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MAREK GOLASINSKI

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SOME REMARKS ON THE RATIONAL HOMOTOPY

TYPE OF DIAGRAMS AND REDUCED K_0 .

by

Marek GOLASINSKI

1 - THE RATIONAL HOMOTOPY TYPE OF DIAGRAMS

Let $\mathcal{C}$ be a closed model category in the Quillen's sense (see 5). If $\mathcal{I}$ is a small category $\mathcal{C}^{\mathcal{I}}$ denotes the functors category. A map in $\mathcal{C}^{\mathcal{I}}$ $f : C \rightarrow C'$ is a fibration, respectively a weak equivalence if $f(i)$ is a fibration, respectively a weak equivalence for every $i \in \text{ob } \mathcal{I}$. A cofibration is a map that has the left lifting property with respect to all trivial fibrations. We have the following result of Quillen-Bousfield-Kan (see 1, p. 313).

THEOREM 1.1.- $\mathcal{C}^{\mathcal{I}}$ equipped as above is a closed model category.

Let $\mathcal{G}$ be a discrete group. $\mathcal{G}\text{-Set}$ is the category of left $\mathcal{G}$ -sets and $\mathcal{I}$ is the full subcategory of $\mathcal{G}\text{-Set}$ determined by $\mathcal{G}/\mathbb{H}$ as $\mathbb{H}$ varies over all subgroups of $\mathcal{G}$. Denote by $\mathcal{G}\text{-SS}$ the category of left $\mathcal{G}$ -simplicial sets and $\mathcal{G}\text{-Top}$ the category of left $\mathcal{G}$ -topological spaces. Define functors $J : \mathcal{G}\text{-SS} \rightarrow \text{SS}^{\mathcal{I}}$ by $J(X)(\mathcal{G}/\mathbb{H}) = X^{\mathbb{H}}$, where $X^{\mathbb{H}} = \{x \in X ; h x = x, \text{ for all } h \in \mathbb{H}\}$ and $T : \text{SS}^{\mathcal{I}} \rightarrow \mathcal{G}\text{-SS}$ by $T(F) = F(\mathcal{G})$ provided with its natural $\mathcal{G}$ -action acquired from $\mathcal{G}\text{-Set}(\mathcal{G}, \mathcal{G}) = \mathcal{G}$.

Let $f : T(F) \rightarrow X$ be a map in $\mathcal{G}\text{-SS}$. Define $f' : F \rightarrow J(X)$ by $f'(\sigma) = f F(q)(\sigma)$ for $\sigma \in F(\mathcal{G}/\mathbb{H})$ and $q : \mathcal{G} \rightarrow \mathcal{G}/\mathbb{H}$ the natural quotient map. It is routine to check that f' is natural. Furthermore if $h : F \rightarrow J(X)$ then $h(\sigma) = h^{\vee} F(q)(\sigma)$ where $h^{\vee} : F(\mathcal{G}) \rightarrow X$ is the $\mathcal{G}$ -component of h , i.e. h is determined by $h^{\vee}$. We have thus established :

PROPOSITION 1.2.- J is full and faithful and right adjoint to T . Furthermore T preserves limits and both T and J preserve tensor products over SS .

Using J we view $G\text{-}SS$ as a subcategory of SS^I . A map $f : X \rightarrow X'$ of $G\text{-}SS$ is said to be a fibration, respectively a weak equivalence if $J(f)$ is a fibration, respectively a weak equivalence of SS^I . A cofibration in $G\text{-}SS$ is a map of $G\text{-}SS$ that has the left lifting property with respect to all trivial fibrations in $G\text{-}SS$. We have

PROPOSITION 1.3.- $G\text{-}SS$ equipped as above is a closed model category. Furthermore each monomorphism of $G\text{-}SS$ is a cofibration and thus any object of $G\text{-}SS$ is cofibrant.

Consider the adjoint pair $S : \text{Top} \rightleftarrows SS : | |$, where S is the singular functor and $| |$ is the geometric realization. The functors yield by naturality an adjoint pair $S_G : G\text{-}Top \rightleftarrows G\text{-}SS : | |_G$ with natural isomorphism $G\text{-}Top(|F|_G, X) \simeq G\text{-}SS(F, S_G(X))$.

Let $Q\text{-}DGA$ be the category of differential graded Q -algebras and $A^* : SS \rightleftarrows Q\text{-}DGA : F^*$ the pair of de Rham adjoint functors (see 6). These functors determine an adjoint pair $A^{*\mathbb{I}} : SS^{\mathbb{I}} \rightleftarrows Q\text{-}DGA^{\mathbb{I}} : F^{*\mathbb{I}}$. If $fQ\text{-}SS_N^{\mathbb{I}} \subset SS^{\mathbb{I}}$ is the full subcategory given by functors $X \in SS^{\mathbb{I}}$ such that $X(G/H)$ is nilpotent, rational and of finite Q -type for every subgroup $H \subset G$ and $fQ\text{-}DGA^{\mathbb{I}} \subset Q\text{-}DGA^{\mathbb{I}}$ is the full subcategory given by those functors $A \in Q\text{-}DGA^{\mathbb{I}}$ that $A(G/H)$ is equivalent to a minimal algebra with finitely many multiplicative generators in each dimension for every subgroup $H \subset G$. Then we obtain a generalization of the Sullivan-de Rham result (cf. 6) :

THEOREM 1.4.- Let G be a finite group. The adjoint pair $A^{*\mathbb{I}} : SS^{\mathbb{I}} \rightleftarrows Q\text{-}DGA^{\mathbb{I}} : F^{*\mathbb{I}}$ induces an equivalence of homotopy categories

$$Ho(fQ\text{-}SS_N^{\mathbb{I}}) \xrightarrow[\sim]{\quad} Ho(fQ\text{-}DGA^{\mathbb{I}}).$$

Let $fQ\text{-}SS_N$ be the full subcategory of $G\text{-}SS$ given by nilpotent, rational and of finite Q -type G -simplicial sets. The functor $J : G\text{-}SS \rightarrow SS^I$ is full and faithful, then we have.

COROLLARY 1.5.- The above equivalence induces a bijection between equivariant rational homotopy types of $fQ\text{-}SS_N$ on the one hand and isomorphism classes of minimal systems of DGA 's in the Triantafillou sense (see 7) on the other.

2 - REDUCED K_0 OF O-FORMS ON A FINITE SIMPLICIAL COMPLEX

Sullivan has proved (see 6, cf. also 4) that for a finite simplicial complex X with vertices $v_1, \dots, v_n$ and corresponding barycentric coordinates $b_1, \dots, b_n$ the algebra of rational forms on X

$$A_Q^0 X = Q[b_1, \dots, b_n] \otimes \Lambda(db_1, \dots, db_n) / I,$$

where $Q[b_1, \dots, b_n]$ is the ring of rational polynomials in $b_1, \dots, b_n$, $\Lambda(db_1, \dots, db_n)$ is the exterior algebra on $db_1, \dots, db_n$ and I is the ideal generated by $b_1 + \dots + b_n - 1$, $db_1 + \dots + db_n$, $b_{i_1} \dots b_{i_p} db_{j_1} \dots db_{j_q}$ if there is no $p+q$ -simplex of X with vertices $v_{i_1}, \dots, v_{i_p}, v_{j_1}, \dots, v_{j_q}$.

Kan and Miller have shown (see 3) that the weak homotopy type of a finite simplicial set X can be reconstructed from R -algebra $A_R^0 X$ of O -forms on X , when R is a unique factorization domain.

If $\text{pro } R\text{-}\mathcal{A}$ denote the pro-category of R -algebras then Jardine has proved (see 2) that there are functors $\hat{A} : \text{SS} \xrightarrow{\sim} \text{pro } R\text{-}\mathcal{A} : \hat{F}$ inducing an equivalence of suitable homotopy categories

$$\text{Ho}(\text{SS}) \xrightleftharpoons[\sim]{} \text{Ho}(\text{pro } R\text{-}\mathcal{A}).$$

Our purpose is to show that there exists a simplicial set $G_\infty(\omega)$ (the simplicial Grassman variety) such that for a finite simplicial complex X , $\tilde{K}_0(A_k^0 X) = [\tilde{X}, G_\infty(\omega)]$ where $\tilde{K}_0$ is the reduced Grothendieck's group of $A_k^0 X$ and k is a field.

The Grassman variety $G_m(n)$ is defined as a functor from the K -algebras category $k\text{-}\mathcal{A}$ to the category of sets, for $1 \leq n < m$ and R in $k\text{-}\mathcal{A}$ by

$$G_m(n)(R) = \{Q \subset R^m; Q \text{ is } R\text{-split projective of rank } n\}.$$

The assignment $Q \mapsto Q \otimes_R S$ associated to the k -algebra homomorphism $\theta : R \rightarrow S$ defines the function $\theta_* : G_m(n)(R) \rightarrow G_m(n)(S)$.

Let $P(R)$ ($P_n(R)$) be the set of isomorphism classes of R -modules finitely generated and projective over R (of rank n), $\tilde{K}_0(R)$ the Grothendieck's group of $P(R)$ and $K_0(R)$ reduced K_0 .

The natural embedding $R^m \rightarrow R^{m+1}$ induces a map $G_m(n) \rightarrow G_{m+1}(n)$.

Put $G_\infty(n) := \text{colim}_n G_m(n)$.

Then there is a natural surjective function $\tau_R : G_\infty(n)(R) \rightarrow P_n(R)$ which is induced by the assignment $(P \rightarrow R^m) \mapsto P$.

Let X be a finite simplicial complex, thought of a member of as a member of the category SS of simplicial sets, and let k be an arbitrary field. Recall that there is a natural simplicial set map

$$\eta_X : X \longrightarrow \text{Spec}(A_k^0 X) (A_k^0 \Delta_*) = k\text{-}A(A_k^0 X, A_k^0 \Delta_*)$$

where Δ_n is the standard n -simplex.

Let Sch_k denotes the category of schemes over k , thought of as a full subcategory of the functors category from $k\text{-}A$ to Set .

η_X may be used to define a function

$$\psi : \text{Sch}_k(\text{Spec } A_k^0 X, Y) \longrightarrow SS(X, Y(A_k^0 \Delta_*))$$

for arbitrary k -schemes Y in such a way that ψ associates to a k -scheme map $f : \text{Spec } A_k^0 X \longrightarrow Y$ the composition

$$X \xrightarrow{\eta_X} \text{Spec}(A_k^0 X) (A_k^0 \Delta_*) \xrightarrow{f_*} Y(A_k^0 \Delta_*) .$$

PROPOSITION 2.1.- ψ induces a bijection

$$\psi_* : \text{Sch}_k(\text{Spec } A_k^0 X, Y) \xrightarrow{\sim} SS(X, Y(A_k^0 \Delta_*))$$

for all finite simplicial complexes X and all schemes Y .

Then the above map τ gives rise to a natural surjective function

$$\tau_X : SS(X, G_\infty(n) (A_k^0 \Delta_*)) \longrightarrow P_n(A_k^0 X)$$

in view of above theorem and the Yoneda lemma.

THEOREM 2.2.- The map $\tau_X : SS(X, G_\infty(n) (A_k^0 \Delta_*)) \longrightarrow P_n(A_k^0 X)$ factors through a bijection

$$(\tau_X)_* : [X, G_\infty(n) (A_k^0 \Delta_*)] \xrightarrow{\sim} P_n(A_k^0 X) .$$

The map $P_n(A_k^0 X) \longrightarrow P_{n+1}(A_k^0 X)$ which is defined by $P \longmapsto A_k^0 X \oplus P$ clearly fits into a commutative diagram

$$\begin{array}{ccc} [X, G_\infty(n) A_k^0 \Delta_*] & \xrightarrow{\quad} & [X, G_\infty(n+1) A_k^0 \Delta_*] \\ (\tau_X)_* \downarrow & & \downarrow (\tau_X)_* \\ P_n(A_k^0 X) & \xrightarrow{\quad} & P_{n+1}(A_k^0 X) \end{array}$$

Formal nonsense now shows that

$$[X, G_\infty^{(\infty)}(A_k^0 \Delta_*)] = \text{colim} (P_n(A_k^0 X) \longrightarrow P_{n+1}(A_k^0 X))$$

may be identified with $\tilde{K}_0(A_k^0 X)$ via a map which is induced by the assignments

$$P \longmapsto P - (A_k^0 X)^{\text{rk} P}, \text{ where } G_\infty^{(\infty)} = \text{colim}_n G_\infty^{(n)}.$$

There is also a similar result for finite G -simplicial complexes.

Let G be a finite group such that $\chi(k) \chi|G|$, $\chi(k)$ is the characteristic of k , $V_1, \dots, V_\ell$ all irreducible G -modules over k and $V := \bigoplus_{i=1}^{\ell} V_i$. The G -Grassman variety $G_m^G(n)$ is defined as a functor from the k -algebras category $k\text{-}\mathcal{A}$ to the category of sets, for $1 \leq n < m$ and R in $k\text{-}\mathcal{A}$ by

$$G_m^G(n)(R) := \{Q \subset R^m \otimes V; Q \text{ is } R\text{-split projective of rank } n\}.$$

Remark that for a k - G -algebra R the category of R - G -modules is equivalent to the category of $R * G$ -modules, where $R * G$ is the twisted product of R and G . Let $P^G(R)$ denotes the set of isomorphism classes of $R * G$ -modules finitely generated and projective over R , $K_0^G R$ the Grothendieck's group of $P^G(R)$ and $\tilde{K}_0^G(R)$ reduced K_0 . Then we have.

THEOREM 2.3.- For a finite group G such that $\chi(k) \chi|G|$ and a finite G -simplicial complex X

$$\tilde{K}_0^G(A_k^0 X) = [X, G_\infty^{(\infty)}(A_k^0 \Delta_*)]_G.$$

REFERENCES.

- [1] A.K. BOUSFIELD, D.M. KAN - Lecture Notes in Math. 304 1972.
- [2] J.F. JARDINE - Algebraic homotopy theory, Can. J. Math. 33 1981, pp. 302-319.
- [3] D.M. KAN, E.Y. MILLER - Homotopy types and Sullivan's algebras of 0-forms, Topology vol. 16 1977, pp. 193-197.
- [4] M.A. PENNA - On a theorem of Sullivan, Can. Math. Bull. Vol. 21 1978 pp. 201-206.
- [5] D.G. QUILLEN - Lecture Notes in Math. 43 1967.
- [6] D. SULLIVAN - Differential forms and the topology of manifolds, Proc. of Congress on Manifolds, Tokyo, Japan 1973 pp. 37-49.
- [7] G. TRIANTAFILLOU - Equivariant minimal models, Trans. of A.M.S. Vol. 274 2 1982, pp. 509-532.