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$p$-adic theta series with integral coefficients


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P-ADIC THETA SERIES WITH INTEGRAL COEFFICIENTS

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0. INTRODUCTION.

Let \( R \) be the ring of the integers of a local field \( K \), let \( k \) be its residue field, and assume \( k \) be perfect of characteristic \( p \neq 0 \). If \( A \) is an abelian variety over \( K \) with good reduction mod \( p \), we will denote by \( A_0 \) its reduced variety, by \( e \) and \( e_0 \) the identity of \( A \) and \( A_0 \) respectively, by \( \theta_0 \) the local ring of \( A \) at \( e_0 \) and by \( S \) its completion. So, if \( A \) has dimension \( n \), \( S = R[t_1, \ldots, t_n] \), where \( (t_1, t_2, \ldots, t_n) \) is a set of uniformizing parameters of \( A \) at \( e_0 \).

Now, if \( X \) is a divisor of \( A \), rational over \( K \), and if we denote by \( \theta_X \) a theta of \( X \) in \( S_K = K[t_1, \ldots, t_n] \) (we are assuming that the polar part of \( X \) doesn't go through \( e \)), a natural question arises : is it possible to choose \( \theta_X \) in \( S \)? The answer, in general, is no. In fact, if \( \theta_X \in S \), the image of \( \theta_X \) in \( S_0 = S \otimes k \) would be a theta of the image \( X_0 \) of \( X \) in \( A_0 \). But, as shown in [7], if \( A_0 \) is not ordinary, or if \( X_0 \not\equiv 0 \), the thetas of \( X_0 \) live in a ring quite bigger than \( S_0 \). So, if we are looking for a positive answer to our former question, we must assume \( A_0 \) be ordinary. In fact, with this assumption, denoted by \( D = (D_1, \ldots, D_n) \) a basis of the \( R \)-module of the invariant derivation of \( A \), and by \( (n_{1,X}, \ldots, n_{n,X}) \) the \( n \)-uple of integrals of the second kind corresponding to the couple \( (X, D) \) (see section 3. for a precise definition), we will show that the system of differential equations

\[
0.1. \quad D_i \theta - \theta n_{i,X} = 0, \quad i = 1, 2, \ldots, n,
\]

has solutions in \( S \). However, we will not use a direct approach to \( 0.1 \). In fact, if we denote by \( p_i, \ i = 1, 2, 3 \), the projections from \( A^3 \)
to $A$, and if $p^*_i$ are the corresponding applications from $S$ to $S \hat{\otimes} S \hat{\otimes} S$, the system 0.1 is equivalent to the functional equation

$$0.2 \quad \frac{(p_1^* + p_2^* + p_3^*)(p_1^* \theta)(p_2^* \theta)(p_3^* \theta)}{(p_1^* + p_2^*)(p_1^* + p_3^*)(p_2^* + p_3^*) \theta} = F,$$

where $F$ is an equation of the divisor

$$Y = (p_1^* + p_2^* + p_3^*)^{-1}X + p_1^* p_2^* p_3^* \theta,$$

of $A^3$. Now, in view of the cohomological properties of $F$, the equation 0.2 is not only much more easier to solve than 0.1, but also allows to understand that 0.1 has solutions even in a more general situation.

After the construction of the solutions of 0.2, we will show how these are related to the canonical decomposition of $H^1_{DR}(A)$ (see [9] and [4]), and finally we'll give some explicit computation for the elliptic curves.

An analogous construction has been done by P. Norman using different techniques; here I'd like to thank him for the useful conversations we had on these topics.

1. SPLITTING OF BI-MULTIPLICATIVE CO-CYCLES.

Let $R$ be a commutative ring with identity, $t = (t_1, \ldots, t_n)$ a set of indeterminates over $R$, and let $S = R[t]$ be a $R$-bi-algebra. For short, the image of $t$ in $S \hat{\otimes} S$ given by the coproduct will be denoted by $t_1 \hat{\otimes} t_2$.

1.1. DEFINITION. An element $H = H(t_1, t_2, t_3) \in S \hat{\otimes} S \hat{\otimes} S$ is called a symmetric, bi-multiplicative (resp. bi-additive) co-cycle of $S$ if

i) $H(0, t_2, t_3) = 1$ (resp. $H(0, t_2, t_3) = 0$);

ii) $H(t_1, t_2, t_3) = H(t_{\sigma_1} t_{\sigma_2} t_{\sigma_3})$, for each permutation $\sigma \in S_3$;

iii) $H(t_1 \hat{\otimes} t_2, t_3, t_4)H(t_1, t_2, t_4) = H(t_1, t_2, t_3, t_4)H(t_2, t_3, t_4)$ (resp. $H(t_1 \hat{\otimes} t_2, t_3, t_4)H(t_1, t_2, t_4) = H(t_1, t_2, t_3, t_4)H(t_2, t_3, t_4)$).
Moreover, if there exists an element $h \in S$ such that

$$1.2 \quad \frac{h(t_1^2 + t_2^2 + t_3)h(t_1)h(t_2)h(t_3)}{h(t_1^2 + t_2^2)h(t_1 + t_2)h(t_1 + t_3)h(t_2 + t_3)} = H$$

(resp. $h(t_1^2 + t_2^2 + t_3) + h(t_1) + \ldots + h(t_2 + t_3) = H$) the co-cycle $H$ is called a co-boundary of $S$. Later on the left hand side of 1.2 will be denoted by $\mathcal{Q}_u^2 h$ (resp. $\mathcal{Q}_a^2 h$).

For instance, if $R$ and $t$ have the same meaning as in the introduction, and if $X$ is a divisor of $A$ such that its reduced mod $p$ doesn't go through $e_Q$, one can choose for $F$ (cfr. 02) a symmetric, bi-mult. co-cycle of $S$ (see [7] and [5]). So our main goal in this section will be the proof of the following.

1.3. THEOREM. Let $R$ be the ring of the Witt vectors with components in the algebraically closed field $k$ of characteristic $p \neq 0$, and $S = R[[t]]$ be a multiplicative bi-algebra. Then each symmetric, bi-multiplicative co-cycle of $S$ is a co-boundary of $S$.

The assumption about the algebraic closure of $k$ seems necessary if we like results which can be applied to each divisor. Later on we will show that symmetric divisor possess theta series with integral coefficients even if $k$ is only a perfect field.

In order to prove 1.3 we need some results which are given in theorem A.4 and section 2 of [5]. With our actual language they can be formulated in the following way:

1.4. THEOREM. If $R$ is a $\mathbb{Q}$-algebra, each symmetric, bi-multiplicative (resp. bi-additive) co-cycle $H$ of $S$ is a co-boundary of $S$.

1.5. THEOREM. If $R$ is an algebraically closed field of characteristic $p \neq 0$, and if $S$ is a multiplicative bi-algebra, then each bi-multiplicative co-cycle $H$ of $S$ is a co-boundary of $S$.

If $R$ is algebraically closed field of characteristic $0$, and if $A$ is an abelian variety over $R$, result 1.4, under the assumption that $H$ be a rational function on $A^3$, was first proved in [2].
Since the symmetric, bi-additive co-cycles are more easy to use, we start with the following result:

1.6. PROPOSITION. Let $S$ be as in 1.3; then each symmetric bi-additive co-cycle of $S_0 = S \hat{\otimes} k$ is a co-boundary of $S_0$.

In fact, as the following arguments will show, from 1.6 we deduce the following

1.7. PROPOSITION. Let $S$ be as in 1.3; then each symmetric, bi-additive co-cycle of $S$ is a co-boundary of $S$.

An finally, from 1.7 we can get 1.3.

Proof of (1.6 $\Rightarrow$ 1.7). Let $H \in S \hat{\otimes} S \hat{\otimes} S$ be a symmetric, bi-additive co-cycle of $S$. Denote by $H_0$ the image of $H$ in $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$; now as $H_0$ is a co-boundary of $S_0$, there exists an element $h_0 \in S_0$, such that $\delta_a^2 h_0 = H_0$. If $h$ is an element of $S$ whose image in $S_0$ is $h_0$, and if $H_1 = \delta_a^2 h$, we have $H \equiv H_1 \mod p$; and then, since

$$\frac{1}{p} (H-H_1)$$

is a symmetric, bi-additive co-cycle of $S$, our procedure may be repeated. As a consequence,

$$H = H_1 + pH_2 + p^2 H_3 + \ldots ,$$

is a co-boundary of $S$, Q.E.D.

Proof of (1.7 $\Rightarrow$ 1.3). Let $F \in S \hat{\otimes} S \hat{\otimes} S$ be a symmetric, bi-multiplicative co-cycle of $S$, and denote by $F_0$ its canonical image in $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$. By 1.4 we know that $F_0$ is a co-boundary of $S_0$, so there exists $\theta_0$ in $S_0$, s.t. $\delta^2_{\mu} \theta_0 = F_0$. Now, denote by $S^+$ the kernel of the coidentity of $S$, and let $\theta'$ be an element of $S$, $\theta' \equiv 1 \mod S^+$, whose image in $S_0$ is $\theta_0$. If we denote by $F_1$ the co-boundary $\delta^2_{\mu} \theta'$ of $S$, we have

$$F/F_1 \equiv \mod p ,$$

and therefore

$$\log F = \log F_1 + pH ,$$

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where $H$ is a symmetric, bi-additive co-cycle of $S$. Now, by 1.7 there exists an element $h \in S$, s.t. $\mathcal{D}_h^2 = H$; and it is clear that $\theta = \theta' \exp ph$ is an element of $S$ which satisfies the equation $\mathcal{D}_h^2 \theta = F$.

Q.E.D.

Now will give a lemma which will be used the proof of 1.6.

1.8. LEMMA. Let $B$ be an integral domain of characteristic $p \neq 0$, $B[t_1, \ldots, t_n]$ a multiplicative bi-algebra; then each symmetric additive co-cycle of $B[t]$ is a co-boundary.

Proof. This result is probably well known; nevertheless we'll give here a direct proof. Let $g$ be such a co-cycle. Using the co-cycle property $g(t_1 + t_2, t_3) + g(t_1, t_2) = g(t_1, t_2 + t_3) + g(t_2, t_3)$, it is immediate to see that $(pt \circ p \cdot) g$ is a co-boundary ($p t =$ multiplication by $p$), i.e. there exists an element $\tau \in B[t]$ such that

$$\tau(t_1 + t_2) - \tau(t_1) - \tau(t_2) = g(pt_1, pt_2).$$

From the last formula we deduce that

$$1.9. \quad DT - \varepsilon(DT) = 0$$

for each invariant derivation $D$ of $B[t]$, where $\varepsilon$ is the co-identity. But, as $B[t]$ is multiplicative, 1.9 implies that $DT = 0$, and so $\tau = pt \circ \sigma$, for $\sigma \in B[t]$. In conclusion $\sigma(t_1 + t_2) - \sigma(t_1) - \sigma(t_2) = g(t_1, t_2)$,

Q.E.D.

Proof of 1.6. Let $H \in S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$ be a symmetric bi-additive co-cycle of $S_0$; then by 1.8 there exists a (unique) element $\varphi$ in $S_0 \hat{\otimes} S_0$, such that

$$1.10. \quad \varphi(t_1, t_2 + t_3) - \varphi(t_1, t_2) - \varphi(t_1, t_3) = H.$$

Now, if $\mu \in S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$ is the element defined by

$$\mu(t_1, t_2, t_3) = \varphi(t_1, t_2 + t_3) + \varphi(t_2, t_3) - \varphi(t_1 + t_2, t_3) - \varphi(t_1, t_2),$$

as a consequence of the co-cycle properties of $H$ (see def. 1.1) we have
\[ \mu(t_1, t_2, t_3 + t_4) = \mu(t_1, t_2, t_3) + \mu(t_1, t_2, t_4). \]

But, since \( S_0 \hat{\otimes} k[t_1, t_2] \) is a multiplicative \( k[t_1, t_2] \)-bi-algebra, from the last formula we deduce that \( \mu = 0 \). As a consequence, recalling also point ii) of 1.1, we conclude that \( \varphi \) is a symmetric, additive co-cycle of \( S_0 \); so using 1.8 again we have

\[ \varphi(t_1, t_2) = \lambda(t_1 + t_2) - \lambda(t_1) - \lambda(t_2), \]

and finally \( \mathcal{D}^2_{\varphi} = H \), Q.E.D. .

1.11. Remark. Let \( S \) as in 1.3, and \( u = (u_1, \ldots, u_n) \) be a basis of the integrals of the first kind of \( S \), i.e. a basis of the \( R \)-module of the additive elements \( u \) of \( S_K = S \hat{\otimes} K \) such that \( Du \) is in \( S \) for each invariant derivation \( D \) of \( S \). If \( F \) is a symmetric, bi-multiplicative co-cycle of \( S \), and if \( \vartheta \in S \) is a solution of the equation

1.12.

\[ \mathcal{D}^2_{\vartheta} = F, \]

each solution of 1.12 in \( S_K \) is of the form \( \vartheta \exp(L(u) + Q(u)) \), where \( L(u) \) and \( Q(u) \) are linear and respectively quadratic forms of the \( u_1 \)'s. Now, since in \( S \) there is no element of the form \( \exp Q(u) \) (see [MA]), we conclude that all solutions of 1.12 in \( S \) are of the form \( \vartheta \exp L(u) \).

Now we'll show that, if 1.12 admits a even solution, i.e. invariant with respect to the inversion of \( S_K \), it is sufficient to assume \( k \) perfect; more precisely we have the following.

1.13. THEOREM. Let \( k \) be a perfect field of characteristic \( p \neq 0,2 \), let \( R \) be the ring of the Witt vectors with components in \( k \), and \( S = R[t_1, \ldots, t_n] \) be a bi-algebra of multiplicative type. Then if \( F \) is a symmetric, bi-multiplicative co-cycle of \( S \), such that \( F(t_1, t_2, t_3) = F(\zeta t_1, \zeta t_2, \zeta t_3) \) (\( \zeta \) is the image of \( t \) given by the inversion of \( S \)), the equation 1.12 has a unique solution \( \hat{\vartheta} \in S \) which satisfies the relation

1.14. \[ \hat{\vartheta}(t) = \hat{\vartheta}(-t). \]

Moreover if \( \vartheta \in S_K (K = \text{Frac} R) \) is a solution of 1.12 which satisfies 1.14 we have...
1.15. \[ \hat{\vartheta}_n = \lim_{n \to \infty} \vartheta / (p^n)^{2n} \]

Finally, the direct relation between \( \hat{\vartheta}_n \) and \( F \) is the following:

1.16. \[ \hat{\vartheta}_n = \lim_{n \to \infty} (p^n)^{-n} \left( \prod_{j=1}^{n-1} F(t, z^j t, j^j t)^{p^n-j} \right) \]

where the limits 1.15 and 1.16 are considered in the topology of \( \lim(S \rightarrow S_{pightarrow} \ldots) \) given by the system \( I_{m,n} = t^m S + p^n S \) of ideals of \( S \).

Proof. Let \( \overline{R} \) be the ring \( W(\overline{K}) \) of the Witt vectors with components in the algebraic closure \( \overline{K} \) of \( k \). By 1.3 we known that there exists a solution \( \vartheta(t) \) of 1.12 in \( \overline{R}[t] \); but in view of the properties of \( F \), also \( \vartheta(-t) \) is a solution of 1.12, and so \( \vartheta(t)^{1/2} \vartheta(-t)^{1/2} \) is a solution of 1.12 which satisfies 1.14. Now, each element of \( \overline{R}[t] \) which satisfies 1.14 is a even power series of \( u \) (see 1.11); therefore it can’t be multiplied by an exponential of a linear form \( L(u) \) without losing the property 1.14. As a consequence 1.12 has a unique solution \( \hat{\vartheta}_n \) in \( \overline{R}[t] \) such that \( \hat{\vartheta}_n(t) = \hat{\vartheta}_n(-t) \). Now we'll show that \( \hat{\vartheta}_n \) is in \( R[t] \). In fact by 1.1, if \( \vartheta \) is a solution of 1.12 in \( S_K \) which satisfies 1.14, we have

1.17. \[ \vartheta(t)^{p^{2n}} / (p^n)^{n \vartheta(t)} = \prod_{j=1}^{p^n-1} F(t, z^j t, j^j t)^{p^n-j} \]

for each \( n > 1 \). So the remaining part of the theorem will be proved if we verify that

1.18. \[ \hat{\vartheta}_n(t) = \lim_{n \to \infty} \vartheta(t) / (p^n)^{n \vartheta(t)}^{p^{2n}} \]

Now the relation between \( \vartheta \) and \( \hat{\vartheta}_n \) must be \( \vartheta = \hat{\vartheta}_n \exp Q(u) \), where \( Q(u) \) is a quadratic form.

But \( \lim_{n \to \infty} (p^n)^{-n \vartheta}^{2n} = 1 \), and \( \lim_{n \to \infty} (p^n)^{-n}(\exp Q(u))^{p^{2n}} = \exp W(u) \), Q.E.D.

1.19. Remark. With the notation of 1.17 also the limit

1.20. \[ \lim_{n \to \infty} \prod_{j=1}^{p^n-1} F(t, z^j t, j^j t)(p^n-1)/p^{2n} \]
exists in $S_K$: it gives the (unique) solution $\theta_0$ of 1.12 in $S_K$ which satisfies 1.14 and the initial condition
\[
\varepsilon(D'D \log \theta) = 0,
\]
for each couple $(D, D')$ of invariant derivations of $S_K$. In fact, in order to show that 1.20 exists, we remark that by 1.14, $\theta = 1 + Q(u) + \ldots$, where $Q$ is a quadratic form; as a consequence
\[
\lim_{n \to \infty} \frac{1}{2^n} \log(p^N)^n = Q(u),
\]
and finally
\[
\theta_0 = \theta / \exp Q(u).
\]
This is the procedure used in [10]; but in general $\theta_0$ isn't in $S$.

1.15. Remark. Let $\mathcal{S}$ be the completion of the perfect closure of $S_Q = S \otimes k$ and $Biv(\mathcal{S})$ the completion of the ring of Witt bivectors with components in $\mathcal{S}$. Using the methods described in [12] (see in particular th. 8.1) one can define a canonical embedding $j$ of a subring of $S_K$, containing all solutions of 1.12, in $Biv(\mathcal{S})$. In such situation $\mathcal{S}$ is characterized by the property $j \in W(\mathcal{S})$. Since 1.12 has solutions with this peculiarity also when $S_Q$ is an affine algebra of a general B-T group, it would be interesting to describe the functions (series) which correspond to them.

2. THETA SERIES.

In this section we'll translate the previous results in a geometric language.

2.1. THEOREM. If $k$, $K$ and $R$ have the same meaning as in 1.13, if $A$ is an abelian variety over $K$ with good reduction $\bmod p$, and if the reduced variety $A_0$ is ordinary; then each divisor $X$ of $A$, rational over $K$, has a theta series in $R((t))$, where $R$ is the ring of the Witt vectors of the algebraic closure $\overline{k}$ of $k$, and $t = (t_1, \ldots, t_n)$ is a set of uniformizing parameters of $A$ at the identity point $e_0$ of $A_0$. Moreover if $X$ is totally symmetric, i.e. if there exists $X'$, s.t.
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X = X' + (-t)^{-1}X', X possesses a theta series in \( R((t)) \), \( \hat{\theta}_X \) which satisfies the relation \( \hat{\theta}_X(t) = \hat{\theta}_X(-t) \). The series \( \hat{\theta}_X \) is determined up to a constant.

Proof. We'll assume that the support of the reduced divisor \( X_O \) doesn't intersect \( e^O \); in fact the result in general is an immediate consequence of this particular situation (see remark 2.2). If \( Y \) has the same meaning as in the introduction, as remarked in section 1, we can choose as an equation of \( Y \) a symmetric, bi-multiplicative cocycle \( F \) of \( S \). At this point it is clear that the first part of the theorem is a consequence of 1.3. Now if \( X \) is totally symmetric, \( F(\bar{t}_1, \bar{t}_2, \bar{t}_3) \) is, as \( F \), an equation of \( Y \), which satisfies i) of 0.1, and so \( F(\bar{t}_1, \bar{t}_2, \bar{t}_3) = F(t_1, t_2, t_3) \). As a consequence, the second part our theorem follows immediately by the first part of 1.13, Q.E.D.

2.2. Remark. The assumption \( \text{supp } X_O \cap e^O = \emptyset \) used in the proof of 2.1 is not necessary. In fact each divisor \( X' \) of \( A \) rational over \( K \) can be written as \( X' = X'' + (f) \), where \( X'' \) satisfies the assumption and \( f \) is in \( R((t)) \). In this case we define \( \hat{\theta}_X = \hat{\theta}_{X''} f \). In particular, if \( X = X' + (-t)^{-1}X \), \( \hat{\theta}_X(t) = \hat{\theta}_{X''} (t) f(t) f(-t) \) is determined up to a multiplicative constant. Finally, if the polar part \( X'' \) of \( X \) satisfies the previous assumption, i.e. \( \text{supp } X'' \cap e^O = \emptyset \), \( R((t)) \) and \( R((t)) \) can be replaced by \( \overline{R[t]} \) and \( \overline{R[t]} \) respectively.

3. THE CANONICAL SPLITTING OF \( H^1_{DR}(A) \) ASSOCIATED TO \( \hat{\theta} \).

With the notations and assumptions of 2., we recall that to \( A \) and \( S \) are associated the free \( R \)-modules \( H^1_{DR}(A) \) and \( H^1_{DR}(S) \) of rank \( 2n \) and \( n \) respectively. For our purposes, the more convenient description of them is the following (see [3] and [4]):

we start with two sub-\( R \)-modules of \( S_X = K[t] \) : the first is

\( I_2(A) = \{ f \in S_X \mid df \text{ is a diff. of } S \, \text{and } \, f(t_1 t_2) - f(t_1) - f(t_2) \in K(A^2) \} \);

the second is
\[ I_2(S) = \{ f \in S_K \mid df \text{ is a diff. of } S, \text{ and } f(t_1 + t_2) - f(t_1) - f(t_2) \in S \otimes S \}. \]

Clearly \( I_2(A) \) contains the local ring \( \mathcal{O}_0 \) of \( A \) at \( e_0 \), and \( I_2(S) \) contains \( S \). With these notations, we have:

\[ H^1_{\text{DR}}(A) = I_2(A)/\mathcal{O}_0 \quad \text{and} \quad H^1_{\text{DR}}(S) = I_2(S)/S. \]

Now let \( I_1 \) be the sub-R-module of \( I_2(A) \) (and of \( I_2(S) \)) given by the additive elements:

\[ I_1 = \{ f \in I_2(A) \mid f(t_1 + t_2) - f(t_1) - f(t_2) = 0 \}. \]

It is well known that \( I_1 \) is a free \( R \)-module of rank \( n \), and that \( I_1 \cap S = \{0\} \). Therefore, by a comparison of the ranks, we conclude that the canonical map of \( I_2(A) \) in \( H^1_{\text{DR}}(S) \) is surjective, that \( I_2(A) = I_1 \ast (I_2(A) \cap S) \), and finally that

\[ 3.1. \quad H^1_{\text{DR}}(A) = I_1 \ast (I_2(A) \cap S)/\mathcal{O}_0 : \]

this is the canonical splitting of \( H^1_{\text{DR}}(A) \). Now we'll show how the sub-R-module \( N = (I_2(A) \cap S)/\mathcal{O}_0 \) of \( H^1_{\text{DR}}(A) \) is related to the theta series.

3.2. THEOREM. Let \( A \) be an abelian variety as in 2.1; let \( X > 0 \) be a totally symmetric, ample divisor of \( A \) rational over \( K \), and \( \hat{\delta} \) (one of) its theta series in \( S \). If \( \text{Lie}(S) \) denotes the \( R \)-module (dual of \( I_1 \)) of the invariant derivations of \( S \), then the image of the map \( \lambda : D \longrightarrow D\log\hat{\delta} \) of \( \text{Lie}(S) \) in \( S \) is contained in \( I_2(A) \). Moreover, if \( N_\delta \) denotes the image of \( \lambda(\text{Lie}(S)) \) in \( H^1_{\text{DR}}(A) \), we have

\[ N = \{ f \mid f \in H^1_{\text{DR}}(A), p^n f \in N_\delta, \text{ for some } n \in \mathbb{N} \}. \]

**Proof.** As in the proof of 2.1 we'll assume, for simplicity, that \( \text{Supp } X_0 \cap e_0 = \emptyset \). So we can assume \( \hat{\delta} \equiv 1 \mod S^+ \), and therefore

\[ 3.3. \quad F = \bigotimes_{\mu}^{2 \hat{\delta}} \]

is a symmetric, bi-multiplicative co-cycle of \( S \) which is in \( K(A^3) \). Now, if we transform both terms of 3.3 by the operator \( (\bigotimes (\otimes \delta \in D)) \log, \]

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and successively by \((t \in D')\), where \(D, D' \in \text{Lie}(S)\), we have

\[3.4. \quad (L \in D') \log F(t_1, t_2, t_3) = (D \log \theta)(t_1, t_2) - (D \log \eta')(t_1) - (D \log \eta')(t_2) + \varepsilon(D \log \theta), \quad \text{and} \]

\[3.5. \quad (t \in D) \log F(t_1, t_2, t_3) = (D' \log \theta)(t_1) - \varepsilon(DD' \log \eta), \]

which say precisely that \(D \log \theta\) is in \(I_2(A)\). Since \(\theta \in S\), \(\lambda(\text{Lie}(S))\) is contained in \(S\), and so \(N_{\theta}/N\) is contained in \(N\). Finally, since \(X\) is ample \(\lambda(\text{Lie}(S))\) is a free \(R\)-module which doesn't intersect \(\theta\) (see [1] and [6]), so by comparing the ranks we conclude that \(N_{\theta}/N\) is isogenous to \(N\), Q.E.D.

\[3.6. \quad \text{Remark. If } A \text{ is the determinant of the map } \text{Lie}(S) \to N, \frac{1}{||1/A||_p} \text{ is the separable degree of the polarization associated to } X_{\theta} \text{ (cfr. [MA]). So, in particular, if } A \text{ is principally polarized, one can choose } X \text{ in such a way that } N = N_{\theta}. \]

\[3.7. \quad \text{Remark. Let } G_{\text{et}} \text{ be the local an the } \text{étale} \text{ component of the Barsotti-Tate group } G \text{ of the reduced abelian variety } A_{\text{et}}. \text{ By results on the crystalline cohomology (see [3] and [9\alpha]), } H^1_{\text{DR}}(A) \text{ is canonically isomorphic to the Dieudonné module } D(G) \text{ and } H^1_{\text{DR}}(S) \text{ is canonically isomorphic to the Dieudonné module } D(G_{\text{et}}). \text{ Moreover the canonical map from } H^1_{\text{DR}}(A) \text{ onto } H^1_{\text{DR}}(S) \text{ corresponds to the projection } D(G) = D(G_{\text{et}}) \to D(G_{\text{et}}); \text{ therefore } N_{\theta} \text{ as Dieudonné module, is isogenous to } D(G_{\text{et}}). \text{ As a consequence, if } V \text{ denotes the Verschiebung of } H^1_{\text{DR}}(A) \text{ and } \overline{\text{Dlog}}\theta \text{ the image of } \text{Dlog}\theta \text{ in } H^1_{\text{DR}}(A), \text{ we have } \lim_{i \to \infty} V^i(\overline{\text{Dlog}}\theta) = 0, \text{ for each } D \in \text{Lie}(S). \]

\[4. \quad \text{AN EXAMPLE.} \]

Let \(\mathbb{F}_p\) be the Galois field with \(p\) elements, \(p \neq 2\), and let \(\lambda_0\) be an indeterminate over \(\mathbb{F}_p\). We shall denote by \(k\) the perfect field \(\mathbb{F}_p(\lambda_0, \lambda_0^{1/p}, \lambda_0^{1/p^2}, \ldots)\), by \(R = W(k)\) the ring of the Witt vectors
with components in \( k \), and by \( \lambda \) an element of \( R \) whose image in \( k \) is \( \lambda_0 \). Now we consider the cubic \( E_\lambda \) over \( K = \text{Frac}R \), whose affine equation is

\[ i) \quad y^2 = (1 - x^2)(1 - \lambda x). \]

If we choose as identity the point \( e \) of coordinates \( x=0, y=1 \), \((E_\lambda,e)\) is an abelian variety which satisfies the request of 3.2. Moreover \( 2e \) is a totally symmetric divisor which gives a principal polarization, and so the image of \( \text{Dlog}^\alpha 2e \) in \( H^1_{DR}(E_\lambda) \) spans \( N \) (see th. 3.2). This, in view of 3.7, is equivalent to saying that the image \( \text{Dlog}^\alpha 2e \) is an eigenvector of the Frobenius of \( H^1_{DR}(E_\lambda) \) corresponding to a unit eigenvalue. So, as remarked also by Norman (see [11]), \( \text{Dlog}^\alpha 2e \) spans the Dwork's sub-crystal of \( H^1_{DR}(E_\lambda) \) (see [8] and [9]).

The aim of this example is to give an explicit computation for \( \text{Dlog}^\alpha 2e \).

Since \( x \) is a uniformizing parameter of \( E_\lambda \) at \( e_0 \), a basis of \( H^1_{DR}(E_\lambda) \) is given by the canonical images of two series \( u \) and \( v \) of the following type:

\[ u = \sum_{i=1}^{\infty} (c_i/i)x^i \quad \text{and} \quad v = \sum_{i=1}^{\infty} (b_i/i)x^i, \]

where \( c_i \) and \( b_i \) are in \( R \).

In particular we can choose \( u \) and \( v \) in such a way that \( du = dy/y \) and \( dv = xdx/y \); with this choice \( b_i = c_{i-1} \), if \( i > 1 \), and \( b_1 = 0 \).

Now let \( \theta(x) \in R[x] \) be a theta series of \( 2e \) (see th. 3.2) and let \( D \) be the derivation of \( S \) defined by \( Du = 1 \). By 3.2

\[ ii) \quad D\theta / \theta = v + au, \quad \text{mod } R((x)), \]

where \( a \) is in \( R \). Since \( D\theta / \theta - 2Dx \in R[x] \), we deduce that

\[ iii) \quad v + au \equiv 0, \quad \text{mod } R[x]. \]

The relation \( iii) \), as shown in [4], allows to compute \( a \):

\[ a = - \lim_{i \to \infty} \frac{c_i}{c_i} \quad \text{for } i \equiv p \pmod{p^2-1}. \]
To finish, let us show how the image of \( v + au \) may be recovered from each theta, \( \theta \), of \( 2e \) which satisfies the property \( \theta(x) = \theta(-x) \). As we have shown in 1.13, there exists a constant \( b \in K \), such that

\[
\text{iv)} \quad D\theta/\theta + bu \equiv D\theta/\theta \mod R((x)),
\]

and so

\[
\text{v)} \quad D\theta/\theta + bu \equiv 0 \mod R((x)).
\]

The relation \( \text{v)} \) determines \( b \). In fact if \( z = \exp u - 1 \), and if

\[
\log(\theta/x^2) = \sum_{i=1}^{\infty} a_i z^i, \quad \text{v)} \text{ is equivalent to }
\]

\[
\text{vi)} \quad (1+z) \sum_{i=1}^{\infty} i a_i z^{i-1} + b \sum_{i=1}^{\infty} (-1)^{i-1} z^{i}/i \equiv 0 \mod R[[x]],
\]

and therefore

\[
b = -\lim_{i \to \infty} p^i((p^{i}+1)a_{i} + p^i a_{i+1}).
\]

\[\text{BIBLIOGRAPHY}\]


M. CANDILERA e V. CRISTANTE, Biextensions associated to divisors on abelian varieties and theta functions, to appear.


W. MESSING, The crystals associated to Barsotti-Tate groups: with applications to abelian schemes, LNM 264, Springer (1970).


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