+6)^2}$</td>
<td>$\frac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4) (n+6)}$</td>
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<tr>
<td>$\frac{2n^2 + 25n + 48}{120n^3 (n+2)^2 (n+4) (n+6)}$</td>
<td>$\frac{33n^5 + 792n^4 + 8292n^3 + 38208n^2 + 83520n + 69120}{12960n^3 (n+2)^2 (n+4)^3 (n+6)^2}$</td>
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References