H. W. Alt
E. Di Benedetto

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FLOW OF OIL AND WATER THROUGH POROUS MEDIA

by H.W. ALT (Universität Bonn) and E.DI BENEDETTO (Indiana University)

We shall prove existence and regularity for the flow of two immiscible fluids through a porous medium. It is described by the following system of degenerate elliptic parabolic equations (see [2], [3]).

\[ \partial_t s_i - \nabla \cdot (k_i (\nabla p_i + e_i)) = 0 \text{ in } \Omega_T := \Omega \times [0,T] \]

for \( i = 1,2 \), with side condition \( s_1 + s_2 = 1 \).

The porous body \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with Lipschitz boundary. \( s_i \) is the fluid content of the \( i \)-th fluid depending on \( p_1 - p_2 \), \( k_i \) its conductivity depending on \( s_i \). The hydrostatic pressure is denoted by \( p_i \) and \( e_i \) is the gravity. \( s_i \) and \( k_i \) are continuous functions as in the Figure, \( s_i \) strictly monotone in \( [p_{\text{min}}, p_{\text{max}}] \), where \( -\infty < p_{\text{min}} < 0 < p_{\text{max}} < \infty \), and \( k_i \) positive in \( ]0,1[ \). Therefore we have the additional side condition

\[ p_{\text{min}} \leq p_1 - p_2 \leq p_{\text{max}} \]
As initial condition we pose

\[ s_1(p_1 - p_2)(x,0) = s_1^0(x) \quad \text{for} \quad x \in \Omega , \]

where \( s_1 \) are nonnegative measurable functions with \( s_1^0 + s_2^0 = 1 \). We assume that \( \psi(s_1^0) \in L^1(\Omega) \) where \( \psi \) is defined below. The boundary conditions are induced by a partition of \( \partial \Omega \) into three measurable sets \( \Gamma_1, \Gamma_2 \) and \( \Gamma_0 \). We consider Neumann data

\[ k_1(p_1 + e_1) \cdot v = 0 \quad \text{on} \quad \Gamma_0 \times 0, T \]

and mixed Dirichlet and overflow conditions

\[ p_1 = p_1^D \]

\[ k_2(p_2 + e_2) \cdot v = 0 \quad \text{if} \quad p_1 - p_2 > p_{\min} \]

\[ k_2(p_2 + e_2) \cdot v = 0 \quad \text{if} \quad p_1 - p_2 = p_{\min} \]

on \( \Gamma_1 \times 0, T \) and similar conditions on \( \Gamma_2 \times 0, T \).

Here

\[ p_1^D \in L^\infty(\Omega_T) \cap L^2(0,T;H^{1,2}(\Omega)) \]

with

\[ p_{\min} \leq p_1^D - p_2^D \leq p_{\max} \]

and

\[ \partial_\Gamma p_1^D \in L^1(0,T;L^2(\Omega)) \cap L^r(\Omega_T) \quad \text{for some} \quad r > 1 . \]

Common Dirichlet conditions for \( p_1 \) and \( p_2 \) are easier to handle.

Multiplying (1) by \( p_1 - p_1^D \) we see that

\[ \sum_{i=1,2} \int_0^T \int_\Omega k_i(s_1(p_1 - p_2))(\nabla p_1)^2 \]

determines the natural topology of the problem. Therefore since \( k_i \) degenerates we cannot work in function spaces for \( p_1 \). But if we define
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\[ u_1 := \Phi_1(p_1, p_2) := p_2 + \int_0^{p_1-p_2} \frac{k_1(s_1(\min(\xi, 0)))}{k_1(s_1(\xi))} d\xi , \]

\[ u_2 := \Phi_2(p_1, p_2) := p_1 - \int_0^{p_1-p_2} \frac{k_2(s_2(\max(\xi, 0)))}{k_2(s_2(\xi))} d\xi , \]

then (2) is equivalent to the \( L^2 \)-Norm of \((\nu_1, \nu_2)\). Also

\[ \begin{bmatrix} k_1(s_1) \nu_1 \\ k_2(s_2) \nu_2 \end{bmatrix} = K(s_1) \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} , \]

where in the set \( \{p_1 \geq p_2\} \) the matrix \( K \) is given by

\[
K(s_1) = \begin{bmatrix} k_1(s_1) \\ k_2(s_2) - \sqrt{k_2(s_2(0)k_2(s_2))} \\ \sqrt{k_2(s_2(0)k_2(s_2))} \end{bmatrix}
\]

and similarly in \( \{p_1 \leq p_2\} \). Introducing the notation

\[
K := \{(v_1, v_2) \in L^2(0,T; H^{1,2}(\Omega)) \}; \quad v_1 = p^D_1 \text{ and } v_1 - v_2 \geq p_{\min} \text{ on } \Gamma_1 \times ]0, T[ ,
\]

\[
v_2 = p^D_2 \text{ and } v_1 - v_2 \leq p_{\max} \text{ on } \Gamma_2 \times ]0, T[ ,
\]

we can formulate the properties of a weak solution \((p_1^*, p_2^*)\) as follows. \( p_1^*: \bar{\Omega} \to \mathbb{R} \)

with \( p_{\min} \leq p_1 - p_2 \leq p_{\max} \) and the transformation \((u_1^*, u_2^*)\) obtained by (3) (in \( \bar{\Omega} \))

is of class \( L^2(0,T; H^{1,2}(\Omega)) \). Furthermore for \((v_1, v_2) \in K \) with \( \partial_t v_1 \in L^1(\Omega_T) \) the following inequality holds for almost all \( t \), where \( s_1 = s_1(p_1 - p_2) \):

\[
\left( \psi(s_1(t)) - \psi(s_1^0) \right) - \int_0^t \left( s_1(t)(v_1 - v_2)(t) - s_1^0(v_1 - v_2)(0) \right) + \int_0^t \int_\Omega s_1 \partial_t (v_1 - v_2) +
\]

\[ + \Sigma_i \int_0^t \int_\Omega \left( \Sigma_j k_{ij}(s_1) \nu_{ij} + k_{ij}(s_1) \nu_i \right) \leq 0 . \]

Here by convention

\[ k_{ij}(0) = 0 \text{ and } \frac{k_{ij}}{\sqrt{k_{ij}}}(0) = 0 , \]

and the convex function \( \psi \) is defined by
\[ \psi(s_1(z)) := \int_0^z (s_1(z) - s_1(\xi)) \, d\xi. \]

Hence formally \( \partial_t \psi(s_1(p_1 - p_2)) = (p_1 - p_2) \partial_t s_1(p_1 - p_2) \), therefore the variational inequality (4) formally is equivalent to the above stated initial boundary value problem. We prove

1. Existence Theorem. Suppose that \( H^{N-1}(\Gamma_1) > 0 \), \( p_{\min} > -\infty \), and
\[ u_{\max} := -\Phi_2(0, -p_{\max}) < \infty, \text{ or that } H^{N-1}(\Gamma_2) > 0, \text{ } p_{\max} < \infty, \text{ and} \]
\[ u_{\min} := \Phi_1(p_{\min}, 0) > -\infty. \]
Then there exists a weak solution.

Proof. We approximate the conductivity \( k_i \) by positive functions
\[ k_{\varepsilon i} := \max(\varepsilon^2, k_i), \]
and the water content by adding a penalizing term
\[ s_{\varepsilon 1}(z) := s_1(z) + \varepsilon z, \text{ } s_{\varepsilon 2}(z) := s_2(z) - \varepsilon z. \]
Furthermore we approximate the time derivative \( \partial_t \) by backward difference quotients \( \partial_{-h} \). Thus we start with solutions \( (p_{h\varepsilon 1}, p_{h\varepsilon 2}) \in K_h \) of \( (p_{h\varepsilon} := p_{h\varepsilon 1} - p_{h\varepsilon 2}) \)
\[ \Sigma_i \int_\Omega \left( \partial_{-h} s_{\varepsilon i}(p_{h\varepsilon})(p_{h\varepsilon} - v_i) + \nabla(p_{h\varepsilon} - v_i)k_{\varepsilon i}(s_{\varepsilon i}(p_{h\varepsilon}))(\nabla p_{h\varepsilon} + e_i) \right) \leq 0 \]
for all times and for every \( (v_1, v_2) \in K_h \). Here \( K_h \) is defined as \( K \) with \( p_i^D \) replaced by
\[ p_{h\varepsilon i}^D(t) := \int_{(j-1)h}^{jh} p_i^D(\tau) \, d\tau \text{ for } (j-1)h < t < jh. \]
The initial condition is
\[ s_{\varepsilon i}(p_{h\varepsilon i})(t) = s_{i}^0 \text{ for } -h < t < 0. \]
The solution \( p_{h\varepsilon i}^D \) of these inductively defined elliptic problems exists since
\( H^{N-1}(\Gamma_1 \cup \Gamma_2) > 0 \). Setting \( v_i = p_{h\varepsilon i}^D \) we obtain for the parabolic part since \( s_{\varepsilon i} \) is monotone.
Together with the elliptic part we obtain the a priori estimate

$$
\varepsilon \sup_{0 \leq t \leq T} \int_\Omega (|p_h e^2 + \Sigma_i e^2 \int_\Omega k_{e_i} (s_{e_i} (p_h e) ) |\nabla \phi h e^2 |^2 \leq C .
$$

Therefore if $u_{e_i} h$ are defined as in (3) with respect to $k_{e_i}$ we can conclude

that $\nabla u_{e_i} h$ are bounded in $L^2 (\Omega_T)$. Now in the set $\{ p_{e_i} h \geq p_{e_2} h \}$ by definition

of $k_{e_i} (\text{write } u_{e_i} := u_{e_i} h - u_{e_2} h )$

$$
0 \leq u_{e_i} h \leq u_{\text{max}} + C \varepsilon |p_h e^2 | .
$$

Thus if $u_{\text{max}} \to \infty$ by the a priori estimate

$$
\max(u_{e_i} h - u_{\text{min}}, 0) \to 0 \text{ in } L^\infty (0,T;L^2 (\Omega) ) .
$$

Similarly if $u_{\text{min}} \to -\infty$

$$
\min(u_{e_i} h - u_{\text{max}}, 0) \to 0 \text{ in } L^\infty (0,T;L^2 (\Omega) ) .
$$

Together with the boundary condition and the assumptions made this implies that

$u_{e_i} h$ are bounded in $L^2 (0,T;H^1(\Omega))$. Hence for a subsequence $h \to o , \varepsilon \to o$

$$
u_{e_i} h \to v_i \text{ weakly in } L^2 (0,T;H^1(\Omega))
$$

and

$$
u_{\text{min}} \leq v_i \leq u_{\text{min}} .
$$

Consequently $p_1$ and $p_2$ are well defined by (3).

The next step is to prove compactness of $s_{e_i} (p_h e)$. We multiply the equation

in the time interval $J_{(j-m), jh}$ by the time independent function

$$
v = p_{e_i} + \eta^2 (u_{e_i} h (t) - u_{e_i} h (t-mh) ) .
$$
where \( \eta \in C^\infty_c(\Omega) \), \( j \geq m \), and \((j-1)h < t < jh\). Using the a priori estimate we obtain
\[
\int_{\Omega}^T \int_{ mh \Omega} \eta^2 \left( s_{\varepsilon_1}(p_{h\varepsilon}(t)) - s_{\varepsilon_1}(p_{h\varepsilon}(t-mh)) (u_{h\varepsilon}(t) - u_{h\varepsilon}(t-mh)) \right) \leq C mh .
\]

Since \( p_{h\varepsilon} \) is a monotone function of \( u_{h\varepsilon} \) and since \( \varepsilon |p_{h\varepsilon}| \to 0 \) in \( L^1(\Omega_T) \) it follows as in [11] that \( s_{\varepsilon_1}(p_{h\varepsilon}) \) is relative compact in \( L^1(\Omega_T) \), hence for a subsequence convergent to \( s_{1}(p_1 - p_2) \) in \( L^1(\Omega_T) \) and almost everywhere.

Then also \( u_{h\varepsilon_1} - u_{h\varepsilon_2} \to u_1 - u_2 \) almost everywhere in \( \Omega_T \). Moreover the boundary condition on \( \Gamma_i \), \( i = 1, 2 \), is of the form
\[
u_1 + \nu_2 = \gamma_\varepsilon(u_{h\varepsilon_1} - u_{h\varepsilon_2}) ,
\]
where \( \gamma_\varepsilon \) are continuous functions converging uniformly to some \( \gamma \). This implies that
\[
u_1 + \nu_2 = \gamma(u_1 - u_2) ,
\]
that is, \( (u_1, u_2) \) is of class \( \mathcal{K} \).

Finally we have to show that \( (u_1, u_2) \) satisfies the variational inequality. For this write (5) (omitting unessential positive terms on the left) in the form
\[
\frac{1}{h} \int_{t-h}^t \int_{\Omega} \left( \psi(s_1(p_{h\varepsilon})) - \psi(s_0) \right) + \sum_i \int_{t-h}^t \int_{\Omega} k_{\varepsilon_1}(s_1(p_{h\varepsilon})) |\nabla P_{h\varepsilon_1}|^2 + k_{\varepsilon_1}(s_1(p_{h\varepsilon})) \nabla P_{h\varepsilon_1} \cdot e_i
\]
\[
- \frac{1}{h} \int_{t-h}^t \int_{\Omega} s_{\varepsilon_1}(p_{h\varepsilon}) \nu_1 - \frac{1}{h} \int_{t-h}^t \int_{\Omega} s_{\varepsilon_1}(p_{h\varepsilon}) \nabla \nu_1 + \frac{1}{h} \int_{t-h}^t \int_{\Omega} s_{\varepsilon_1}(p_{h\varepsilon}) \nabla \nu_1 + \frac{1}{h} \int_{t-h}^t \int_{\Omega} s_{\varepsilon_1}(p_{h\varepsilon}) \nabla \nu_1
\]
\[
+ \sum_i \int_{t-h}^t \int_{\Omega} \left( k_{\varepsilon_1}(s_1(p_{h\varepsilon})) \nabla \nu_1 \cdot e_i + k_{\varepsilon_1}(s_1(p_{h\varepsilon})) \nabla \nu_1 \cdot e_i \right)
\]

Here \( (v_{h1}, v_{h2}) \in \mathcal{K}_h \) is a suitable approximation of a given function \( (v_1, v_2) \) with the properties as in (4). Since \( s_1(p_{h\varepsilon}) \) converges almost everywhere the first integral on the left and all terms on the right except the last one converge to the desired limit. Since
\[
k_{\varepsilon_1}(s_1(p_{h\varepsilon})) \nabla P_{h\varepsilon_1} = \sum_j k_{ij}(s_1(p_{h\varepsilon})) \nabla u_{h\varepsilon_1}
\]
also the last term on both sides converge. The second term on the left is \((\varepsilon \leq \varepsilon_o)\)

\[
\geq \sum_{j} \int_0^t \int_{\Omega} \frac{1}{k_{\varepsilon_o}^i(s_i(p_{\varepsilon_o}))} \left| \sum_j k_{ij}^i(s_i(p_{\varepsilon_o})) \nabla u_i \right|^2,
\]

which in the limit \(\varepsilon \to 0\), \(h \to 0\) is

\[
\geq \sum_{j} \int_0^t \int_{\Omega} \frac{1}{k_{\varepsilon_o}^i(s_i(p_1-p_2))} \left| \sum_j k_{ij}^i(s_i(p_1-p_2)) \nabla u_i \right|^2.
\]

Then let \(\varepsilon_o \to 0\).

That weak solutions satisfy the differential equation is stated in the next Lemma.

2. Lemma. For any weak solution \(\partial_t s_i(p_1-p_2) \in L^2(0,T; H_0^1,2(\Omega))\) with initial values \(s^0_i\), that is,

\[
\int_0^T \int_{\Omega} \partial_t s_i(p_1-p_2) \zeta + \int_0^T \int_{\Omega} (s_i(p_1-p_2) - s^0_i) \partial_t \zeta = 0
\]

for \(\zeta \in C_0^\infty(\Omega \times [0,T])\). Moreover in the above space

\[
\partial_t s_i(p_1-p_2) - \nabla \cdot \left( \sum_j k_{ij}^i(s_i(p_1-p_2)) \nabla u_j + k_i^i(s_i(p_1-p_2)) u_i \right) = 0.
\]

Proof. Formally this follows by setting \(v_i = p_i + \zeta\) in (4). But since we do not know whether \(p_i\) is regular enough to do so, we have to approximate these functions. Choose \(u_1^o \to u_{\min}\) and \(u_2^o \to u_{\max}\) and define

\[
u_{1,2}^o := \frac{u_1^o + u_2^o}{2} \pm \frac{1}{2} \max \left( u_{\min}^o, \min(u_{\max}^o, u_1^o - u_2^o) \right).
\]

Then the corresponding pressures \(p_i^o\) belong to \(L^2(0,T; H_0^1,2(\Omega))\). Similarly define \(p_i^D\). Then

\[
w_i := p_i^D + (p_i^o - p_i^D)
\]

satisfy \(p_{\min} \leq w_1 - w_2 \leq p_{\max}\) and the Dirichlet condition on \(\Gamma_i\). As test function in (4) we use
where

\[ w_{1,2}^\varepsilon (t) := p_1^D (t) + (p_1^D - p_1^P) (t) + \]

\[ + \max \left( 0, 1 - \frac{(j+1)h - \tau - \varepsilon}{\varepsilon} \right) \left( (p_1^P - p_1^D) ((j+1)h - \tau) - (p_1^P - p_1^D) (jh - \tau) \right) \]

whenever \( jh - \tau \leq t \leq (j+1)h - \tau \), \( j = 0, \ldots, j_h \), \( t_h = j_h \cdot h \), \( t_h - h \leq t_0 \leq t_h \) for given \( t_0 < T \). In this definition \( p_1^D (t) := p_1^D (0) \) and \( p_1^P (t) := p_1^{op} \) for \( t < 0 \), where \( p_1^{op} \in H^{1,2} (\Omega) \) is chosen such that \( p_1^{op} - p_1^{op} \leq \max \) and

\[ \int_\Omega \left( \psi(s_1^o) - \int_0^{p_1^D - p_1^P} (s_1^o - s_1^o (\xi)) d\xi \right) \to 0 \text{ as } \rho \to 0 . \]

Then the \( \varepsilon_1 \) terms in (4) give the assertion provided we can show that for \( \varepsilon_1 = 0 \) the right side in (4) does not exceed the left in the limit \( \varepsilon \to 0 \), \( h \to 0 \), and \( \rho \to 0 \).

Let us consider the parabolic terms. For almost all \( \tau \) almost everywhere in \( \Omega \) we have (writing \( s_1 (t) \) for \( s_1 (x, (p_1 - p_2) (x, t)) \), \( v_{1,2}^\varepsilon \) for \( v_{1,2}^\varepsilon \), etc.)

\[ \int_{jh - \tau}^{(j+1)h - \tau} s_1 \partial_t v_{1,2}^\varepsilon = \int_{jh - \tau}^{(j+1)h - \tau} \chi (\{ p_{\min} < w_{1,2}^\varepsilon < p_{\max} \}) \cdot (s_1 - s_1 ((j+1)h - \tau)) \partial_t w_{1,2}^\varepsilon + \]

\[ + s_1 ((j+1)h - \tau) \left( w_{1,2}^\varepsilon ((j+1)h - \tau) - w_{1,2}^\varepsilon (jh - \tau) \right) \geq \]

\[ \geq - \int_{jh - \tau}^{(j+1)h - \tau} |s_1 - s_1 ((j+1)h - \tau)| \left| \partial_t p_1^D - \frac{1}{\varepsilon} \right| \int_{jh - \tau}^{(j+1)h - \tau - \varepsilon} \left| s_1 - s_1 ((j+1)h - \tau) \right| \cdot \]

\[ \cdot |h \partial_t ^h (p_1^P - p_1^D) (jh - \tau) - s_1 ((j+1)h - \tau)| \left| \partial_t (p_1^P - p_1^D) \right| + \]

\[ + s_1 ((j+1)h - \tau) \left( p_1^P ((j+1)h - \tau) - p_1^P (jh - \tau) \right) . \]

The second term tends to zero as \( \varepsilon \to 0 \), hence summing over \( j \) and integrating
over \( \Omega \) we obtain

\[
(8) \lim_{\varepsilon \to 0} \int_{\Omega} s_1(\tau) v^{TE}(t, \tau) - s_1^0 \cdot v^{TE}(0) \geq R_1 + \sum_{j=0}^{J} \int_{\Omega} s_1((j+1)h - \tau) \left( p^0((j+1)h - \tau) - p^0(jh - \tau) \right).
\]

For the second term on the left of (4) we have

\[
(9) - \int_{\Omega} s_1(t_h - \tau) v^{TE}(t_h - \tau) - s_1^0 \cdot v^{TE}(0) \geq R_2 - \int_{\Omega} s_1(t_h - \tau) p^0(t_h - \tau) - s_1^0 \cdot p^0(0).
\]

Thus the sum of the left sides in (8) and (9) is

\[
\geq R_3 - \sum_{j=0}^{J} \int_{\Omega} s_1((j+1)h - \tau) - s_1(jh - \tau) \cdot p^0(jh - \tau).
\]

Integrating over \( \tau \) from 0 to \( h \) and dividing by \( h \) the last integral converges to the first term in (4). The remainder \( R_3 \) tends to zero with \( h \) and \( p \) after performing the mean over \( \tau \). In the elliptic term we first can go to the limit with \( \varepsilon \). After that it is not hard to complete the proof.

3. Remark. In order to show that the weak solution \( p_1, p_2 \) satisfies the original problem, we have to show that \( s_1(p_1 - p_2) \) are continuous in space and time. This would imply that \( \nabla p_i \) is well defined in the open set \( \{ k_i(s_1(p_1 - p_2)) > 0 \} \).

We need

4. Assumptions. \( s_i \) is continuous differentiable with respect to the \( z \) variable in \( \Omega \times \{ p_{\min} < z < p_{\max} \} \) and

\[
(10) \frac{\partial s_i}{\partial z} > 0,
\]

\[
(11) \frac{k_i(s_1(z))}{\frac{\partial s_i}{\partial z} s_i(z)} \geq c(\sigma) > 0 \text{ for } z \leq p_{\max} \Rightarrow p_{\max}.
\]
Let us consider the transformation (see [8], [13])

\[ v = s_1, \quad \text{and} \quad u = \frac{p_1 - p_2}{\partial_z s_1(z)} k_1(s_1(z)) + \int_{\Omega} \frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi. \]

Then formally \( v \) and \( u \) locally in \( \Omega \) are solutions of the system

\[ 0 = \nabla \cdot (k(v)\nabla u + e(v)) \quad (\text{define} \quad \nu := -(k(v)\nabla u + e(v))) \]

\[ \partial_\tau v = \nabla \cdot \left( a(v)\nabla v + b(v) + d(v)\nu \right), \]

where

\[ k(z) = k_1(z) + k_2(1 - z), \]
\[ e(z) = k_1(z)e_1 + k_2(1 - z)e_2, \]
\[ a(z) = \frac{k_1(z) k_2(z)}{k(z)} \partial_z s_1(z), \]
\[ b(z) = \frac{k_1(z) k_2(z)}{k(z)} (e_1 - e_2), \]
\[ d(z) = \frac{k_2(z)}{k(z)} \quad \text{or} \quad = -\frac{k_1(z)}{k(z)}. \]

The assumptions made imply that these coefficients are bounded and

\[ c \leq k \leq C, \quad |\partial_z| \leq C, \]

\[ \phi_0(\omega) := \inf_{z \leq 1 - \omega/4} a(z) > 0 \quad \text{for every} \quad \omega > 0. \]

Then \( u \) satisfies an elliptic equation and \( v \) a degenerate parabolic equation, coercive near 0.
5. Remark. \( u \) and \( v \) are solutions of (14) and (15) with \( Vv \) replaced by
\[
\lim_{\rho \to 0} V \min (v, 1 - \rho) \quad \text{and} \quad \lim_{\rho \to 0} Vu^0,
\]
where \( \min (v, 1 - \rho) \) and \( u^0 \) are in \( L^2(0,T; H^{1,2}(\Omega)) \). \( u^0 \) is defined as \( u \) with \( p_1 \) replaced by \( p^0_1 \), which is the transformation of \( u^0_1 \) according to (3), and
\[
\begin{align*}
  u^0_1 := \frac{u_1^0 + u_2^0}{2} & \pm \frac{1}{2} \max \left( u^0_{\text{min}}, \min(\max(u^0, u_1^0), u_2^0) \right)
\end{align*}
\]
with \( u^0_{\text{min}} \leftarrow u_{\text{min}} \) and \( u^0_{\text{max}} \leftarrow u_{\text{max}} \).

Next we show

6. Lemma. In addition to the assumptions in theorem 1 suppose that if \( \mathbf{H}^{N-1}(\Gamma_1) > 0 \) then \( p_{\text{min}} > -\infty \) and
\[
\int_0^{p_{\text{max}}} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} \, d\xi \leq C,
\]
(similar if \( \mathbf{H}^{N-1}(\Gamma_2) > 0 \)). Then \( u \) is locally bounded in \( \Omega_T \).

Proof. The assumptions imply that the functions \( u^0 \) defined as above are uniformly bounded on \( \Gamma_1 \cup \Gamma_2 \) by some \( C \). Then
\[
\phi(u^0) := \min(u^0 + C, \max(u^0 - C, 0))
\]
can be used as test function for the equation (14). This gives that
\[
\lim_{\rho \to 0} \|\phi(u^0(t))\|_{H^{1,2}(\Omega)}
\]
is bounded in \( t \). Then multiplying (14) by \( \eta^2 u^0 \) with \( \eta \in C_0^\infty(\Omega) \) we obtain that
\[
\lim_{\rho \to 0} \|u^0(t)\|_{H^{1,2}_{\text{loc}}(\Omega)}
\]
is bounded in \( t \). Therefore \( u^0 \) has a weak limit, which is a bounded function satisfying (14).

Now we are able to prove

7. Regularity theorem. Suppose that the assumptions in 1. and 4. hold and that \( u \) is bounded. Then \( s_1(p_1 - p_2) \) are continuous in \( \Omega_T \), and the modulus of
continuity can be estimated.

Proof. This follows by an iterative procedure from the two propositions below, and they are proved using the De Giorgi techniques, where the special features here are the degeneracy of the coefficient \( a \) in the parabolic equation for \( v \) and the coupling to the elliptic equation for \( u \).

8. Notation. Let \((x^0, t^0) \in \Omega_t\). For \( R > 0 \), \( \alpha > 0 \), and \( 0 < \sigma_1 < \sigma_2 < 1 \) we let

\[
Q^\alpha_R(\sigma_1, \sigma_2) := B((1-\sigma_1)R(x^0) \times]t^0 - (1-\sigma_2)\alpha R^2, t^0[ \]

and \( Q^\alpha_R = Q^\alpha_R(0,0) \), \( Q^\alpha = Q^1_R \). We define

\[
\|w\|^2_{Q_R} = \text{ess sup}_{t^0 - R^2 < t < t^0} \int_{B_R(x^0)} |w|^2 + \int_{Q_R} |\nabla w|^2,
\]

and similar for \( Q^\alpha_R(\sigma_1, \sigma_2) \). In the following \( 0 < R \leq R_0 \) with \( Q_{R_0} \subseteq \Omega_t \) and \( u^+ \), \( u^- \) are any numbers with

\[
\text{ess sup} v \leq u^+ \leq 1, \quad \text{ess inf} v \geq u^- \geq 0,
\]

hence \( \text{ess osc} v \leq u^+ - u^- \leq 1 \). Furthermore \( \omega \) is any positive number satisfying \( u^+ - u^- \leq \omega \leq 2(u^+ - u^-) \).

9. Proposition. There is a small constant such that if

\[
\text{meas}(Q_R \cap \{v > u^+ - \frac{\omega}{2}\}) \leq c \phi_1(\omega) \text{meas}(Q_R),
\]

then

\[
\text{ess osc} v \leq \frac{5}{6} \omega.
\]

Here \( \phi_1(\omega) := (\omega \phi_0(\omega))^{N+2} \).

Proof. Let \( v_\omega := \min(v, u^+ - \frac{\omega}{2}) \) and \( u^+ - \frac{\omega}{2} \leq k \leq u^+ - \frac{\omega}{4} \) and multiply (15) by \( (v_\omega - k)^+ \eta^2 \) in the time interval \( ]t^0 - R^2, t^0[ \) with \( t < t^0 \). Here \( \eta \) is a cut off
function with \( \eta = 1 \) in \( Q_R(\sigma_1, \sigma_2) \), \( \eta = 0 \) on the parabolic boundary of \( Q_R \), and
\[
|\nabla \eta| \leq C(\sigma_1, R)^{-1}, \quad |\nabla \eta| \leq C(\sigma_1, R)^{-2},
\]
\[
0 \leq \partial_t \eta \leq C(\sigma_2, R)^{-1}.
\]

We obtain
\[
\int_{B_R} \eta(t)^2 \Phi(v(t)) + \int_0^t \int_{B_R} a(v) \eta^2 |\nabla (v - k)^+|^2 = \int_0^t \int_{B_R} (\Phi(v) \partial_t \eta^2 - a(v)(v - k)^+ \nabla \eta^2 - (b(v) + d(v) \nabla)((v - k)^+ \eta^2))
\]
where
\[
\Phi(v) = \frac{1}{2} |(v - k)^+|^2 + (\mu^+ - \omega - k)(v - (\mu^+ - \omega))^+.
\]

Since \( a(v) \geq \phi_o(\omega) \) in \( \{(v - k)^+ \neq 0\} \) and
\[
a(v)(v - k)^+ \nabla v = (v - k)^+ \nabla \left( \int a(\xi) d\xi \right) - \int a(\xi) d\xi \nabla(v - k)^+
\]
we derive using the various properties of the coefficients
\[
c \int_{B_R} \eta(t)^2 |(v - k)^+|^2 + \frac{1}{2} \phi_o(\omega) \int_0^t \int_{B_R} \eta^2 |\nabla (v - k)^+|^2 \leq \]
\[
\leq C \left( \frac{\sigma_1^2}{\phi_o(\omega)} + \frac{\sigma_2^2}{(\sigma_2 R)^2} \right) \int \chi(\{v > k\}) - \int_{B_R} v d(v) \nabla((v - k)^+ \eta^2).
\]

Using the fact that \( \nabla \) is divergence free the last term equals
\[
= - \int_0^t \int_{B_R} (v - k)^+ (d(v) - d(\xi)) d\xi \nabla \eta^2 \leq \]
\[
\leq \delta \int_0^t \int_{B_R} |\nabla u|^2 (v - k)^+ \eta^2 + \frac{C}{\delta} \int_0^t \int_{B_R} \chi(\{v > k\}) |\nabla \eta|^2.
\]

Multiplying (14) with \( u((v - k)^+ \eta)^2 \) we see that the integral involving \( |\nabla u|^2 \) is estimated by
Substituting this estimate we obtain
\[ \| (v - k) \|_{Q_R(\sigma_1, \sigma_2)}^2 \leq \frac{C}{\phi_0(\omega)} \left( \frac{\sigma_1 R}{2} + \frac{\sigma_2 R^2}{2} \right) \cdot \text{meas}(Q_R \cap \{ v > k \}). \]

Now we use this over a sequence
\[ R_n := \frac{R}{2} + \frac{R}{2^{n+1}} \quad \text{and} \quad k_n = \mu^+ - \frac{\omega}{2} + \frac{\omega}{2^n} + \frac{\omega}{2^n + 3}. \]

Using an embedding Lemma [10; II(3.9)] we get
\[ \int_{Q_{R_n+1}^{n+1}} \| (v - k_n) \|_{Q_R}^2 \leq \frac{C}{\phi_0(\omega)} \left( \frac{\sigma_1 R}{2} + \frac{\sigma_2 R^2}{2} \right) \cdot \text{meas}(Q_R \cap \{ v > k_n \}) + \frac{2n}{R^{n+2}}. \]

But the left side controls \( (k_{n+1} - k_n)^2 \text{meas}(Q_R \cap \{ v > k_{n+1} \}) \), hence
\[ y_{n+1} \leq \frac{C}{\omega^2 \phi_0(\omega)} \left( 1 + \frac{2}{R^{n+2}} \right) y_n \quad \text{where} \quad y_n := \frac{1}{R^{n+2}} \text{meas}(Q_R \cap \{ v > k_n \}). \]

Since by assumption \( y_0 \) was small enough, \( y_n \to 0 \) as \( n \to \infty \) by [9; 2 Lemma 4.7], which proves the Lemma.

From below we will only assume that
\[ \text{(16)} \quad \text{meas}(Q_R \cap \{ v < \mu^- + \frac{\omega}{4} \}) \leq (1 - c_\phi \phi_1(\omega)) \text{meas}(Q_R). \]

But since \( a \) is coercive near \( 0 \), we can derive a similar statement to

Proposition 9. First we show an uniform estimate in time.

10. Lemma. Let \( k \leq \mu^+ + \frac{\omega}{4} \) and \( p \geq 3 \). Then for \( t_o - R^2 < t_1 < t < t_o \)
\[ \int_{B_{(1-\sigma_1)R}} \psi^2((v(t) - k) - k) \leq \int_{B_{R}} \psi^2((v(t_1) - k) - k) + \frac{C}{\phi_0(\omega)} \left( \frac{1}{2} + \left( \frac{2P_0 R}{\omega} \right)^2 \right) R^N, \]
where
\[ \psi(z) := \max(0, \log \frac{\omega/4}{\omega/4 - z + \omega/2^p}). \]
Proof. Multiply (14) by \(-(\psi^2)'(v-k)^-\)\(\eta^2\) where \(\eta\) is a cut off function in space with \(\eta = 1\) in \(B(1-\sigma_1)\). We obtain

\[
\int_{B_R} \eta^2 \psi^2 ((v(t) - k)^-) + \int_{t_1}^t \int_{B_R} a(v) \eta^2 \psi^2'((v-k)^-) |\nabla (v-k)^-|^2 =
\]

\[
= \int_{B_R} \eta^2 \psi^2 ((v(t_1) - k)^-) + \int_{t_1}^t \int_{B_R} a(v) \psi^2'((v-k)^-)\nabla v\eta^2 +
\]

\[
+ \int_{t_1}^t \int_{B_R} (b(v) + d(v) v) \nabla (\psi^2'((v-k)^-)\eta^2).
\]

Since \(a(v) \geq \phi_0(\omega)\) in \(\{\psi^2^-((v-k)^-) \not\in 0\}\) and

\[(\psi^2)^n = 2(1+\psi)^2\], hence \(\frac{(\psi^2)^{2n}}{(\psi^2)^n} \leq 2\psi\),

we derive that

\[
\int_{B_R} \eta^2 \psi^2 ((v(t) - k)^-) + \frac{c}{\phi_0(\omega)} \int_{t_1}^t \int_{B_R} \eta^2 \psi^2''((v-k)^-) |\nabla (v-k)^-|^2 \leq \int_{B_R} \eta^2 \psi((v(t_1) - k)^-) +
\]

\[
+ \frac{c}{\phi_0(\omega)} \int_{t_1}^t \int_{B_R} ((1+\psi)\psi^2 + \psi|\nabla |^2) + \int_{t_1}^t \int_{B_R} d(v) \nabla (\psi^2''((v-k)^-)\eta^2).
\]

Since \(v\) is divergence free the last term equals

\[
= -\int_{t_1}^t \int_{B_R} v\partial_z d(v) \nabla (v-k)^- \psi^2'((v-k)^-)\eta^2 \leq
\]

\[
\leq \delta \int_{t_1}^t \int_{B_R} \eta^2 \psi^2''((v-k)^-)(|\nabla (v-k)^-|^2 + 1) + \frac{c}{\phi_0(\omega)} \int_{t_1}^t \int_{B_R} \eta^2 (|\nabla |^2 + 1)\psi.
\]

Using

(17) \(\psi((v-k)^-) \leq (\log 2)(p-2)\) and \(\psi'((v-k)^-) \leq \frac{1}{\omega/2^p}\)

the assertion follows, where the integral with \(|\nabla u|^2\) can be estimated by multiplying (14) with \(u\eta^2\).

As a consequence we obtain
11. Lemma. There is a $p = p(u)$ such that if $R \leq 2^{-p} \omega$ and (16) hold then
\[
\text{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p} \omega\}) \leq (1 - \alpha^2) \text{meas}(B_R)
\]
for $t_0 - \alpha R^2 < t < t_0$. Here $\alpha := \frac{c}{2} \phi_1(\omega)$.

Proof. By the previous lemma ($k = \mu^- + \frac{\omega}{4}$)
\[
(18) \quad \int_{B_R} \psi^2((v(t) - k)^-) \leq \int_{B_R} \psi^2((v(t_1) - k)^-) + \frac{C(p - 2)}{\phi_0(\omega)\sigma_1^2} R^N
\]
and by (16) for some $t_1 \in [t_0 - R^2, t_0 - \alpha R^2]$
\[
\text{meas}(B_R \cap \{v(t_1) < k\}) \leq \frac{1 - 2\alpha}{1 - \alpha} \text{meas}(B_R)
\]
hence using (17)
\[
\int_{B_R} \psi^2((v(t_1) - k)^-) \leq (\log 2)^2 (p - 2)^2 \frac{1 - 2\alpha}{1 - \alpha} \text{meas}(B_R)
\]
The left side of (18) is
\[
\geq \int_{B_R \cap \{v(t) < \mu^- + 2^{-p} \omega\}} \psi^2((v(t) - k)^-) \geq \\
\geq \max(0, \log 2) \sigma_0^{-3} \text{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p} \omega\}) \geq \\
\geq (\log 2)^2 (p - 3)^2 (\text{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p} \omega\}) - \sigma_1 N \text{meas}(B_R))
\]
Substituting these estimates in (18) we get
\[
\frac{\text{meas}(B_R \cap \{v(t) < \mu^- + 2^{-p} \omega\})}{\text{meas}(B_R)} \leq \left( \frac{p - 2}{p - 3} \right)^2 \frac{1 - 2\alpha}{1 - \alpha} + \frac{C(p - 2)}{\phi_0(\omega)\sigma_1^2(p - 3)^2} + \sigma_1 N
\]
Now choose $\sigma_1 = 3\alpha^2/(2N)$ and $p$ large enough so that
\[
\frac{C(p - 2)}{\phi_0(\omega)\sigma_1^2(p - 3)^2} \leq \frac{3}{2} \alpha^2 \quad \text{and} \quad \left( \frac{p - 2}{p - 3} \right)^2 \leq (1 - \alpha)(1 + 2\alpha)
\]
We also need the following estimate
12. Lemma. There is a constant $C$ such that for $k < \mu - \frac{\omega}{4}$ and $0 < \beta < 1$

$$
\| (v - k) \|_2^2 \leq \frac{C}{\phi_0(\omega)^2} \left( \sigma_1^2 + (\sigma_2^2 R^2)^{-1} \right) \int_{Q_R^0} \| (v - k) \|_2^2 +
$$

$$
+ \frac{C}{\phi_0(\omega)^2} \text{meas}(Q_R^\beta \cap \{v < k\}) .
$$

Proof. This follows similarly to the first part of the proof of Proposition 9 by multiplying (15) with $-(v - k)^{-2}$, where $\eta$ is a suitable cut off function.

Now we are able to show

13. Lemma. For $\theta > 0$ there is a $q = q(\omega, \theta) > p(\omega)$ such that if $R \leq 2^{-P_\omega}$ and

(16) hold, then

$$
\text{meas}(Q_R^\alpha \cap \{v < \mu - 2^{-q_\omega}\}) < \theta \text{meas}(Q_R^\alpha) .
$$

Proof. Let $q \geq p(\omega)$, $\xi = \mu - 2^{-q_\omega}$, and $k = \mu - 2^{-q_\omega}$. By Lemma 11 for

$$
t_o - aR^2 < t < t_o
$$

$$
\text{meas}(B_R \cap \{v(t) \geq \xi\}) \geq cR^2 N ,
$$

therefore using [9; 2 Lemma 3.5]

$$
\frac{\omega}{2^{q+1}} \text{meas}(B_R \cap \{v(t) < k\}) \leq \frac{CR}{\alpha^2} \int_{B_R \cap \{k < v(t) < \xi\}} |\nabla ((v(t) - \xi))| .
$$

Integrating over $t$ yields

$$
\left( \frac{\omega}{2^{q+1}} \right)^2 \text{meas}(Q_R^\alpha \cap \{v < k\})^2 \leq \frac{CR^2}{\alpha^4} \text{meas}(Q_R^\alpha \cap \{k < v < \xi\}) \int_{Q_R^\alpha} |\nabla (v - \xi)|^2
$$

and by lemma 12

$$
\int_{Q_R^\alpha} |\nabla (v - \xi)|^2 \leq \frac{C}{\phi_0(\omega)^2} \left( \left( \text{ess sup}_{Q_R^2} (v - \xi) \right)^2 + R^2 \right)^N \leq \frac{C}{\phi_0(\omega)^2} (2^{-q_\omega})^2 R^N .
$$

Thus (19) becomes

$$
\text{meas}(Q_R^\alpha \cap \{v < k\})^2 \leq \frac{CR^N}{\alpha^4 \phi_0(\omega)^2} \text{meas}(Q_R^\alpha \cap \{k < v < \xi\}) .
$$

Adding this unquality for $q = p(\omega), \ldots, q_0 - 1$ we obtain the lemma, if $q_o$ is
large enough (depending on $\omega$ and $\theta$).

14. Proposition. There is a $q = q(\omega)$ such that if (16) holds and $R \leq 2^{-q}\omega$, then

$$\text{ess}_{Q_{R^*}} \text{osc } v \leq \omega(1 - 2^{-q})$$

where $R^* = c_1 R^{7/6}$. Here $c_1$ is a small constant independent of $R$ and $\omega$.

Proof. Consider the cylinders $Q_{R_n}^\alpha$ and the levels $k_n$ defined by

$$R_n := \frac{R}{2} + \frac{R}{2^{n+1}}$$

and

$$k \colon= \frac{R^*}{2} + \frac{\omega}{2^{q+1}} + \frac{\omega}{2^{q+n+1}},$$

where $q = q(\omega, \theta)$, $\theta$ to be chosen. By the embedding lemma \[10; II (3.9)\]

$$\int_{Q_{R_n}^\alpha} |v - k_n|^2 \leq C \text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\}) \frac{2}{\phi_0(\omega)} \text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\})$$

The left side controls

$$\frac{1}{2} \text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\})$$

and by Lemma 12

$$\|v - k_n\|^2_{Q_{R_n}^\alpha} \leq C \frac{\phi_0(\omega)}{2} \left( \left( \frac{\omega}{R} \text{ ess sup } (v - k_n) \right)^2 + 1 \right) \cdot \text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\}).$$

Since $(v - k_n)^- \leq 2^{-q}\omega$ on $Q_{R_n}^\alpha$, we get the recursive estimate

$$y_{n+1} \leq \frac{2^{N+2}}{\phi_0(\omega)^2} 2^{2n+1} \frac{1}{y_n} + \frac{2}{y_{n+1}} \text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\})$$

$$y_n := \frac{\text{ meas}(Q_{R_n}^\alpha \cap \{v < k_n\})}{\text{ meas}(Q_{R_n}^\alpha)}.$$

By \[10; II Lemma 5.6\] we infer that $y_n \to 0$ as $n \to \infty$ if

$$y_0 < C \frac{\phi_0(\omega)^{N+2}}{\alpha}.$$
FLOW OF OIL AND WATER...

But if we choose \( \theta \) to be the right side of this inequality, this is just the statement in Lemma 13.

15. Remark. In [8] the existence of a classical solution is proved in the case that the equation (15) is strictly parabolic. The paper also contains uniqueness and stability results, but the overflow condition is not included. Some of the arguments are restricted to the two dimensional case.

Recently in [7] the existence of a weak solution was shown for the Dirichlet-Neuman problem. The assumption is that the initial and boundary data stay away from one side of the degeneracy, so that the solution contains only one pure fluid besides the mixture.

In the article presented here the statement of Lemma 6 in connection with the assumption in Theorem 7 is not quite satisfactory, since if \( k_1(z) \leq Cz \) condition (11) implies that \( p_{\min} = -\infty \), but then Lemma 6 does not cover the case \( H^{N-1}(\Gamma_1) > 0 \).

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Hans Wilhelm ALT
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstrasse 6
D-5300 Bonn
West Germany