ROBERT A. BLUMENTHAL

Transverse curvature of foliated manifolds


<http://www.numdam.org/item?id=AST_1984__116__25_0>
Let $M$ be a smooth manifold and let $\nabla$ be a linear connection on $M$. A fundamental problem in differential geometry is to find relations between the curvature of $\nabla$ and the topology of $M$. We consider the analogue of this fundamental problem for foliated manifolds and basic connections.

Let $(M,\mathcal{F})$ be a foliated manifold. Let $\mathcal{F}$ be the normal bundle of $\mathcal{F}$ and let $\nabla$ be a basic connection on $\mathcal{F}$. Our fundamental problem is then to study the relationship between the curvature of $\nabla$ and the structure of the foliated manifold $(M,\mathcal{F})$.

We recall a few basic concepts. Let $T(M)$ be the tangent bundle of $M$ and let $E \subset T(M)$ be the subbundle tangent to the leaves of $\mathcal{F}$. Let $\mathcal{Q} = T(M)/E$ be the normal bundle. Let $\pi:T(M) \rightarrow \mathcal{Q}$ be the natural projection and let $\chi(M), \Gamma(E), \Gamma(\mathcal{Q})$ denote the sections of $T(M), E, \mathcal{Q}$ respectively. A connection $\nabla:\chi(M) \times \Gamma(\mathcal{Q}) \rightarrow \Gamma(\mathcal{Q})$ is basic [3], transverse [9], adapted [7] if $\nabla^Y_X = \pi([X,Y])$ for all $X \in \Gamma(E), Y \in \Gamma(\mathcal{Q})$ where $\bar{Y} \in \chi(M)$ satisfies $\pi(\bar{Y}) = Y$. The parallel transport which $\nabla$ induces along a curve lying in a leaf of $\mathcal{F}$ coincides with the natural parallel transport along the leaves. Let $R:\chi(M) \times \chi(M) \times \Gamma(\mathcal{Q}) \rightarrow \Gamma(\mathcal{Q})$, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ be the curvature of $\nabla$.

Question. **What influence does $R$ exert on the structure of $(M,\mathcal{F})$?**

We consider this question in the particular case where $\mathcal{F}$ is Riemannian and $\nabla$ is the unique torsion-free metric-preserving basic connection on $\mathcal{Q}$.

Let $M$ be a compact manifold and let $\mathcal{F}$ be a codimension-$q$ Riemannian foliation of $M$. There is a metric $g$ on $\mathcal{Q}$ such that the natural parallel transport along a curve lying in a leaf of $\mathcal{F}$ is an isometry. This is equivalent to the existence of a bundle-like metric in the sense of Reinhart [11].
Lemma [9]. There is a unique metric-preserving basic connection \( \nabla \) on \( Q \) with zero torsion \( T(X,Y) = \nabla_X Y - \nabla_Y X - \pi [X,Y] = 0 \) for all \( X, Y \in \chi(M) \).

Remark. \( \nabla \) is transversely projectable [9], basic [7] \( (R(X,Y) = 0 \) for all \( X \in \Gamma(E), Y \in \chi(M) \)).

Let \( p \in M \). Let \( \pi_p \) be a two-dimensional subspace of \( Q_p \) and let \( \{ X, Y \} \) be an orthonormal basis of \( \pi_p \). The transverse sectional curvature of \( \pi_p \) is defined by \( K(\pi_p) = -g_p(R(X,Y)X,Y) \) where \( X, Y \in T_p(M) \) satisfy \( \pi(X) = X, \pi(Y) = Y \).

Let \( \tilde{\mathcal{F}} \) be the universal cover of \( M \) and let \( \tilde{\mathcal{F}} \) be the lift of \( \mathcal{F} \) to \( \tilde{\mathcal{F}} \).

Theorem A [2]. If \( VR = 0 \) and \( K \leq 0 \), then \( \tilde{M} \) is diffeomorphic to a product \( \tilde{L} \times \tilde{\mathcal{F}} \) where \( \tilde{L} \) is the common universal cover of the leaves of \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) is the product foliation.

Application to Reeb's structure theorem [10] for codimension-one foliations defined by a closed one-form: Let \( M \) be a compact manifold and let \( \mathcal{F} \) be a codimension-one foliation of \( M \) defined by a nonsingular closed one-form \( \omega \). Then \( E = \ker(\omega) \). Let \( \bar{Y} \in \chi(M) \) be such that \( \omega(Y) = 1 \). Then \( Y = \pi(\bar{Y}) \in \Gamma(Q) \).

Define a metric \( g \) on \( Q \) by requiring \( g(Y,Y) = 1 \). Define a connection \( \nabla \) on \( Q \) by requiring \( \nabla_X Y = 0 \) for all \( X \in \chi(M) \).

Lemma. \( g \) is parallel along the leaves of \( \mathcal{F} \) and \( \nabla \) is the unique torsion-free metric-preserving basic connection on \( Q \).

Proof: Let \( X \in \Gamma(E) \). Then \( 0 = d\omega(X,\bar{Y}) = Xu(\bar{Y}) - \bar{Y}u(X) - u[X,Y] \) and so \( [X,\bar{Y}] \in \Gamma(E) \). Let \( f \in C^\infty(M) \). Then \( \nabla_X fy = f\nabla_X y + (xf)Y = (xf)Y = \pi((xf)\bar{Y}) = \pi([X,f\bar{Y}] - f[X,\bar{Y}]) = \pi([X,f\bar{Y}]) - \pi([X,\bar{Y}]) \) is basic.

Clearly \( \nabla \) preserves \( g \) and so \( g \) is parallel along the leaves. Let \( Z_1, Z_2 \in \chi(M) \). Then \( T(Z_1, Z_2) = T(h\bar{Y}, k\bar{Y}) \) where \( h, k \in C^\infty(M) \) and so \( T(Z_1, Z_2) = hkT(\bar{Y}, \bar{Y}) = 0 \) proving the lemma.

Since \( Y \) is a nowhere zero parallel section, it follows that \( R = 0 \). Hence \( VR = 0 \).
and $K = 0$. Thus Theorem A implies that $\mathbb{R} = \mathbb{L} \times \mathbb{R}$ and $\mathbb{F}$ is the product foliation which is Reeb's result.

**Remark.** We may rephrase Theorem A in terms of foliages [8]: If the foliage $\mathcal{W} = \mathcal{M}/\mathcal{F}$ admits a Riemannian structure with parallel curvature and non-positive sectional curvature, then $\mathcal{W}$ will have (in terms of foliages) a "covering" which will be a smooth manifold diffeomorphic to $\mathbb{R}^q$.

We now consider the relationship between curvature and cohomology. The relevant cohomology theory here is base-like cohomology [11], [12]. A differential $r$-form $\omega$ on $\mathcal{M}$ is called base-like if on each coordinate neighborhood $U$ with coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^q)$ respecting the foliation $\mathcal{F}$, the local expression of $\omega$ is of the form

$$
\sum_{1 \leq i_1 < \ldots < i_r \leq q} a_{i_1 \ldots i_r} (y^1, \ldots, y^q) dy^{i_1} \wedge \ldots \wedge dy^{i_r}.
$$

Equivalently, $i_X \omega = i_X dw = 0$ for all $X \in \Gamma(E)$ [13]. Since $d$ preserves such forms, we obtain the base-like cohomology algebra $H^*_\text{bas} (\mathcal{M}) = \bigoplus_{r=0}^q H^r_\text{bas} (\mathcal{M})$.

**Theorem B.** If $\nabla R = 0$ and $K > 0$, then $H^*_\text{bas} (\mathcal{M})$ is finite dimensional and $H^1_\text{bas} (\mathcal{M}) = 0$.

**Remark.** We may rephrase Theorem B in terms of foliages [8]. Let $\mathcal{W} = \mathcal{M}/\mathcal{F}$ be the space of leaves (a foliage). We can think of $H^*_\text{bas} (\mathcal{M})$ as the "De Rham cohomology" of $\mathcal{W}$, $H^*_\text{De R} (\mathcal{W})$. Of course, if $\mathcal{W}$ is a smooth manifold, this agrees with the De Rham cohomology algebra of $\mathcal{W}$. In this terminology, Theorem B states: If $\mathcal{W}$ admits a Riemannian structure with parallel curvature and positive sectional curvature, then $H^*_\text{De R} (\mathcal{W})$ is finite dimensional and $H^1_\text{De R} (\mathcal{W}) = 0$.

**Example.** Let $G$ be a compact connected Lie group of dimension $q$ and let $g$
be the Lie algebra of $G$. Let $M$ be a compact manifold and suppose $w$ is a
smooth $q$-valued one-form of rank $q$ on $M$ satisfying $d w + \frac{1}{2} [w, w] = 0$. Then
$w$ defines a smooth codimension-$q$ foliation $\mathcal{F}$ on $M$ which is a Lie foliation
modeled on $G$ [5]. Let $\langle , \rangle$ be a bi-invariant Riemannian metric on $G$. Then
$\langle , \rangle$ induces a holonomy-invariant metric on $Q$ with parallel curvature and
$K > 0$. For example, if $G = S^1$ then $\mathcal{F}$ is a codimension-one foliation defined
by a nonsingular closed one-form. If $\pi_1(G)$ is finite (e.g., if $G$ is semi-
simple), then $H^*_\text{bas}(M) \cong H^*(G)$ [1].

Example. This example uses the suspension construction of Haefliger [6].
Define a left action of $\pi_1(S^1) = Z$ on $S^2$ by

$$
1 \mapsto \begin{pmatrix}
\cos 2\pi \alpha & \sin 2\pi \alpha & 0 \\
-\sin 2\pi \alpha & \cos 2\pi \alpha & 0 \\
0 & 0 & 1
\end{pmatrix} \in SO(3)
$$

where $0 < \alpha < 1$ is irrational. Let $M = \mathbb{R} \times Z S^2$ be the associated bundle over
$S^1$ with fiber $S^2$. The foliation of $\mathbb{R} \times S^2$ whose leaves are the sets $\mathbb{R} \times \{x\}$,
$x \in S^2$ passes to a foliation $\mathcal{F}$ of $M$. Since $Z$ acts on $S^2$ by isometries,
the normal bundle of $(M, \mathcal{F})$ admits a transverse metric with $K = 1$. There are
exactly two compact leaves. If $L$ is a non-compact leaf, then $\overline{L}$ is diffeomor-
phic to the two-dimensional torus and the foliation of $\overline{L}$ by the leaves of $\mathcal{F}$ is
Riemannian with $K = 0$.

We now prove Theorem B. Since $\nabla R = 0$, we have that $N = \overline{\mathbb{R}}/\mathcal{F}$ is a complete,
Riemannian, Hausdorff manifold and the natural map $f:\overline{\mathbb{R}} \to N$ is a fiber bundle [2].
Each covering transformation $\sigma$ of $\overline{\mathbb{R}}$ induces an isometry $\Upsilon(\sigma)$. We thus obtain
a homomorphism $\Upsilon: \pi_1(M) \to I(N)$ such that $f \circ \sigma = \Upsilon(\sigma) \circ f$ for all $\sigma \in \pi_1(M)$
where $I(N)$ denotes the isometry group of $N$. Let $\Sigma = \text{image } (\Upsilon)$ and let
$K = \overline{\Sigma} \subseteq I(N)$. Let $A^r_K(N)$ be the space of $K$-invariant $r$-forms on $N$ and let
$A^r_{\text{bas}}(M)$ be the space of base-like $r$-forms on $M$. 

28
TRANSVERSE CURVATURE

Lemma. There is an isomorphism of cochain complexes

\[ \cdots \to A^r_{\text{bas}}(M) \overset{d}{\to} \cdots \]

\[ \cdots \to A^r_K(N) \overset{d}{\to} \cdots . \]

Thus \( H^*_\text{bas}(M) = H^*_K(N) \).

Proof: Let \( p: \tilde{M} \to M \) be the covering projection. Let \( \omega \in A^r_{\text{bas}}(M) \). Then \( p^*\omega = f^*\eta \) for a unique \( r \)-form \( \eta \) on \( N \). Since \( p^*\omega \) is \( \pi_1(M) \)-invariant, it follows that \( \eta \) is \( \Sigma \)-invariant and hence \( K \)-invariant. Conversely, let \( \eta \in A^r_K(N) \). Then \( f^*\eta \in A^r_{\text{bas}}(\tilde{M}) \). Since \( \eta \) is \( \Sigma \)-invariant, it follows that \( f^*\eta \) is \( \pi_1(M) \)-invariant and hence \( f^*\eta = p^*\omega \) for a unique \( \omega \in A^r_{\text{bas}}(M) \) proving the lemma.

Lemma. \( N \) and \( K \) are compact.

Proof: Let \( Q \) be the normal bundle of \( \tilde{F} \) and let \( \bar{g} \) be the lift of \( g \) to \( \tilde{G} \). The Riemannian metric on \( N \) is the one induced by \( \bar{g} \). Since \( \nabla R = 0 \), it follows that \( N \) has parallel curvature. Thus \( N \) is a complete, simply connected, Riemannian locally symmetric space and hence \( N \) is Riemannian symmetric. Since \( K > 0 \), it follows that \( N \) has positive sectional curvature. Thus \( N \) is compact [14] and \( K \) is compact proving the lemma.

Since \( K \) is compact, the inclusion \( A^*_K(N) \to A^*(N) \) induces an injection \( H^*_K(N) \to H^*(N) \) [4]. Since \( N \) is compact, \( H^*(N) \) is finite dimensional and hence \( H^*_\text{bas}(N) \) is finite dimensional. Since \( \pi_1(N) = 0 \), we have that \( H^1_{\text{bas}}(M) = 0 \).

References


Robert A. Blumenthal
Department of Mathematics
St. Louis University
St. Louis, Missouri 63103