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VARIATIONAL PRINCIPLES FOR EQUILIBRIUM FIGURES OF FLUIDS WITHOUT SYMMETRY ASSUMPTIONS

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1. INTRODUCTION.

In this article we shall discuss some of the classic problems of nonlinear analysis and applied mathematics that deal with rotating figures of equilibrium under various physical forces.

I wish to point out four classic problems in this connection. The first problem concerns the question of rotating figures of equilibrium held together under gravitational forces. This problem dates back to Newton, MacLaurin, Jacobi, Poincaré, and Lichtenstein. The problem arose as an attempt to explain the shape of astronomical objects such as stars and the planets of our own solar system. This study involved very subtle problems in mathematics and we shall discuss a few of these in the sequel. The second classic problem is closely related to the first. It is the problem of rotating figures of equilibrium under surface tension with no gravitational forces acting. This problem dates back to the classic studies of J. Plateau and I refer to it as the second Plateau problem in contradistinction to finding minimal surfaces on a given wire frame. This problem has recently become very important because of the important experimental work done at JPL in Pasadena by a group headed by Taylor Wang and the detailed numerical studies of Scriven and Brown. The third classic problem in this connection is once more rotating figures of equilibrium, but this time with the combined action of surface

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tension and electric charge. This is the so-called "liquid drop model" of Wheeler and Bohr that arises in nuclear fission. The fourth classic problem concerned the steady vortex motion of an ideal fluid under various external forces. In this problem the inner motion of the fluid becomes extremely important and this problem dates back to Helmholtz and Riemann.

2. THE MATHEMATICAL SETTING.

The mathematics associated with these problems has never been clarified. However, it is clear that each of them can be treated by the calculus of variations and that because multiple solutions of interesting geometric structure are known to arise, it is clear that these problems are a challenge for the modern theories of analysis and the calculus of variations.

Before proceeding it is a good idea, I believe, to review the physical forces and energies that are associated with the problems. First, there is the notion of rotation of a fluid. This rotation is generally measured by a parameter of either angular velocity or angular momentum. The next two physical quantities involved are gravity and electric charge acting on the fluid. These two quantities differ only in sign and thus can be treated from a unified point of view. Another physical force involved is surface tension which is taken as proportional to surface area of the fluid involved. This surface area leads to new classes of variational problems when closed surfaces of fluids are considered. Next, there is the internal motion of the fluid itself. This is generally measured by its vorticity. Finally, there is the distinction between compressible and incompressible fluids which becomes very important in contemporary astronomy.

Associated with my studies in this area are two closely related axioms. The first can be written succinctly.

<u>Axiom A</u>.

(i) *Good Science* implies good mathematics (but not necessarily conversely).

(ii) *Good Modern Science* implies good variational problems.

Axiom B.

If one cannot solve a hard problem, it is helpful to solve an easier one by general methods that might generalize.

The mathematical aspects of the four nonlinear problems discussed above are classic and I wish to describe the key issues raised. First, these problems are defined by nonlinear elliptic partial differential equations and they are doubly nonlinear because the boundaries of the fluids involved are not known a priori. Such problems are global free-boundary problems. The calculus of variations aspects are also very interesting. Potential theory and Hilbert space approaches are very useful for dealing with problems involving fluids under gravitational and vorticity forces, whereas geometric measure theory and its modern variants seem to be essential to handle surface tension effects. Parameter dependence is very crucial in all the four problems raised because the equilibrium shape observed depends on the magnitude of the rotation involved. This effect gives rise to bifurcation phenomena as is well known, but it also leads to new ideas and global linearization; for example, the phenomenon I call "nonlinear desingularization" (see [3]). A fifth important mathematical consideration is the classification of the free boundaries observed. This leads to fascinating questions of global geometry and topology. A sixth key point here is that the critical points of a functional associated with equilibrium figures are often of saddle point type or at least not absolute minima. This is because various physical quantities are conserved in the associated fluid problem. Finally, all these fluid problems involve consideration of stability. Here, new ideas are needed and I shall outline one that I have recently discovered in a special case.

The special case I have in mind (Example 1 below) is easily described. It is related to a problem in pure geometry and requires us to find the simplest closed geodesic on an ovaloid in three dimensions that has no self-intersections. In keeping with Axiom B I claim this problem is a simple version of the second classic problem involving fluid tension as mentioned at the beginning of this article. My reasons for saying this are as follows. First, the form of the closed geodesic is not known in advance but has a closed geometric shape. Secondly, the functional to be minimized is of (surface tension) surface area type, namely arc length. As we shall see, this variational problem possesses a volume constraint and the desired solution of the original variational problem is of saddle point type. All the arguments that I shall give for this problem can be extended to higher dimensions. Finally, determining closed geodesics on an ovaloid is a problem with multiple solutions of high complexity and we wish to find the simplest one, namely, "the one of shortest nonzero length" and to determine its stability. More formally, we note the following examples (mentioned in the sequel) relating to the classic problems mentioned above. Example 2 shows that for vortex ring problems, axisymmetry assumptions seem necessary at the moment for any possible analytical program. On the other hand, examples 3 and 4 show that for rotating figures of equilibrium under gravity an easier situation prevails, since equilibrium forms can be studied without symmetry assumptions. Indeed the classic Jacobi ellipsoids and Poincaré pear-shaped figures are important examples.

3. SOME EXAMPLES.

Example 1. Determine the shortest nonconstant closed geodesic, without self-intersections, on an ovaloid (M^2,g).

(a) *Standard variational problem.* Minimize the arc length functional in a homotopy class of smooth closed curves on (M^2g).

(b) *Modified variational problem* (cf. Klingenberg [6]). Utilize the critical point theories of Morse and Ljusternik-Schnirelmann [via Hilbert manifolds] to find the saddle point of the arc length functional that corresponds to the closed geodesic as described above.

(c) *Variational problem via natural constraints.* Minimize the arc-length functional among all simple closed curves $\{C\}$ on (M^2,g) such that C besects the total (integral) curvature of (M^2,g) [i.e. C divides the surface $(M^2 \cdot g)$ into 2 pieces $\Sigma_1(C)$ and $\Sigma_2(C)$ and

$$\int_{\Sigma_1(C)} K(x)dV = 2\pi$$] ($K(x)$ denotes the Gauss curvature of (N^2,g)).

<u>Comments on the various approaches to the problem</u>. The approach (a) leads to the wrong answer for the ovaloid (viz. the trivial solutions the point geodesic $x(t) \equiv$ point because an ovaloid has $\pi_1(M^2) = \{0\}$). The approach (b) has difficulties: in higher dimensional generalizations since it uses arc length parametrizations of curves, requires special arguments for eliminating the possibility of self-intersections, the approach is basically not analytically effective so that computational and stability questions are not natural in this context.

In the approach (c) differential geometry instead of topology is used to introduce a *strict analytic* approach to the problem that enables one to study stability questions. Moreover, this approach requires moving outside the usual Hilbert space of curves to geometric measure theoretic arguments, since point sets $\Sigma_1(C)$ and their boundaries on a Riemannian manifold are the essential variables.

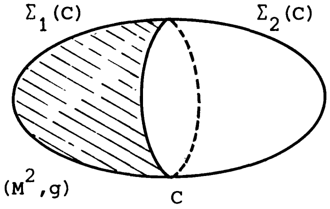
<u>Discussion of viewpoint (c)</u>. In my language the constraint

$$\int_{\Sigma_1(C)} K(x) = 2\pi$$

is a "natural" one. This fact means (i) all the smooth simple closed geodesics desired have this property and (ii) adding the natural constraint to the problem of miniming arc length of curves does not affect the fact that its smooth solutions are geodesics.

<u>Proof of (i)</u>. Apply the Gauss-Bonnet theorem to the curve C as in the diagram to find

$$(2.1) \qquad \int_{\Sigma_1(C)} K + \oint_C Kg = 2\pi$$



Then since C is a geodesic $Kg \equiv 0$ on C so the natural constraint is satisfied. (Here Kg is the geodesic curvature on C on (M^2,g)).

<u>Proof of (ii)</u>. The Euler-Lagrange equation for the new isoperimetric problem is

$$(2.2) \qquad Kg = \lambda K$$

where λ is the Lagrange multiplier. Integrating (2.2) over C we find

$$\oint_C Kg = \lambda \oint_C K \quad .$$

Applying (2.1) and the fact C satisfies the constraint we find

$$\lambda \oint_C K = C \quad \text{which implies} \quad \lambda = 0$$

since $K > 0$ on an ovaloid.

The details of the facts that (i) variational problem of (c) attains its infimum, (ii) the infimum is positive, (iii) the minimizing curve C for (c) has no self-intersections (i.e. C is simple) and (iv) C is smooth followed by the theory of integral currents as developed by Bombieri [4] provided one restricts (M^2,g) to be not too distant from the metric of the standard sphere. This restriction has been removed in recent research by C. Croke and W. Allard. The details of the approach via Bombieri's theory of integral currents can be found in our paper Berger and Bombieri [2] and my review Berger [1]. The problem was first brought up by Poincaré [7] almost 80 years ago and remained unresolved until we began our work.

A key virtue of the approach (c) by geometric measure theory is that it extends to higher dimensional problems involving surface tension, such as outlined at the beginning of this article. In fact, this example reflects my application of Axiom B, mentioned earlier. I hope to treat surface tension problems mentioned earlier for large magnitudes on rotation by utilizing the idea in this example. The methods of geometric measure theory we intend to use do not require symmetry assumptions for the solution.

In [2] the following two results are proven.

Theorem 1. *Poincaré's isoperimetric variational problem has a smooth solution* T *without self-intersections for all* C^3 *ovaloids* (M^2, g). *Provided* $\left|\frac{\partial K}{\partial n}\right| < 2K^{3/2}$ *along* ∂T, ∂T *is a connected one dimensional manifold and thus a simple closed geodesic of shortest nonzero length on* (M^2, g).

Stability Theorem. *A* C^3 *perturbation* $\tilde{g}$ *of the standard metric* g *of the sphere* S^2 *leads to a simple closed geodesic* $\tilde{c}$ *whose Hausdorff distance from a great circle* c *can be controlled by* $\|g - \tilde{g}\|_{C^3}$. *(See* [2] *for the precise statement).*

This result is a new type of nonlinear stability for free boundary problems, since it relies on geometric arguments. I intend to continue this type of result to higher dimensional examples.

Example 2. Vortex motion of an ideal fluid via Euler equations.

Find an analogue of steady vortex rings in an ideal incompressible fluid without assuming axisymmetry on the solution. The Euler Equations for the problem are

$$(2.3) \qquad \underline{V} \times \text{curl } \underline{V} = \text{grad } H$$

$$(2.4) \qquad \text{div } \underline{V} = 0$$

Here $\underline{V}$ represent the velocity vector of the fluid and H the "modified pressure".

Introducing the vector potential $\underline{V} = \text{curl } \underline{A}$ $\text{div } \underline{A} = 0$ this problem can be reduced to the complicated quasilinear system

$$- \operatorname{curl} \underline{A} \times \Delta\underline{A} = \operatorname{grad} H$$

$$\operatorname{div} \underline{A} = 0 \quad .$$

This problem has proved very difficult analytically but Garabedian using Clebsch potentials has made considerable progress numerically recently by variational methods.

Key Remarks. Before beginning the third example on self-gravitating rotating fluids and their equilibrium figures it is important to remark that this problem possesses one simplifying feature that distinguishes it from the study of steady vortex motion. In particular, *it can be studied without symmetry assumptions in terms of semilinear elliptic partial differential equations.* To see this we begin with the Euler momentum equation for an ideal fluid in $\mathbb{R}^3$ moving with constant angular velocity ω, about the z-axis.

To see this I now briefly mention:

Example 3. Figures of equilibrium of a rotating fluid under gravity, (incompressible).

Here the Euler momentum equation for an ideal fluid in $\mathbb{R}^3$ moving with constant angular velocity ω, about the z-axis with density ρ and pressure p is given by the formula

$$-\frac{1}{2}\omega^2\nabla(x^2+y^2) = -\frac{p}{\rho}\nabla p + \nabla\psi \quad .$$

Here ψ denotes the Newtonian potential. Under appropriate assumptions concerning the dependence of the pressure p and density variable we find by the first integral of this relation

$$-\frac{p}{\rho} + \psi + \frac{1}{2}\omega^2(x^2+y^2) = \text{const} \quad .$$

Thus on the free boundary ∂B where $p = \text{constant}$ and $\rho = \text{const.}$ we find

$$(*) \qquad \psi + \frac{\omega^2}{2}(x^2+y^2) = \text{const} \quad .$$

On the other hand in the interior of B we find

(**) $\Delta\psi = -4\pi\gamma\rho$

(γ = gravitational constant).

Thus we need to solve (*) and (**) for various assumptions on ρ the density. The classic homogeneous case is ρ = constant in B.

Instead of treating this classic problem in this article we consider the following simpler model problem. In passing however we note the interesting two dimensional example of Kirckoff [Lamb, Hydrodynamics, p. 232] of a two dimensional rotating ellipsoid of permanent form with constant vorticity.

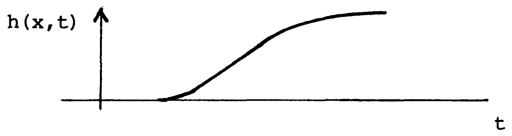
Example 4. Model problem for a rotating ideal fluid under gravity without symmetry assumptions.

Here we consider the following analogue for the problem (*), (**) on the bounded domain Ω

(1*) $\Delta\psi + h(x,\psi) = 0$ with $H_\psi(x,\psi) = h$

(1**) $\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = -\frac{\alpha}{|\partial\Omega|}$

where $|\partial\Omega|$ denotes the measure of $\partial\Omega$. Here $H(x,t)$ is a strictly convex function of its argument, possibly depending on parameters and possible vanishing whenever $t \le 0$. Limiting processes can treat the cases of $h(x,t)$ the Heaviside function or $h(x,t)$ merely continuous. Roughly speaking a graph of h looks for fixed $x = x_o$ like



and represent a generalized density with $H(x,t)$ mass.

The associated variational problem consists in studying the critical points of the functional

$$I(\psi) = \int_\Omega \frac{1}{2}|\nabla\psi|^2 - \int_\Omega H(x,\psi) - \frac{\alpha}{|\partial\Omega|}\int_{\partial\Omega}\psi \quad .$$

Lemma. *This variational problem possesses a natural constraint*

$$\mathfrak{M} = \left\{\psi \,\middle|\, \int_\Omega h(x,\psi) = \alpha\right\} \quad .$$

Proof. We adapt the definition given previously in Example 1.

(i) First, assume we have a smooth critical point ψ of $I(\psi)$ then standard results imply ψ satisfies

$$\Delta\psi = -h(x,\psi) \quad \text{in } \Omega$$

and the boundary condition

$$\frac{\partial\psi}{\partial n} = -\frac{\alpha}{|\partial\Omega|} \quad .$$

(ii) Now suppose ψ is a smooth critical point of $I(\psi)$ restricted to the set $\mathfrak{M}$ then ψ satisfies the Euler-Lagrange equation

$$\Delta\psi + h(x,\psi) + \beta h'(x,\psi) = 0 \quad .$$

Integrating over Ω we find utilizing the condition (1^{**})

$$\beta\int_\Omega h'(x,\psi) = 0 \quad .$$

This fact implies $\beta = 0$ by the positivity convexity properties of h.

Now we note that the problem (1^*) - (1^{**}) has a solution obtained by minimizing $I(\psi)$ on $W_{1,2}(\Omega) \cap \mathfrak{M}$.

The solution is smooth by elliptic regularity theory and thus we assert after a fairly straightforward argument of the calculus of variations in the Sobolev space $W_{1,2}(\Omega)$:

Theorem. *The problem* (1^*) - (1^{**}) *has a smooth nontrivial solution obtained by* *minimizing* $I(\psi)$ *over* $\mathfrak{M} \cap W_{1,2}(\Omega)$.

Remark. To get a solution to the physical problem of rotating fluids we let $\Omega \to \mathbb{R}^3$, keeping careful bounds on the solutions ψ_Ω so that ψ_Ω converges in the limit. Also $h(x,\psi)$ is chosen so that the level surfaces $\psi = \text{const}$ determine the free boundary of the rotating fluid. Finally it is more convenient to utilize angular momentum as a parameter measuring the magnitude of rotation rather than angular velocity as described above. This approach is being used in joint research in progress with Prof. L.E. Fraenkel. This type of argument is a generalization of the methods we used in the study of global vortex rings [5].

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