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The behavior of capillary surfaces when gravity goes to zero


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THE BEHAVIOR OF CAPILLARY SURFACES WHEN GRAVITY GOES TO ZERO

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1. PRELIMINARY.

Let \( \Omega \) be a bounded (smooth or piecewise smooth) domain in \( \mathbb{R}^n \), \( n \geq 2 \).

Consider the following boundary value problem:

\[
\begin{align*}
\text{div}(Tw) &= \text{div}\left(\frac{Dw}{\sqrt{1 + |Dw|^2}}\right) = H + Bw \quad \text{in } \Omega \\
Tw \cdot \nu &= \cos \gamma \quad \text{on } \partial \Omega \\
\end{align*}
\]

(1.1)

where \( B > 0 \), \( \pi/2 > \gamma \geq 0 \) are constants, \( H = \frac{|\partial \Omega|}{|\Omega|} \cos \gamma \) and \( \nu \) is the outward normal of \( \partial \Omega \).

The solution of (1.1) corresponds to capillary surface with gravity. We are interested in the behavior of \( w \) when gravity goes to zero, i.e. when \( B \) tends to zero. So we compare \( w \) with the solution of

\[
\begin{align*}
\text{div}(Tv) &= H \quad \text{in } \Omega \\
Tv \cdot \nu &= \cos \gamma \quad \text{on } \partial \Omega . \\
\end{align*}
\]

(1.2)

(1.2) may not have a solution. If (1.2) has a bounded solution, \( \gamma > 0 \) and \( \Omega \) is smooth, then it is proved by Siegel in [18] that there exists a constant \( C \) which is independent of \( B \) such that \( \sup_{\Omega} |w - v| \leq C \cdot B \) where \( v \) is the solution of (1.2) normalized by \( \int_{\Omega} v \, dx = 0 \).

In this paper we are going to investigate the case when \( \Omega \) is piecewise smooth, the case when \( \Omega \) is smooth but \( \gamma = 0 \) and the case when (1.2) has no solution. We shall use the idea of generalized solutions introduced by Miranda [17], see also Giusti [10].
It is known that if $v$ is a bounded solution of (1.2) where $H$ is replaced by any bounded measurable function $H(x)$, then $v$ is a variational solution of
\begin{equation}
F(\overline{\Omega};v) = \int_{\overline{\Omega}} \sqrt{1+|Dv|^2} + \int_{\Omega} H(x)v(x)dx - \cos \gamma \int_{\partial \Omega} v(x)d\mathcal{H}^{n-1}
\end{equation}
for $v \in BV(\overline{\Omega})$.

We introduce another functional:
\begin{equation}
F(\overline{\Omega};U) = \int_{\Omega \times \mathbb{R}} |D\chi_U| + \int_{\Omega \times \mathbb{R}} H(x)\chi_U(x,t)dt - \cos \gamma \int_{\partial \Omega \times \mathbb{R}} \chi_U(x,t)d\mathcal{H}^{n}
\end{equation}
where $U \subset \overline{\Omega} \times \mathbb{R}$ is a Caccioppoli set, $\chi_U$ is the characteristic function of $U$.

In (1.3) and (1.4) we do not assume $\overline{\Omega}$ to be bounded.

**Definition 1.1.** $U \subset \Omega \times \mathbb{R}$ is said to be a solution of (1.4) if and only if for any compact set $K$ in $\mathbb{R}^{n+1}$ and any Caccioppoli set $V$ of $\Omega \times \mathbb{R}$ such that $\text{spt}(\chi_U - \chi_V) \subset K$, then $F_K(\overline{\Omega};U) \leq F_K(\overline{\Omega};V)$ where
\begin{equation}
F_K(\overline{\Omega};W) = \int_{\Omega \times \mathbb{R} \cap K} |D\chi_W| + \int_{\Omega \times \mathbb{R} \cap K} H(x)\chi_W(x,t)dt - \cos \gamma \int_{\partial \Omega \times \mathbb{R} \cap K} \chi_W(x,t)d\mathcal{H}^{n}
\end{equation}

We also introduce two subsidiary functionals:
\begin{equation}
G_1(\overline{\Omega};A) = \int_{\Omega} |D\chi_A| + \int_{\Omega} H(x)\chi_A(x)dx - \cos \gamma \int_{\partial \Omega} \chi_A(x)d\mathcal{H}^{n-1}
\end{equation}
and
\begin{equation}
G_2(\overline{\Omega};A) = \int_{\Omega} |D\chi_A| - \int_{\Omega} H(x)\chi_A(x)dx + \cos \gamma \int_{\partial \Omega} \chi_A(x)d\mathcal{H}^{n-1}
\end{equation}
for $A \subset \overline{\Omega}$. Solutions of (1.6) and (1.7) are defined similarly.

**Definition 1.2.** A function $u : \overline{\Omega} \to [-\infty, +\infty]$ is a generalized solution of (1.3) if its subgraph $U = \{(x,t) \in \Omega \times \mathbb{R}| t < u(x)\}$ is a solution of (1.4).

**Theorem 1.1.** Let $\Omega$ be a bounded piecewise smooth domain, and $u \in BV(\overline{\Omega})$, then $u$ is a solution of (1.3) if and only if $u$ is a generalized solution of (1.3).
2. CASE WHEN $\Omega$ IS PIECEWISE SMOOTH.

In this section we make the following assumptions:

(2.1) $\Omega$ is a bounded piecewise smooth domain in $\mathbb{R}^2$;

(2.2) let $2\cdot \bar{\alpha} =$ minimum of interior angles of $\Omega$, then $\pi/2 - \gamma < \bar{\alpha} < \pi/2$;

(2.3) (1.2) has a bounded solution $v$ which is normalized by $\int_{\Omega} v(x) dx = 0$.

We also assume $0 < B < 1$.

**Theorem 2.1.** Under the above assumptions, there exists a constant $C$ which is independent of $B$ such that

$$\sup_{\Omega} |w - v| \leq C \cdot B.$$ 

Before we prove the theorem, we have several lemmata. In what follows $C_1$ will denote constants independent of $B$.

**Lemma 2.1.** There is a constant $C_1$ such that

$$|w| \leq C_1.$$ 

Proof. Use comparison principle as in [18].

**q.e.d.**

The next crucial step is to obtain a uniform bound for the gradients of $w$ and $v$. If $\Omega$ is smooth, then it immediately follows from [7]. If $\Omega$ is only piecewise smooth, then by [7], [13] and [20] we can always get uniform bound for the gradients away from the corners. So it remains to find a bound near the corners. Without loss of generality we may assume a corner is at $(0,0)$ and near it $\Omega$ consists of two segments on $\theta = -\alpha$ and $\theta = \alpha$. Let $\bar{w} = w + \text{constant}$ such that $(0,0,0) \in \mathbb{R}^3$ belongs to the closure of the graph of $\bar{w}$. Here $w$ is a solution of (1.1) or (1.2).

**Lemma 2.2.** Let $\bar{U}$ be the subgraph of $\bar{w}$. There exists constants $C_2 > 0$ and $R_0 > 0$ which are independent of $B$, such that for any $(x_o, t_o) \in \bar{U} \times \mathbb{R}$ and let $C_r(x_o, t_o) = \{(x,t) \in \mathbb{R}^3 | |x-x_o| < r \text{ and } |t-t_o| < r\}$ the following are true:
Lemma 2.3. There exists a constant $C_3$ such that

$$|\nabla w| \leq C_3,$$

where $w$ is the solution of (1.1) or (1.2).

Proof. Take any sequence $B_k \geq 0$ (not necessarily distinct) and take any sequence of positive numbers $\varepsilon_k > 0$. Let

$$w_k, \varepsilon_k = \frac{1}{\varepsilon_k} w_k(\varepsilon_k x)$$

where $w_k$ is the solution corresponding to $B_k$. We can then find a subsequence of $w_k, \varepsilon_k$ which tends to a generalized solution $u$ of (1.3) with $H(x) \equiv 0$ in the domain

$$\Omega_\infty = \lim_{k \to \infty} \frac{1}{\varepsilon_k} \Omega_k.$$  

Let $P = \{x|u(x) = \infty\}$ and let $N = \{x|u(x) = -\infty\}$. Then $P$ is a solution of $G_1(\Omega_\infty, A)$ with $H(x) \equiv 0$. Use assumption (2.2) we can prove that $P = \phi$ or $\Omega_\infty$. By Lemma 2.2 we conclude that $P = \phi$. Similarly $N = \phi$. From these and lemma 2.2 we can prove that $w_k, \varepsilon_k$ are uniformly bounded in $\{x \in \Omega_\infty |1 \leq |x| \leq 2\}$ if $k$ is large enough. From [7], [13] and [20], the lemma follows.

q.e.d.

Now we can proceed as in [18] to get a proof of Theorem 2.1.

3. CASE WHEN $\gamma = 0$.

Let $\Omega$ be a smooth domain in $\mathbb{R}^n$, $n \geq 2$. If $\gamma = 0$, solution of (1.2) may not exist, or may exist but fail to be bounded. See [9]. Suppose (1.2) has a solution $v$, then we have the following

Theorem 3.1. Either (1) $v \in L^1(\Omega)$ and $\lim_{B \to 0} w = v + C$ in $\Omega$ for some constant $C$; or
(2) \( v \in L^1(\Omega) \) and \( \lim_{B \to 0} w = -\infty \) in \( \Omega \).

The proof of Theorem 3.1 is obtained by using the idea of generalized solution and comparison principle.

Theorem 3.2. We can find a function \( C(B) \) such that \( \lim_{B \to 0} (w + C(B)) = v \) in \( \Omega \).

The proof of Theorem 3.2 is also obtained by using the idea of generalized solution and the following lemma.

Lemma 3.1. For any \( B_k \to 0 \), we can find a subsequence \( B_{k_j} \) such that \( \lim_{j \to \infty} w_{k_j} = 0 \), where \( w_k \) is the solution of (1.1) corresponding to \( B_k \).

Corollary 3.1. \( \lim_{B \to 0} \partial w = \partial v \) in \( \Omega \).

Note that all convergences are uniform in compact subset of \( \Omega \).

4. CASE WHEN (1.2) DOES NOT HAVE A SOLUTION.

We make the following assumptions:

(3.1) \( \Omega \) is a piecewise smooth domain in \( \mathbb{R}^2 \) such that every interior angle \( 2\alpha \) satisfies \( \pi/2 > \alpha > \pi/2 - \gamma \);

(3.2) \( \Omega \) satisfies internal sphere condition for some radius \( \delta > 0 \) and angle \( \gamma \) in the sense of [6];

(3.3) \( G_1(\Omega; A) \geq 0 \) for all \( A \subset \Omega \) where \( H(x) \equiv H \), and there is a unique set \( P \) such that \( P \neq \emptyset \) or \( \Omega \) and \( G_1(\Omega; P) = 0 \).

Lemma 3.1 is still true in this case and we have:

Theorem 4.1. There are functions \( C_1(B) \) and \( C_2(B) \) such that:

(1) \( w + C_1(B) \) tends to a classical solution of \( \text{div}(Tu) = H \) in the interior of \( N \) and tends to positive infinity in the interior of \( P \);

(2) \( w + C_2(B) \) tends to a classical solution of \( \text{div}(Tu) = H \) in the interior of \( P \) and tends to negative infinity in the interior of \( N \).
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REFERENCES


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