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Some estimates in $\bar{\partial}_b$ Neumann boundary value problem for strongly pseudo-convex CR structures


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SOME ESTIMATES IN $\overline{\partial}_b$ NEUMANN BOUNDARY VALUE PROBLEM FOR STRONGLY PSEUDO-CONVEX CR STRUCTURES

by MASATAKE KURANISHI

Introduction. We consider a system of partial differential equations of the first order

(1) $Xf = 0$ for all $X \in E$,

where $f$ is an unknown complex valued function and $E$ is a subbundle of the bundle $\mathbb{C} TM$ of the complex tangent vectors to a compact manifold $M$, possibly with boundary. We denote by $C^\infty(M, E)$ the vector space of smooth sections of $E$. We assume that, for any $X$ and $Y \in C^\infty(M, E)$, their bracket $[X, Y]$ is also a section of $E$. We also assume that $E \cap \overline{E} = \{0\}$.

Let $M \to \mathbb{C}^n$ be a smooth embedding. Denote by $E$ the set of all $X \in \mathbb{C} TM$ such that, when considered as a complex tangent vector to $\mathbb{C}^n$ via the embedding, it is of type $(0,1)$. Then $E$ satisfies the above conditions, provided $E$ is a subbundle. This always happens when the codimension of $M$ is 1. We say that $E$ is embeddable when it is locally obtained by embedding in $\mathbb{C}^n$.

The nature of the equation (1) depends very much on its Levi-form. Namely, we consider

$$\mathbb{C}^\infty(M, E) \times \mathbb{C}^\infty(M, E) \ni (X, Y)$$

$$(2) \to \frac{1}{i} [X, \overline{Y}] \mod E + \overline{E} \in \mathbb{C}^\infty(M, \mathbb{C} TM/ (E + \overline{E})).$$

We see easily that the map does not involve differentiation and actually comes from hermitian quadratic forms on the fibers of $E$ valued in $\mathbb{C} TM/ (E + \overline{E})$. Here, by hermitian we mean with respect to the bar operation.
ration induced on $\mathcal{C}TM/ (E + \overline{E})$ by that of $\mathcal{C}TM$.

When the complex fiber dimension of $\mathcal{C}TM/ (E + \overline{E})$ is 1 and the Levi-form is non-degenerate, we have another interesting example. Namely, we now consider the principal bundle on which its normal Cartan connection is defined. Then the vertical complex tangent vectors over $E$ of the connection form a subbundle which satisfies our conditions.

The equation (1) is closely related to the complex it induces. Namely, denote by $\Lambda^q(E)$ the bundle of skew-symmetric multi-linear maps $E \times \cdots \times E$ (q factors) $\to \mathbb{C}$. Then we have the exterior derivative

$$D : C^\infty(M, \Lambda^q(E)) \to C^\infty(M, \Lambda^{q+1}(E))$$

just as in the case of de Rham complex, i.e.

$$Du(X_0, \ldots, X_q) = \sum_{a=0}^{q} (-1)^a X_a u(X_0, \ldots, X_a, \ldots, X_q)$$

$$\quad + \sum_{a < b} (-1)^{a+b} u([X_a, X_b], X_0, \ldots, X_a, \ldots, X_b, \ldots, X_q).$$

Introduce hermitian metrics on the fibers of $E$ and a volume element of $M$. Then they induce a pre-Hilbert structure on $C^\infty(M, \Lambda^q(E))$. We wish to exhibit two formulas related to a semi-norm

$$||XDu||^2 + ||X^* Du||^2$$

where $X$ is an arbitrary real valued smooth function compactly supported in the interior of $M$. When the complex fiber dimension of $\mathcal{C}TM/ (E + \overline{E})$ is 1, dim $M = 2n - 1$ with $q(n - 2 - q) > 0$, and the boundary of $M$ satisfies rather strict conditions (cf. (31)) these formulas can be combined to find an estimate of (5). When we let $X$ converge to a function $\mu$ which may not be zero on the boundary, in the limit estimate we find terms which involve integrals on the boundary of $M$. Our main concern is to find an estimate of (5) such that, under D-Neumann...
boundary value condition (cf. (3) and 3) (48)), these boundary integrals are non-negative. The formulas are improved versions of those in (I) [4], where one finds the details which are omitted here. These formulas are not strong enough to solve the D-Neumann boundary value problem. In the last section we derive estimates from the above. Our hope is to find out eventually if the norm \( \|Du\|^2 + \|D^* u\|^2 + C\|u\|^2 \) is compact with respect to \( L_2 \)-norm, provided \( u \) satisfies conditions (cf. (48)) including the D-Neumann boundary value condition. However it seems that our estimates are not yet strong enough to show the compactness.

Preliminary. We first fix a complex vector subbundle \( F \) of \( \mathbb{C}T\mathbb{M} \) (with \( F = F \)) supplementary to \( E + \overline{E} \). Write for \( X,Y \in \mathcal{C}^{\infty}(M,E) \)

\[
[X,Y] = iC_F(X,Y) + [X,\overline{Y}]_E + [X,\overline{Y}] \bigg|_E,
\]

where

\[
C_F(X,Y) \in \mathcal{C}^{\infty}(M,F), [X,\overline{Y}]_E \in \mathcal{C}^{\infty}(M,E), \quad \text{and}
\]

\[
[X,\overline{Y}] \bigg|_E \in \mathcal{C}^{\infty}(M,\overline{E}) .
\]

We define \( E \)-hessian of a function \( f \) by

\[
H^E_f(X,Y) = X\overline{Y}f - [X,\overline{Y}]_E f
\]

for any \( X,Y \in \mathcal{C}^{\infty}(M,E) \). We find easily that it does not involve differentiation. When \( f \) is real valued

\[
H^E_f(X,Y) = H^E_f(Y,X) + iC_F(X,Y) f .
\]

The exterior product induces an algebra structure on \( \Lambda(E) = \sum \Lambda^q(E) \), and

\[
D(u_A v) = (Du)_A v + (-1)^p u_A D v, \quad u_A v = (-1)^p v u_A
\]

for \( u \in \mathcal{C}^{\infty}(M,\Lambda^p(E)) \) and \( v \in \mathcal{C}^{\infty}(M,\Lambda^q(E)) \). In terms of the metric we in-
introduce the interior product $L v : \Lambda^p(E) \to \Lambda^{p-1}(E)$ by

\begin{equation}
< u \cdot L v, w > = < u, v \wedge w > .
\end{equation}

If $a \in \Lambda^1(E)$ and $u \in \Lambda^p(E)$

\begin{equation}
(u \wedge a) \cdot L v = (u \cdot L a) \wedge v + (-1)^p u \wedge (v \cdot L a).
\end{equation}

This formula plays a crucial role in the proof of (23). We assume that

the support of $\chi$ is so small that we can pick an orthonormal base

$e_1, \ldots, e_m$ of $\Lambda^1(E)$ defined on a neighborhood $U$ of the support of $\chi$.

Let $g : \Lambda^1(E) \to \Lambda^1(E)$ be a homomorphism of vector bundles over the identity map of $M$. Then we let $g$ also operate on $\Lambda^q(E)$ by

\begin{equation}
g u = \sum_k (g e_k) \wedge (u \cdot L e_k).
\end{equation}

Thus $g f = 0$ for a scalar valued function $f$. We see easily that the above $g$ coincide with the given $g$ when $q = 1$. Since the right-hand side of the above is clearly independent of the choice of orthonormal $e_1, \ldots, e_m$, (11) is defined globally. We also see easily that the adjoint of the above $g$ is equal to the map induced by $g^* : \Lambda^1(E) \to \Lambda^1(E)$, where $*$ is in terms of the metric. Moreover

\begin{equation}
g(u \wedge v) = (g u) \cdot L v + u \cdot L g v
\end{equation}

\begin{equation}
g(u \cdot L v) = (g u) \cdot L v - u \cdot L g^* v.
\end{equation}

We denote by $Y_1, \ldots, Y_m$ a base of $E$ dual to $e_1, \ldots, e_m$. We set

\begin{equation}
[Y_j, Y_k] = \sum_\ell r_{jk\ell} Y_\ell
\end{equation}

and define $r_{(k)} : \Lambda^1(E) \to \Lambda^1(E)$ by

\begin{equation}
r_{(k)} e_\ell = \sum_j r_{kj\ell} e_j .
\end{equation}

For $K = (k_1, \ldots, k_q)$ and $u \in C^\infty(U, \Lambda^q(E))$, set
(16) \[ u_K = u(Y_{k_1}, \ldots, Y_{k_q}) \].

If \( G \) is a vector field on \( U \), we let \( G \) operate on \( C^\infty(U, \Lambda^q(E)) \) by

(17) \[ (Gu)_K = Gu_K. \]

By definition we see that the operation of \( G \) commutes with the exterior and the interior product by \( e_k \). We define \( \hat{Y}_K : C^\infty(U, \Lambda^q(E)) \to C^\infty(U, \Lambda^q(E)) \) by

(18) \[ \hat{Y}_K u = Y_K u - \frac{1}{2} r(k) u. \]

It then follows (cf. §.1 I [4])

(19) \[ Du = \sum_k e_k \hat{Y}_K u. \]

Hence by duality

(20) \[ D^* u = \sum_k Y^*_K (u \wedge e_k). \]

Finally, set

(21) \[ [Y_j, Y^*_K] = \frac{1}{2} C_F (Y_j, Y_K) + \sum_k q_{j\ell} \hat{Y}_K + \sum_{\ell} \hat{Y}_K q_{j\ell} + q_{jk}. \]

(cf. (6)). We define \( q(k) : \Lambda^1(E) \to \Lambda^1(E) \) by

(22) \[ q(k) e_\ell = \sum q_{k\ell j} e_j. \]

A priori estimate. Let \( X \) be as indicated in (5). Pick a section \( u \) of \( \Lambda^q(E) \) which is smooth on a neighborhood of the support of \( X \). To find a formula for the semi-norm (5), it is enough to consider \( D^* X^2 D + D X^2 D^* \). For simplicity we consider \( X \) instead of \( X^2 \) for a while.
(23) PROPOSITION 1.

\[(D^* X D + DX D^*)u = A u + B u + C u,\] where \( A = A_1 + A_2 \) with

\[A_1 u = \sum_k \tilde{\gamma}_k \tilde{\chi} \tilde{\gamma}_k u \]
\[A_2 u = \sum_k \tilde{\gamma}_k \sum_j \left( (q^*(k) + \frac{i}{\varepsilon} r^*(k)) e_j \tilde{\chi} \tilde{\gamma}_j (u L e_k) \right) \sum \frac{\varepsilon}{\gamma} \tilde{\gamma}_j e_j \tilde{\chi} (u L r^*(j) e_k) \]
\[B u = \sum_j \sum_k \tilde{\gamma}_j \tilde{\chi} \tilde{\gamma}_k u \]
\[C u = D (u L D X) + D X D^* u.\]

Outline of the proof. First work out the commutator relation between \( \tilde{\gamma}_k \) and the exterior product by \( e_j \) (cf. (12), (13)). Similarly for the interior product by \( e_j \). Write down \((D^* X D + DX D^*)u\) using (19) and (20). Apply (10) and rewrite it as the sum of \( \tilde{\gamma}_k \tilde{\chi} \tilde{\gamma}_k u \) and terms containing \( u L e_k \). Then our formula follows by (21) and (7).

We note that the above formula is a precise version of the one given by J.J. Kohn in [2].

Note that \( < A_1 u , u > \geq 0 \). In view of \( A_1 \), we do not have to worry too much about the term \( < A_2 u , u > \). When we let \( \chi \) converge to a function which may not be zero on the boundary of \( M, D \chi \) will blow up on the boundary. Note that \( D \chi \) appears in \( C u \). The Neumann boundary condition we consider later is exactly the one which makes \( < Cu , u > \) go to zero. When we want to obtain an a priori estimate for \( ||Du||^2 + ||D^*u||^2 + C ||u||^2 \), we see then that the main difficulty comes through \( < Bu , u > \). We try to eliminate this term by taking advantage of the term \( \sum_j (\tilde{\gamma}_j)^* \tilde{\chi} \tilde{\gamma}_j \) in \( A \).
We assume that \( \tilde{M} \) is in a manifold \( \tilde{M} \) and \( E \) extends to \( \tilde{M} \). Let \( t \) be a real valued function on \( \tilde{M} \). For later application we consider the case when the boundary of \( M \) is defined by \( t = 0 \). However, there is no need to do so now. Set

\[
Y_j t = \sigma_j , \quad \omega_j = \sigma_j / b , \quad b = (\sum \sigma_j \bar{\sigma}_j)^{1/2} .
\]

We also define (where \( b \neq 0 \))

\[
Y^O = \sum_k \bar{\omega}_k Y_k , \quad W_j = Y_j - \omega_j Y^O = \sum_k Q_{kj} Y_k ,
\]

\[
Q_{kj} = \delta_{kj} - \bar{\omega}_k \omega_j , \quad W_j = \sum_k Q_{kj} Y_k
\]

so that

\[
W_j t = 0 , \quad \sum \bar{\omega}_j W_j = 0 .
\]

Then we see that

\[
\sum_k \bar{Y}_k X Y_k = (Y^O)^* X Y^O + \sum_k \bar{W}_k X \bar{W}_k .
\]

We set

\[
\bar{W}_j = \sum Q_{jk} \bar{Y}_k .
\]

When \( X \) is a vector field and \( f \) is a function, we often write \( [X,f] \) instead of \( X f \), i.e. we regard a function as a multiplication operator.

\[
(29) \text{PROPOSITION 2. Assume that the support of } X \text{ is contained in } M' = \{ p \in M ; b(p) \neq 0 \}. \text{ Then } \sum \bar{W}_j X \bar{W}_j = A' + B' + C' , \text{ where } A' = A_1' + A_2' + G \text{ with }
\]
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\[ A_1' = \sum_j (\tilde{W}_j^*) \times \tilde{W}_j = \sum_{j,k} \tilde{Y}_k^* \times Q_{jk} \tilde{Y}_j \]

\[ = \sum_j (\tilde{W}_j^*) \times \tilde{W}_j - \sum_{j,k} (\tilde{Y}_j^* \times Q_{jk} \tilde{Y}_j) + [\tilde{Y}_k^* \times Q_{jk} \tilde{Y}_j] + [Y_j, Q_{jk}] Y_j \]

\[ + \tilde{Y}_k^* [Y_j, Q_{jk}] + R, \quad \text{where} \]

\[ R = - \sum_{j,k} [Y_k, [Y_j, Q_{jk}]] \]

\[ = - 2 \sum_k [Y_k, [W_k, Y_j]] + \sum_{j,k} ([Y_k, [Y_j, Q_{jk}]] - [Y_k, Y_j Q_{jk}]) \]

\[ = - 2 \sum_k [Y_k, [W_k, Y_j]] + \text{(a purely imaginary number)} \]

\[ + \tilde{Y}_j, [Y_k, [Y_j, Y_j] Q_{jk} - \tilde{Y}_j, k \times [Y_k, [Y_j, Q_{jk}]] . \]

Outline of the proof. Since the matrix \( (Q_{jk}) \) defines a projection operator

\[ \sum_j (\tilde{W}_j^*) \times \tilde{W}_j = \sum_{j,k} \tilde{Y}_k^* \times Q_{jk} \tilde{Y}_j \]

\[ = \sum_j (\tilde{W}_j^*) \times \tilde{W}_j - \sum_{j,k} (\tilde{Y}_j^* \times Q_{jk} \tilde{Y}_j) + [\tilde{Y}_k^* \times Q_{jk} \tilde{Y}_j] + [Y_j, Q_{jk}] Y_j \]

\[ + \tilde{Y}_k^* [Y_j, Q_{jk}] + R, \quad \text{where} \]

\[ R = - \sum_{j,k} [Y_k, [Y_j, Q_{jk}]] \]

\[ = - 2 \sum_k [Y_k, [W_k, Y_j]] + \sum_{j,k} ([Y_k, [Y_j, Q_{jk}]] - [Y_k, Y_j Q_{jk}]) \]

\[ = - 2 \sum_k [Y_k, [W_k, Y_j]] + \text{(a purely imaginary number)} \]

\[ + \tilde{Y}_j, [Y_k, [Y_j, Q_{jk}]] Q_{jk} - \tilde{Y}_j, k \times [Y_k, [Y_j, Q_{jk}]] . \]
The last term of the above goes into $G$, and the second from the last (modulo $i\mathcal{C}$) is $\sum_{j,k} [Y_k[Y_j,X]] \partial_{kj}$ which goes into $B'$. 

The case of codimension 1 with definite Levi-form. We fix a generator $S \in C^\infty(M,TM)$ of $F$. Write

$$C_F(X,Y) = C_S(X,Y)S.$$ 

To get rid of the $B$ term in Proposition 1, we put a condition on the boundary of $M$. Pick a real valued function $t$ on $\tilde{M}$ without any critical point on the boundary of $M$ and such that $M$ is defined by $t < 0$ and the boundary of $M$ is defined by $t = 0$.

(31) DEFINITION 1. Assume that $M$ is of dimension $2n-1$ with $n \geq 3$. We say that the boundary of $M$ is admissible when

1) There is a smooth function $\gamma$ on $\tilde{M}$ such that

$$H^t = \gamma C_S$$

at each point on the boundary of $M$, provided $n \geq 4$. If $n = 3$, we assume further that all the first order partial derivatives of $C_S - \gamma H^t$ also vanish at each boundary point of $M$.

2) At each boundary point $p \in M$ such that $b(p) = 0$ (cf. (24))

$$\gamma(p) \neq 0,$$

3) For any $X, Y \in C^\infty(M, E)$ and for any $p$ as above

$$XYt(p) = 0.$$ 

We see easily that the above definition is independent of the choice of $t$ as well as of the choice of a supplementary real vector field $S$. 

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To give an example of $M$ with admissible boundary, consider a real hypersurface $\tilde{M} \subseteq \mathbb{C}^n$ of codimension 1. We write $(z,w) \in \mathbb{C}^n$, where $z \in \mathbb{C}^{n-1}$ and $w \in \mathbb{C}$. We assume that the origin is in $\tilde{M}$ and $\tilde{M}$ is given by an equation:

$$y = h(z, \bar{z}, x)$$

where $x = R_w$ and $y = \partial_m w$. We assume further that

$$h(z, \bar{z}, x) = \sum_{j,k} h_{j,k} z^j \bar{z}^k \mod(y, \bar{y}, x)^3$$

where $(h_{j,k})$ is a positive definite hermitian matrix. As is shown by Chern and Moser in [1] we can always find a holomorphic chart $(z, w)$ so that the above is valid locally, provided $M$ is strongly pseudoconvex.

(34) **PROPOSITION 3.** For a sufficiently small $r > 0$ set

$$t = h(z, \bar{z}, x) + R_w^2 - r.$$ 

Then the equation $t \geq 0$ defines a submanifold $M$ with admissible boundary.

In the following we always assume that the boundary of $M$ is admissible. We pick a smooth real valued function $\varphi(t)$ on $\mathbb{R}$ supported in $\{t \in \mathbb{R}; x < 0\}$ such that $\varphi(t) = 1$ for $t < -c_1$ for a positive number $c_1$. We also pick $\mu \in C^\infty(\overline{M}, \mathbb{R})$ with compact support. We assume that its support is small enough so that we can find an orthonormal base $Y_1, \ldots, Y_{n-1}$ of $E$ on a neighborhood of its support. We set

$$X = \mu \varphi(t).$$

In the following we outline how to get rid of the $B$ term in (23), which is the main obstruction to obtain an a priori estimate.

Note that for functions $f$ and $g$
(36) \[ H^f g(X, Y) = f H^g(X, Y) + g H^f(X, Y) + (Xf)(Yg) + (Yf)(Xg) . \]

Also we see that

(37) \[ H^\varphi(t) (X, Y) = \varphi'(t) H^t(X, Y) + \varphi''(t)(Xt)(Yt) . \]

Write

(38) \[ H^t(Y_j, Y_k) = \gamma C_S(Y_j, Y_k) + r_{jk} , \quad r_{jk} = 0 \quad \text{on bd} \ M . \]

We pick our hermitian metric to be the one defined by \( C_S \) (which we can always assume to be positive definite by replacing \( S \) by \(-S\) if necessary). Then we see that

\[
\langle B_1 u, u \rangle = \langle B_1 u, u \rangle + \langle B_2 u, u \rangle , \quad \text{where}
\]

\[
\langle B_1 u, u \rangle = - \sum_k \langle (i \times S + \mu \gamma \varphi') (u L e_k) , u L e_k \rangle , \quad \text{and}
\]

\[
\langle B_2 u, u \rangle = - \sum_j, k \langle \varphi' (t) r_{jk} (u L e_k) , u L e_j \rangle .
\]

(39)

\[
- 2 \mathcal{M} < \varphi' u L D_t , u L D_t > = - < \varphi'' u L D_t , u L D_t >
\]

\[
- \sum_j, k < \varphi H^\mu (Y_j, Y_k) (u L e_k) , u L e_j > .
\]

Note that

\[
\sum_j H^t (W_j, W_j) = \gamma (n - 2)(1 + r_0) , \quad r_0 = 0 \quad \text{on bd} \ M ,
\]

(40)

\[
\sum_j C_S(W_j, W_j) = (n - 2) .
\]

and

(41) \[ q < v, w > = \sum_k < v L e_k, w L e_k > \]

where \( q \) is the degree of \( v \) and \( w \). In view of (27) we can always take \( \sum_j (q/(n-2)) \hat{W}_j \times \hat{W}_j u \) out of the \( A \) term, and

\[
(1/(n-2)) \sum_k B' (u L e_k) , u L e_k > = \langle B_1 u, u \rangle + \langle B_2 u, u \rangle , \quad \text{where}
\]
\[
< B'_{1}u, u > = \sum_{k} < (i \times S + \mu \gamma \varphi') (u L e_{k}), u L e_{k} > \\
\]
(42) \[
< B'_{2}u, u > = < \mu \varphi' r_{o} u L e_{k}, u L e_{k} > \\
+ \frac{1}{n-2} \sum_{j} \varphi H^{j}(W_{j}, W_{j}) (u L e_{k}), u L e_{k} > .
\]

(cf. (26)). Hence

\[
< B_{u}, u > + \frac{1}{n-2} < B'_{1}(u L e_{k}), u L e_{k} > \\
= < B'_{2}u, u > + < B'_{2}u, u > .
\]

Note that the right-hand side of the above does not cause any trouble at the boundary under the D-Neumann boundary condition. By (23) and (29) we find then that for \( \chi \) as in (35)

\[
< (D^{*} X D + D X D^{*})u, u > = < Y^{*}_{o} X Y_{o} u, u > \\
+ \frac{q}{n-2} \sum_{j} \left( 2 \sum_{j} b^{-1} H^{j}(W_{j}, W_{j}) u > + < G u, u > \right) \\
+ \frac{n-2-q}{n-2} \sum_{j} < \tilde{W}_{j}^{*} X \tilde{W}_{j} u, u > + \frac{q}{n-2} \sum_{j} < \tilde{(W}_{j}^{*})^{*} X W_{j} u, u > \\
+ < B'_{2}u, u > + < B'_{2}u, u > + < (A_{2} + C + (q/(n-2))(A_{2}^{'} + C')u, u > .
\]

Now when we calculate \( G \) more explicitly, we find that

\[
G = \chi b^{-2} \left| \sum_{j} H^{j}(W_{j}, W_{j}) \right|^{2} + G_{1} , \quad \text{with}
\]
(44) \[
G_{1} = - \chi b^{-4} \sum_{k, \ell} \sigma_{\ell} \sigma_{k} \{ Y^{*}_{\ell}, \tilde{Y}_{k} \} \left( \sum_{j} H^{j}(W_{j}, W_{j}) + \sum_{j, i} Q_{ij} q_{j} s \right) \\
+ \chi b^{-1} R
\]

where \( R \) is bounded. Therefore we obtain :
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\[
\| D^2 u \|^2 + \| D^* u \| \cdot \| u \| = \| \frac{1}{n-2} \sum_{j} H_j (w_j, w_j) u \|^2 \\
+ \frac{(n-2-q)q}{(n-2)^2} \| \frac{1}{n-2} \sum_{j} H_j (w_j, w_j) u \|^2 \\
+ \frac{n-2-q}{n-2} \sum_{j} \| \frac{1}{n-2} \sum_{j} H_j (w_j, w_j) u \|^2 \\
\]

(45)

\[
+ \langle G_1 u, u \rangle + \langle B_2 u, u \rangle + \langle B_2^* u, u \rangle \\
+ \langle (A_2 + C + \frac{q}{(n-2)} (A_2^* + C^*) u, u \rangle .
\]

Note by (40), (44), and 3) (31) that $\langle b^2 G_1 u, u \rangle$ vanishes where $b = 0$, provided the support of $X$ is contained in a sufficiently small neighborhood of a boundary point.

$D$-Neumann boundary value problem. The classical method of Kohn and Nirenberg (cf. [3]) to solve the problem is to find a norm $\| u \|$ on $C^2(M, \Lambda^q(E))$ such that

1) $\| u \|$ is compact with respect to $L^2$-norm $\| u \|$,

2) $\| D u \|^2 + \| D^* u \| \cdot \| u \| = C (\| u \|)^2$

for all $u$ satisfying the boundary condition: $u \cdot DT = 0$.

We apply the same method in our case. However, the nature of the formulas in PROPOSITION 1 and 2 forces us to modify it. Firstly, since $b^{-1}$ comes in our picture which is not smooth, it is more natural to enlarge the space $C^2(\mathbb{R}, \Lambda^q(E))$. Secondly, since we localize and use different methods to prove our estimate depending where we are, we replace a single norm $\| u \|$ by a pre-fréchet space structure.

We first study neighborhoods of boundary points $p$ with $b(p) = 0$. They are the characteristic boundary points. By the non-degeneracy of
C, and 1), 2) in (31) we see easily that :

(46) PROPOSITION 4. Let $p$ be a characteristic boundary point. Then, on a sufficiently small neighborhood of $p$, $(t, \sigma)$ is a chart.

(47) COROLLARY. The set of characteristic boundary points is isolated.

(48) DEFINITION. We denote by $C'(M, \Lambda^q(E))$ the vector space of sections $u$ of $\Lambda^q(E)$ on $M$ satisfying the following conditions :

1) $u$ is $C^1$ in the interior of $M$.

2) $D u$ and $D^* u$ are in $L^2$, $b^{-1} u$ is in $L^2$ on a neighborhood of each characteristic boundary point, and $W_j u$ is in $L^2$ in a neighborhood of each boundary point.

3) For each $C^\infty$ function $f$ on $\bar{M}$ whose support is compact and disjoint from the set of characteristic boundary points, $f t^{-1} u |D t$ is in $L^2$.

We prove a priori estimate on $C'(M, \Lambda^q(E))$. The above condition 3) is the D-Neumann boundary condition. We work separately on neighborhoods of interior points of $M$, of characteristic boundary points, and of non characteristic boundary points.

(49) PROPOSITION 5. Let $X$ be a $C^\infty$ function with compact support on $\bar{M}$ which is zero on the boundary of $M$. Then there are constants $C > 0$ and $\sigma > 0$ depending on $X$ such that for any $u \in C'(M, \Lambda^q(E))$

$$\|D u\|^2 + \|D^* u\|^2 + C \|u\|^2 \geq \sigma \left( \sum_j \|Y_j X u\|^2 + \sum_j \|Y_j^* X u\|^2 + | < S X u, X u > | \right).$$

This follows by (23) because we can get rid of $B$ term (without introducing $b^{-1}$) by the well-known method of Kohn. Note in the above
that the term $\sum_j \|Y_j \times u\|^2$ is independent of a choice of a local orthonormal base $Y_1, \ldots, Y_{n-1}$ and hence has a global meaning. Similarly $\sum_j \|Y_j^* \times u\|^2$ has a global meaning modulo a term $\leq C' \|u\|^2$.

We next consider a small neighborhood $U$ of a boundary point $p_0$. In (35) we take $\mu \in C^\infty(\overline{M}, \mathbb{R})$ with support in $U \cap (b \neq 0)$. We also replace $\varphi(t)$ by $\varphi_\epsilon = \varphi(t/\epsilon)$ and let $\epsilon \to 0$. In (45) the term which contains the derivatives of $\varphi(t/\epsilon)$ in $t$ and the derivatives of $\mu$ is

$$E = \langle B_2 u, u \rangle + \langle B_1^* u, u \rangle + \langle (C + (q/(n-2)) C') u, u \rangle.$$

Because $r_{jk} = r_0 = 0$ on the boundary of $M$ we see by 3) (48) that $E$ converges to

$$E'_\mu = 2 \mathcal{R} < D^* u, u \rangle - (2q/(n-2)) \left( \sum_k [Y_k, [W_k, \mu]] u, u \right) + \sum_j < [W_j, \mu] u, W_j u \rangle - \sum_j, k < H^1(Y_j, Y_k) (u \L e_k), u \L e_j \rangle$$

$$+ (q/(n-2)) \sum_j < H^1(W_j, W_j) u, u \rangle$$

because (cf. (26))

$$\mu [W_j, \varphi_\epsilon] = 0 \quad \text{and} \quad [Y_j, \varphi_\epsilon][W_j, \mu] = 0.$$

Assume now that $p_0$ is a characteristic boundary point. We assume that $U$ is sufficiently small so that $(t, \sigma)$ is a chart on $U$ (cf. (46)). We pick $\mu_1 \in C^\infty(\overline{M}, \mathbb{R})$ with support in $U$ and set

$$\mu = \mu_1 \varphi \left( \frac{1}{\epsilon} b \right)$$

and let $\epsilon \to 0$. Because $b^{-1} u$ is in $L_2$ (cf. 2) (48)), we find that $E'_{\mu_1}$ converges to $E'_{\mu_1}$. In view of (40) and 3) (31) we then find by (45) the following:
PROPOSITION 6. Let $p_0$ be a characteristic boundary point of $M$. Assume that $q(n-2-q) > 0$. Then there is a neighborhood $U$ of $p_0$ such that, for any $u \in C^\infty(M, \mathbb{R})$ with support in $U$, there are constants $C, c > 0$ such that

$$
\|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geq c \|b^{-1}u\|^2
$$

for any $u \in C'(\overline{M}, \mathbb{R})$.

We next consider a non-characteristic boundary point $p_0$ and pick a sufficiently small $U$ which does not contain any characteristic boundary point. Let $\mu \in C^\infty(\overline{M}, \mathbb{R})$ with support in $U$. Then the above argument proves that for any $u \in C'(\overline{M}, \mathbb{R})$

$$
||Du\|2 + ||D^*u\|2 + C||u\|2 \geq c \int Y_k \mu \|2 + ||\bar{Y}_k \mu \|2
$$

Looking at terms $B$ and $B'$ in (23) and (29) we also find that

$$
||Du\|2 + ||D^*u\|2 + C||u\|2 \geq c \int_i Y \mu \mu \|y u, y u\|
$$

where $< u, u >_{bd}$ denotes the square of the $L_2$-norm of the restriction of $u$ to the boundary of $M$. With $\varphi$ as in (35), $[Y_0, \varphi] = \varphi'(t)b$. Hence we see easily that

$$
- \int Y \mu \mu \|y u, y u\| = - b^{-1}Y \mu \mu \|y u, y u\| + < \mu, (Y_0^*)^* \mu \gamma b^{-1}u >.
$$

Therefore by (51) and (52)

$$
||Du\|2 + ||D^*u\|2 + C||u\|2 \geq c \int b^{-1}X_0 \mu \mu \|y u, y u\|
$$

where

$$
X_0 = i b S + Y Y O - \gamma Y O
$$

Note that $X_0, W_j, \bar{W}_j$ are tangential to the boundary of $M$. We denote by $(bd)'M$ the set of non-characteristic boundary point.
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(55) PROPOSITION 7. $(bd)'M$ has a foliation of codimension 1 such that $X_0, W_j, \bar{W}_j$ generate the complex tangent vector space of each leaf.

Outline of the proof. It is enough to show that the equation $X_0 = W_j = \bar{W}_j = 0$, when restricted to $(bd)'M$, is completely integrable. This follows by the same calculation made in §2 II [4]. As long as we do not differentiate $H(Y_j, Y_k)$, the calculation there is still valid for our more general $t$. In view of 1) (31), no modification is needed when $n = 3$. For $n \geq 4$ we have to take a little more care for terms containing $[Y_j, Y]$. Set $P_j = [Y_j, \gamma] - \sum_k r_j k \delta_{jk} - \gamma r_j$ and $H(Y_J, Y_k) = \gamma \delta_{jk} + \alpha_{jk}$ with $\alpha_{jk} = 0$ on the boundary. Then instead of the formula $P_j \delta_{kk} = P_k \delta_{jj}$ (cf. the middle of the proof of (2.23) II [4]), we have

$$P_j \delta_{kk} + [Y_j, \alpha_{kk}] = P_k \delta_{jj} + [Y_k, \alpha_{jj}].$$

Apply $\sum_j Q_j \sum_{kk} P_k$. We then find $\sum_j Q_j P_j = 0$ on $(bd)'M$, provided $n \geq 4$. This is what we need. The term containing $[Y_j, \gamma]$ can be also handled similarly.

In view of (51) and (53) we find by the above the following:

(56) COROLLARY. Let $V$ be any complex tangent vector field on $U$ which is tangential at the boundary to the leaves of the foliation in (55). Then

$$\|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geq c < V \mu u, \mu u>.$$

Note that $Y^0 - \bar{Y}^0$ is tangential to the boundary and its restriction together with the restrictions of $X_0, W_j, \bar{W}_j$ generate $\mathcal{C} T (bd M)$.

(57) PROPOSITION 8. The flow generated by $ib^{-1}(Y^0 - \bar{Y}^0)$ preserves the foliation of (55).
Outline of the proof. By the same calculation as in §.2 II [4] we find that \( [b^{-1}(Y^0 - \overline{Y^0}), W_j] = 0 \) mod \( W_k, \overline{W}_k \). Hence \( [b^{-1}(Y^0 - \overline{Y^0}), \overline{W}_j] = 0 \) mod \( W_k, \overline{W}_k \). Since \( W_j, \overline{W}_j \) with bracket generate \( \chi_0 \), our contention follows.

We are now going to prove the following:

(58) PROPOSITION 9. There is a neighborhood \( \tilde{U} \) of \( p_0 \) in \( \tilde{N} \) with a chart \( (x,y_1,y') \), \( y' = (y_2, \ldots, y_{3n-3}) \), centered at \( p_0 \) satisfying the following: 1) \( U = \tilde{U} \cap M \) is given by \( x = 0 \) and \( y_1 \) = constants define the local fibering of (52), 3) \( Y^0 = \frac{1}{b} (\partial / \partial x + i \partial / \partial y_1) + B \) with \( B = 0 \) at each boundary point in \( U \), and 4) for any \( u \in C'(M, \Lambda^2(E)) \) with \( q(n-2-q) > 0 \)

\[ \|Du\| + \|D^*u\| + C \|u\|^2 \geq c \int \|u\|_{1/2}^{-2} \]

where \( \| \|_{1/2}^{-2} \) denotes the integral in \( (x,y_1) \) of the square of the Sobolev norm with respect to the variable \( y' \).

PROOF. Pick a chart \( y' \) centered at \( p_0 \) of the local fiber \( F_0 \) of the foliation in (55). Consider the flow generated by \( i(\overline{Y^0} - Y^0)/b \). Let \( y = (y_1, \ldots, y') \) be the point on the boundary with the parameter \( y_1 \) originating from \( y' \) in \( F_0 \). This gives a chart of \( \partial M \). We now use the flow generated by \( (Y^0 + \overline{Y^0})/b \) to define a chart \( (x,y) \). By the construction

\[ Y^0 - \overline{Y^0} = i b \partial / \partial y_1 + 2 B \]

\[ Y^0 + \overline{Y^0} = b \partial / \partial x \]

with \( B = 0 \) at each boundary point. Note also that \( (Y^0 + \overline{Y^0})/b \equiv 2 \partial / \partial t \) modulo a vector field tangential to the boundary. Hence the inequality \( x \leq 0 \) defines \( U \). Now our contention follows by (56) and (57), q.e.d.
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