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THE PARAMETRIC PROBLEM OF CAPILLARITY: THE CASE OF TWO AND THREE FLUIDS

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I shall speak about the existence of equilibrium configurations in a container filled by two or three non-miscible, homogeneous fluids subjected to surface tension and gravitational energy.

If we denote by $\Omega \subset \mathbb{R}^n$ a bounded open set with Lipschitz-continuous boundary and by $E_1, E_2$ subsets of $\Omega$ occupied by two non-miscible fluids with given densities $\rho_1$ and $\rho_2$, we can write the global energy of the configuration in the following way:

$$E(E_1, E_2) = \gamma_1 \text{meas}_{n-1}(\partial E_1 \cap \partial E_2 \cap \partial \Omega) + \gamma_2 \text{meas}_{n-1}(\partial E_1 \cap \partial \Omega) + \gamma_3 \text{meas}_{n-1}(\partial E_2 \cap \partial \Omega) +$$

$$+ \rho_1 \sum_{i=1}^{2} \int_{\Omega} g_i \phi_i (x) \, dx.$$

We use the $(n-1)$-dimensional measure introduced by E. De Giorgi in 1954 (see [3]). More precisely, if $E$ is a measurable subset of $\Omega$, we define the perimeter of $E$ in $\Omega$ as:

$$\text{meas}_{n-1}(\partial E \cap \partial \Omega) = \int_{\Omega} |D\phi| = \sup \left\{ \int_{\Omega} \text{div}(g(x)) \, dx; \, g \in C^1_c(\Omega, \mathbb{R}^n), \, |g| \leq 1 \right\}.$$

We observe that the perimeter of $E$ is the total variation on $\Omega$ of the vector valued measure $D\phi = (D_1 \phi, D_2 \phi, \ldots, D_n \phi)$ where $D_i \phi$ for $i = 1, 2, \ldots, n$ are the derivatives of the characteristic function of $E$ in the distributional sense.

It is well-known that, if $\int_{\Omega} |D\phi| < +\infty$, then there exists the trace of $\phi$ on the Lipschitz-continuous surface $\partial \Omega$. 
Using the perimeter and the trace of $E$, recalling that $E_2 = \Omega - E_1$, the global energy can be written in the form:

$$E(E_1, E_2) = \gamma_1 \int_\Omega |D\phi| + (\beta_1 - \beta_2) \int_{\partial\Omega} \phi d\mathcal{H}^{n-1} + g(\rho_1 - \rho_2) \int_\Omega \phi \rho_n dx + \mathcal{H}_{n-1}(\partial\Omega) + g\rho_2 \mathcal{H}_n(\Omega)$$

Then, the problem is reduced to minimize the functional:

$$F(E) = \gamma \int_\Omega |D\phi| \leq \frac{1}{\gamma} \left[ F(E) + |\beta| \mathcal{H}_{n-1}(\partial\Omega) + g|\rho| \int_\Omega \phi \rho_n dx \right]$$

in the class $\mathcal{H}$ of all subsets of $\Omega$ having prescribed volume $\nu \in (0, \mathcal{H}_n(\Omega))$.

We observe the following:

a) if $\gamma > 0$, $F$ has a finite lower bound;

b) if $\gamma > 0$, from the inequality

$$\int_\Omega |D\phi_h| \leq \frac{1}{\gamma} \left[ F(E) + |\beta| \mathcal{H}_{n-1}(\partial\Omega) + g|\rho| \int_\Omega \phi \rho_n dx \right],$$

if $\{E_h\}$ is a minimizing sequence, we have:

$$\int_\Omega |D\phi_h| \leq \text{cost}.$$  

From a well-known compactness theorem, there exists a subsequence of $\{E_h\}$ converging in $L_1(\Omega)$ to a set $E$.

c) if $\gamma \geq |\beta|$, the functional $F$ is lower semicontinuous with respect to $L_1(\Omega)$-convergence.

Then we can state the following

**Theorem 1.** If $\gamma \geq |\beta|$ $(\gamma > 0)$, the functional $F$ has a minimum $E$ in the class $\mathcal{H}$.

The regularity results of De Giorgi and M. Miranda can be applied to study the smoothness of $\partial E$ and we obtain that there exists an open subset of $\partial E \cap \Omega : \partial E \cap \Omega$ that is an analytic manifold of dimension $n-1$ and moreover $H_s((\partial E - \partial E) \cap \Omega) = 0$ $\forall s > n-8$. (See [33]).

Let us consider now a container $\Omega$ filled by three fluids: $(E_1, E_2, E_3) = E$.
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If we denote by

\[ |Z_{ij}| = \text{meas}_{n-1}(\partial E_i \cap \partial E_j \cap \Omega) \quad i, j = 1, 2, 3; \quad i \neq j \]

the surface energy of the six interfaces, can be written as:

\[ E(E) = E(E_1, E_2, E_3) = \Sigma_{i,j} |Z_{ij}| + \Sigma_{i,j} \left| \Sigma_{k} \right| \]

Now, if we suppose \( \partial E_i \) (i = 1, 2, 3) Lipschitz continuous and \( H_{n-1}(\partial E_1 \cap \partial E_2 \cap \partial E_3) = 0 \), we have:

\[ \int_{\Omega} |D \phi_{E_i}| = \sum_{j=1}^{3} \left| \Sigma_{ij} \right| \quad i = 1, 2, 3 \]

and then we can write:

\[ E(E) = \sum_{i=1}^{3} \gamma_i \int_{\Omega} |D \phi_{E_i}| + \sum_{i=1}^{3} \beta_i \int_{\partial \Omega} \phi_{E_i} \text{d}H_{n-1} \]

where:

\[
\begin{align*}
\gamma_1 &= \frac{\gamma_{12} + \gamma_{13} - \gamma_{23}}{2} \\
\gamma_2 &= \frac{\gamma_{23} + \gamma_{12} - \gamma_{13}}{2} \\
\gamma_3 &= \frac{\gamma_{13} + \gamma_{23} - \gamma_{12}}{2}
\end{align*}
\]

Therefore the global energy of the configuration is given by:

\[ F(E) = \sum_{i=1}^{3} \left( \gamma_i \int_{\Omega} |D \phi_{E_i}| + \beta_i \int_{\partial \Omega} \phi_{E_i} \text{d}H_{n-1} + g_0 \int_{\Omega} x_n \phi_{E_i} \text{d}x \right) \]

We have now to minimize the functional 3) in the class

\[ K = \left\{ E = (E_1, E_2, E_3); E_i \cap E_j = \emptyset \quad i \neq j; \quad H_n(E_i) = v_i, \quad \Sigma_{i=1}^{3} v_i = H_n(\Omega) \right\} \]

It is easy to see that \( F \) has a finite lower bound if and only if

\[ \gamma_i + \gamma_j \geq 0 \quad i, j = 1, 2, 3 \quad i \neq j \]
In fact, if $\gamma_i > 0 \ \forall \ i = 1,2,3$, we have

$$F(E) \geq \sum_{i=1}^{3} \gamma_i \int_{\Omega} |D\phi_{E_i}| - c$$

where

$$c = \sum_{i=1}^{3} \left( |\beta_i|_{H_{n-1}(\partial\Omega)} + g|p_i| \int_{\Omega} |x_n| \, dx \right).$$

On the other hand, if $\gamma_1 \leq 0$, one has:

$$F(E) \geq \gamma_1 \int_{\Omega} |D\phi_{E_2}| + \int_{\Omega} |D\phi_{E_3}| + \sum_{j=2}^{3} \gamma_j \int_{\Omega} |D\phi_{E_j}| - c = \sum_{j=2}^{3} (\gamma_j + \gamma_1) \int_{\Omega} |D\phi_{E_j}| - c$$

and then

$$\inf_{K} F(E) \geq -c.$$ 

From the last two inequalities, if $\gamma_i + \gamma_j > 0 \ \forall \ i,j = 1,2,3 \ i \neq j$, we obtain:

$$\int_{\Omega} |D\phi_{E_i}| \leq c_1 F(E) + c_2 \ \forall \ i = 1,2,3$$

and then one gets the compactness property we use to prove the existence of a minimum.

We note that 2) implies

$$\gamma_i + \gamma_j = 2\gamma_{ij} \ \forall \ i,j = 1,2,3.$$ 

Physically, condition 4) means that the surface energies of the $i-j$ interfaces are non negative and the fluids do not mix up.

It is easy to see that the conditions

$$\begin{cases} 
\gamma_i \geq 0 \ \forall \ i = 1,2,3 \\
\gamma_i + \gamma_j \geq |\beta_i - \beta_j| \ \forall \ i,j = 1,2,3 \ i \neq j 
\end{cases}$$

are necessary for the lower semicontinuity of the functional $F$. If they are sufficient it isn't clear yet.

We can prove the following:
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**Proposition A.** If \( \Omega \) has the interior sphere condition, \( \gamma_i \geq 0 \), \( \gamma_i + \gamma_j > 0 \) and \( \gamma_i + \gamma_j \geq |\beta_i - \beta_j| \), then \( F \) is lower semicontinuous.

**Proposition B.** If we denote the Lipschitz constant of \( \partial \Omega \) by \( L \) and \( \gamma_i > 0 \),

\[
6) \quad \gamma_i + \gamma_j \geq \sqrt{1 + L^2|\beta_i - \beta_j|} \quad i, j = 1, 2, 3 ;
\]

then \( F \) is lower semicontinuous.

**Proposition C.** Let us suppose \( \beta_1, \beta_2, \beta_3 \). If

\[
7) \quad \gamma_j > \beta_j - \beta_1 \quad j = 2, 3
\]

then \( F \) is lower semicontinuous.

Outline of the proof.

A. We recall that interior sphere condition means that \( \exists \rho > 0 \) and \( \forall x \in \Omega \) a ball of radius \( \rho \) with \( x \in B_\rho \subset \Omega \). If \( \Omega \) has the interior sphere condition, then \( \forall \varepsilon > 0 \) and \( \forall E \subset \Omega \), the following inequality holds:

\[
8) \quad \int_{\partial \Omega} \phi E dH_{n-1} \leq \int_{\Omega_\varepsilon} |D\phi_E| + c \int_{\Omega_\varepsilon} \phi_E dx
\]

where \( \Omega_\varepsilon = \{x \in \Omega, \text{dist}(x, \partial \Omega) < \varepsilon\} \) and \( c \) is a constant depending on \( n, \rho, \varepsilon \) and \( \Omega \). (See [4]). Now, if we suppose \( \beta_1, \beta_2, \beta_3 \), from 8) we have:

\[
F(E) - F(E^n) = \sum_{i=1}^{3} \gamma_i \left( \int_{\Omega} |D\phi_{E_i}| - \int_{\Omega} |D\phi_{E_i}^n| \right) + \sum_{i=1}^{3} \gamma_i \left( \int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i}| - \int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i}^n| \right) +
\]

\[
+ \sum_{i=1}^{3} \beta_i \left( \int_{\partial \Omega} (\phi_{E_i} - \phi_{E_i}^n) dH_{n-1} \right) + \sum_{i=1}^{3} \gamma_i \left( \int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i}| - \int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i}^n| \right) +
\]

\[
+ \sum_{i=1}^{3} \gamma_i \left( \int_{\Omega} |D\phi_{E_i}| - \int_{\Omega} |D\phi_{E_i}^n| \right) + \sum_{i=1}^{3} \gamma_i \left( \int_{\Omega} |D\phi_{E_i}| - \int_{\Omega} |D\phi_{E_i}^n| \right) +
\]

\[
+ \sum_{j=1, 3} (|\beta_j - \beta_2| - \gamma_j) \left( \int_{\Omega_\varepsilon} |D\phi_{E_j}^n| - \gamma_2 \int_{\Omega_\varepsilon} |D\phi_{E_j}^n| \right) + c \sum_{j=1, 3} |\beta_j - \beta_2| \int_{\Omega_\varepsilon} |\phi_{E_j} - \phi_{E_j}^n| dx .
\]
Now it is sufficient to prove

\[ \limsup_h \left( \sum_{j=1,3} |\beta_j - \beta_2| - \gamma_j \right) \leq \limsup_h G(E_h) \leq 0 \]

when \( E_j^h \to E_j \) in \( L^1(\Omega) \).

In fact, if 9) is true, we have:

\[ \limsup_h \left[ F(E) - F(E^h) \right] \leq \sum_{i=1}^3 \gamma_i \int_{\Omega} |D\phi_{E^1}| + \sum_{j=1,3} |\beta_j - \beta_2| \int_{\Omega} |D\phi_{E^j}| \to 0. \]

Inequality 9) is trivial if \( \gamma_j > |\beta_j - \beta_2| \) \( j = 1,3 \). On the other hand, if

\[ \gamma_1 < |\beta_1 - \beta_2| \]

we obtain

\[ G(E^h) \leq \left( |\beta_1 - \beta_2| - \gamma_1 \right) \int_{\Omega} |D\phi_{E^2}| + \left( |\beta_2 - \beta_1| - \gamma_2 \right) \int_{\Omega} |D\phi_{E^3}| \]

\[ = \left( |\beta_2 - \beta_1| - \gamma_1 - \gamma_2 \right) |D\phi_{E^2}| + \left( |\beta_3 - \beta_1 - \gamma_1 - \gamma_3 \right) |D\phi_{E^3}| \leq 0. \]

Proposition B can be proved arguing in the same way. We now use the inequality

\[ \int_{\partial \Omega^h_{n-1}} \phi \leq \sqrt{1 + L^2} \int_{\Omega} |D\phi_E| + c \int_{\Omega} \phi \]

in the place of 8).

Finally, if 7) holds, using the identity

\[ \int_{\mathbb{R}^n} |D\phi_E| = P(E) = \int_{\Omega} |D\phi_E| + \int_{\partial \Omega} \phi \]

we can write the functional \( F \) in the form

\[ F(E) = \gamma_1 \int_{\Omega} |D\phi_{E^1}| + \sum_{j=2}^3 (|\beta_j - \beta_1|) P(E_j) + \sum_{j=2}^3 (|\beta_j - (\beta_j - \beta_1)|) \int_{\Omega} |D\phi_{E^j}| + \sum_{i=1}^3 \rho_i \int_{\Omega} x_n \phi_{E^1} \]

and all the functionals on the right side are lower semicontinuous.

The conditions \( \gamma_i > 0 \) \( i = 1,2,3 \) imply that

\[ \gamma_{12} + \gamma_{13} - \gamma_{23} \geq 0 \]

\[ \gamma_{21} + \gamma_{23} - \gamma_{13} \geq 0 \]
\[ \gamma_{13} + \gamma_{23} - \gamma_{12} \geq 0 \]

Physically these conditions are necessary to have an equilibrium configuration. In fact if \( \gamma_{12} + \gamma_{13} - \gamma_{23} < 0 \) the liquid \( E_1 \) will spread on \( E_2 \) and equilibrium becomes impossible.

The same regularity results can be applied and we obtain that \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) are analytic \((n-1)\)-dimensional manifolds in every ball \( B \) intersecting only two of the three sets \( E_1, E_2, E_3 \). Moreover \( \mathbb{H}(\mathcal{E}_i \cap \mathcal{E}_1 \cap B) = 0 \) \( \forall \ i = 1, 2, 3 \), \( s > n - 8 \).

REFERENCES


