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# QUASI PERIODIC FLOW NEAR A CODIMENSION ONE SINGULARITY OF A DIVERGENCE FREE VECTOR FIELD IN DIMENSION FOUR

by

B.L.J. Braaksma and H.W. Broer

## CHAPTER 1 - INTRODUCTION, RESULTS

### §1.1 Introduction

It is the aim of this paper to prove that within the class of all $C^\infty$ one parameter families of divergence free (or volume preserving) vector fields on $\mathbb{R}^4$, the phenomenon of invariant three-dimensional tori with a quasi-periodic flow, occurs openly.
Our study will be local: we consider generic unfoldings of a specific codimension one singularity, all within the class of vector fields in dimension 4 with divergence zero.
From a slightly different viewpoint such unfoldings are (local) one parameter families in which the unfolded singularity appears as a bifurcation.
For each of our unfoldings it will be shown that, if the parameter ranges over a neighbourhood of the bifurcation value, uncountably many invariant 3-tori with quasi-periodic flow come into existence.

Basically we follow the ideas contained in Moser [12], which have to be adapted for our bifurcation problem.

In Broer [3] a general study was made of bifurcations of singularities in volume preserving vector fields, by investigating generic one parameter families of such vector fields. Also see [5]. It appeared that in our studies with one parameter, compared to the classical bifurcation theory for vector fields without the restriction of volume preservation, only the dimensions 3 and 4 are exceptional. Compare e.g. Arnol'd [1]. The present study treats one of the exceptional bifurcations in dimension 4. It illustrates the fact that in this divergence free case structural stability is not generic. In dimension 3 we found a similar bifurcation, for which in [4] analogous results were obtained.

Below we shall also make some remarks on the same 4-dimensional bifurcation, but now without the restriction to divergence zero. In this case the bifurcation has codimension two. We consider a result claimed by Guckenheimer [8] concerning a normally hyperbolic invariant manifold which occurs in an open set of two parameter unfoldings. This manifold contains uncountably many quasi-periodic orbits and we shall present an indication of a proof for this, illustrating our methods developed from [12].

All the phenomena that we describe, are strongly associated with the fact that by normal form techniques (see e.g. [15, 3])one may assume that the considered unfoldings are symmetric up to some order. This means that, in a certain sense, they are close to cases with infinite codimension. Or, in a different terminology: our unfoldings are nearly integrable. (Also compare Bröer en Van Strien [6].)

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§ 1.2 Statement of the problem

On $\mathbb{R}^4$ consider a divergence free $C^\infty$ vector field, which has the origin as a singular point. Assume that the eigenvalues of the linearized vector field are $\pm i\alpha_1$ and $\pm i\alpha_2$, for some $\alpha_1 > 0$ and $\alpha_2 > 0$.
One easily sees that such singularities have codimension one, i.e. that they may occur in generic one parameter families - or arcs - of divergence free vector fields. This is a consequence of the fact that in the divergence free case the trace of the linear part in a singularity must be zero.
So let us consider such a $C^\infty$ arc $X = X^\mu(x)$, where $\mu$ is a real parameter, which unfolds the above singularity in $x = 0 \in \mathbb{R}^4$ of the vector field $X^0$.
In Broer [3] a normal form theorem was obtained, which implies the following:
If for all $j_1, j_2 \in \mathbb{Z}$ with $1 \leq |j_1| + |j_2| \leq m$ we have that $j_1\alpha_1 + j_2\alpha_2 \neq 0$, i.e. no resonances up to order m, then modulo a $C^\infty$, volume preserving, $\mu$-dependent change of coordinates, one may write $X^\mu = \tilde{X}^\mu + p^\mu$, where

i. Both $\tilde{X}^\mu$ and $p^\mu$ are divergence free,
ii. The Taylor expansion of $p = p^\mu(x)$ vanishes up to order $m-1$,
iii. In toroidal coordinates $(r_1, \varphi_1)$ and $(r_2, \varphi_2)$ on $\mathbb{R}^4$ , the vector field $\tilde{X}^\mu$ has the system form

$$(1.1) \qquad \begin{cases} \dot{\varphi}_i = a_i(r_1^2, r_2^2, \mu) \\ \dot{r}_i = r_i g_i(r_1^2, r_2^2, \mu) \quad , \quad i = 1, 2, \end{cases}$$

where $a_i(0,0,0) = \alpha_i$ and $g_i(0,0,0) = 0$ for $i = 1, 2$.

The last property expresses that $\tilde{X}$ possesses toroidal symmetry. Also compare Takens [15]. The natural number $m \geq 4$ lateron will be fixed sufficiently large. Observe that our resonance condition is open and dense.

The angles $\varphi_1$ and $\varphi_2$ are defined modulo $2\pi$.
If we forget both angular components, from $\tilde{X}^\mu$ we obtain a reduced vector field $\bar{X}^\mu$, defined in the $(r_1, r_2)$-plane. In this plane we blow up, or rescale with $\sqrt{|\mu|}$ , i.e. for $\mu \neq 0$ we introduce new variables $\bar{r}_1$ and $\bar{r}_2$ defined by $r_1 = \bar{r}_1\sqrt{|\mu|}$ and $r_2 = \bar{r}_2\sqrt{|\mu|}$.
Let $Y^\mu$ be the transformed vector field:

$$Y^\mu(\bar{r}_1, \bar{r}_2) = \frac{1}{\sqrt{|\mu|}} \bar{X}^\mu(\bar{r}_1\sqrt{|\mu|}, \bar{r}_2\sqrt{|\mu|}).$$

Note that $Y^\mu(\bar{r}_1, \bar{r}_2) \to 0$ as $\mu \downarrow 0$ or $\mu \uparrow 0$, uniformly on compact sets, but if we consider

$$Z^\mu(\bar{r}_1, \bar{r}_2) = \frac{1}{|\mu|} Y^\mu(\bar{r}_1, \bar{r}_2) ,$$

then for $Z^\mu$ we write

$$(1.2) \qquad \begin{cases} \dot{\bar{r}}_1 = \bar{r}_1(c_1\bar{r}_1^2 + 2c_2\bar{r}_2^2 - c_3 \operatorname{sgn}\{\mu\}) \\ \dot{\bar{r}}_2 = -\bar{r}_2(2c_1\bar{r}_1^2 + c_2\bar{r}_2^2 - c_3 \operatorname{sgn}\{\mu\}) \end{cases} + O(|\mu|)$$

uniformly on compact sets, where $c_1$, $c_2$ and $c_3$ are real constants.

We again impose some generic (and open) conditions:

$$c_1 \neq 0,\ c_2 \neq 0,\ c_3 \neq 0 \text{ and } |c_1| \neq |c_2| \ .$$

In studying the phase portrait of $Z^\mu$ it is no essential restriction to assume that $c_3 \geq 0$ and that $c_1 \geq |c_2|$, otherwise replace $\mu$ by $-\mu$, $Z^\mu$ by $-Z^\mu$, or permute $r_1$ and $r_2$.

Our present study deals with the case where $c_1 > c_2 > 0$, which in [3] was labelled I. Also we restrict ourselves to the situation $\mu > 0$, the parameter $\mu$ varies in a righthand neighbourhood of 0.
Define $Z^0$ to be the limit of $Z^\mu$ as $\mu \downarrow 0$ and observe that the family $\{Z^\mu\}_{\mu \geq 0}$ is smoothly parametrized by $\mu$.
According to [3], theorem 3.5, this vector field $Z^0$ is $C^\infty$-stable within the class of all such reductions of symmetric, divergence free vector fields to the $(\bar{r}_1, \bar{r}_2)$-plane. Fig. 1 depicts the phase portrait of $Z^0$. (cf. [3], fig. 3, case I,-.)

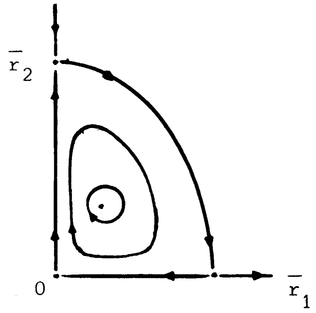


fig. 1

This stability means, that for some $\mu_0 > 0$, sufficiently small, and for all $\mu \in [0, \mu_0]$, the vector field $Z^\mu$ possesses an invariant 'circle'. The inner region of this 'circle' is foliated by an orbit cylinder, which shrinks towards an elliptic singularity.

For the corresponding symmetric vector field $\tilde{X}^\mu$ (now $\mu \neq 0$) we blow down again and add the two rotational components. Thus we find an invariant 3-sphere, the inner region of which is foliated by a 1-parameter family of invariant 3-tori, shrinking towards an invariant 2-torus. (Also the foliation contains two transversal 3-discs.)
Note that, as a consequence of the blow down operation, the characteristic distance in the phase portrait of $\tilde{X}^\mu$ asymptotically equals $\sqrt{\mu}$.
In the 4-disc under consideration we shall study $X^\mu = \tilde{X}^\mu + p^\mu$. Observe that $p^\mu$ can be regarded as a small perturbation for $\mu > 0$ and small: the size of this perturbation is controlled by taking $\mu$ close enough to zero. It is our aim to establish persistence of 3-tori which have sufficiently independent frequencies. The result is formulated in §1.4 as theorem D.

## §1.3 A suitable normal form

We continue our preliminaries by constructing an even more suitable normal form for the system $X^\mu$, restricted to the open 4-disc with the torus-foliation. We shall work in the rescaled coordinates $\bar{r}_1$ and $\bar{r}_2$, such that the size of our 4-disc is of order 1.
Observe that by taking the limit for $\mu \downarrow 0$, in these coordinates we uniformly obtain

$$\tilde{X}^0 = \alpha_1 \frac{\partial}{\partial\varphi_1} + \alpha_2 \frac{\partial}{\partial\varphi_2}$$

and that $\{\tilde{X}^\mu\}_{\mu\geq 0}$ is smoothly parametrized by $\sqrt{\mu}$. For simiplicity we again write $r_1$ and $r_2$ in stead of $\bar{r}_1$ and $\bar{r}_2$ respectively. Observe that now for $j = 1,2$ we have $a_j = a_j(\mu r_1^2, \mu r_2^2, \mu)$.

From the fact that $\tilde{X}^\mu$ is divergence free we conclude that the reduction $Z^\mu$ preserves the 2-form

$$r_1 r_2 dr_1 \wedge dr_2 = \frac{1}{4}\, dr_1^2 \wedge dr_2^2.$$

This means that for $r_1 > 0$ and $r_2 > 0$ the vector field $Z^\mu$ has a hamiltonian function

$$H^\mu(r_1,r_2) = \frac{1}{4} r_1^2 r_2^2 (c_1 r_1^2 + c_2 r_2^2 - c_3) + O(\mu) \quad ,$$

uniformly on compact sets.

For small $\mu \geq 0$ we now introduce action angle variables in the $(r_1,r_2)$-plane, following e.g. Arnol'd & Avez [2], app. 26, i.e. we first define the action integral

$$(1.3) \qquad y_0(h,\mu) := \frac{1}{2\pi} \oint_{H^\mu(r_1,r_2)=h} \frac{1}{2} r_1^2 r_2 dr_2 \quad ,$$

being the $\frac{1}{2\pi}$-th part of the area bounded by the level curve $H^\mu(r_1,r_2) = h$, containing a closed orbit of $Z^\mu$.

Let $\varphi_0$ denote the corresponding phase angle, defined mod $2\pi$.

Observe that, up to multiplicative constants,

$$r_1 r_2 dr_1 \wedge dr_2 = dy_0 \wedge d\varphi_0 \quad ,$$

while $Z^\mu$ now can be written as

$$\begin{cases} \dot{\varphi}_0 = a_0(y_0,\mu) \\ \dot{y}_0 = 0 \ . \end{cases}$$

The following technical proposition will be needed in our further considerations, but first of all it implies that, in the region relevant to us, $(y_0, \varphi_0)$ is a set of smooth coordinates. The proof is postponed to chapter 2.

PROPOSITION 1.1:

i. $y_0$ is a smooth function of h and $\mu$;

ii. $\frac{\partial y_0}{\partial h}(h,0)$ is positive definite;

iii. $\frac{\partial a_0}{\partial y_0}(y_0,0)$ is definite, except for at most finitely many values of the variable $y_0$.

Now consider the frequencies $a_1$ and $a_2$ in the new coordinates $y_0$ and $\varphi_0$. We claim that, up to any order, these functions are independent of $\varphi_0$. A rough argument for this runs as follows: Using normal form techniques as in e.g. [3] near the 2-torus corresponding to the elliptic singularity of Z, one may symmetrize the $a_j$ up to any finite order. This can be achieved using a volume preserving, $\mu$-dependent change of coordinates, which only effects the plane $\varphi_1 = 0$, $\varphi_2 = 0$. Since we are transforming a finite number of terms from the Taylor series, the transformation will be real analytic near the 2-torus. In this way we find coordinates u and v in the $(r_1,r_2)$-plane such that up to a high order both $a_j = a_j(\mu^2(u^2+v^2);\mu)$, $j = 1,2$ and $y_0 = u^2+v^2$. Here we use the unicity of the Birkoff normal form. Since $y_0$ is real analytic in the whole region we consider, see fig. 1, it follows that the transformation $(r_1,r_2) \to (y_0,\varphi_0)$ puts the functions $a_1$ and $a_2$ into the desired normal form, i.e. they are independent of $\varphi_0$ up to any order. Remaining, flat terms will be included in the perturbation p. Note that, because of our blowing up procedure, these terms are flat in $\mu$, uniformly in $y_0$.

We write for j = 1,2 :

$$(1.4) \qquad a_j(\mu^2 y_0,\mu) = \alpha_j + \beta_j\mu + O(\mu^2),$$

uniformly. Here $\beta_1$ and $\beta_2$ are real constants. A new generic condition is imposed by requiring that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Also we shall need one further generic condition: $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

Finally we substitute $x_j = \varphi_j$ $(0 \leq j \leq 2)$ and $y_1 = \sqrt{\mu}$. We consider $X = \tilde{X} + p$ in the coordinates $x_0$, $x_1$, $x_2$, $y_0$ and $y_1$. The perturbation $p$ is of the form

$$(1.5) \qquad \sum_{j=0}^{2} f_j(x,y_0,y_1)\frac{\partial}{\partial x_j} + g_0(x,y_0,y_1)\frac{\partial}{\partial y_0} ,$$

where $x = (x_0,x_1,x_2)$ and where the functions $f_j$ and $g_0$ have period $2\pi$ in the $x_j$. Note that we may write

$$f_j(x,y_0,y_1) = y_1^m \bar{f}_j(x,y_0,y_1) \quad \text{and}$$

$$g_0(x,y_0,y_1) = y_1^m \bar{g}_0(x,y_0,y_1) ,$$

where $\bar{f}_j$ and $\bar{g}_0$ are continuous but only $C^\infty$ for $y_0 \neq 0$: in the expansion near $y_0 = 0$ terms with $\sqrt{y_0}$ show up.

So $X$ now obtains the system form

$$(1.6) \qquad \begin{cases} \dot{x}_0 = y_1^2\, a_0(y_0,y_1^2) + f_0(x,y_0,y_1) \\ \dot{x}_1 = a_1(y_1^4 y_0,y_1^2) + f_1(x,y_0,y_1) \\ \dot{x}_2 = a_2(y_1^4 y_0,y_1^2) + f_2(x,y_0,y_1) \\ \dot{y}_0 = g_0(x,y_0,y_1) \\ \dot{y}_1 = 0 , \end{cases}$$

which is nearly integrable. The factor $y_1^2$ in the first frequency is due to the difference between $Z^\mu$ and $Y^\mu$. See above. From now on we let $\tilde{X}$ denote the integrable part of (1.6), i.e. without the perturbations $f_j$, $g_0$, so changing its connotion slightly.

## § 1.4 Statement of the persistence theorems

The persistence of invariant 3-tori for the vector field $X^{\mu}$ will be investigated by means of its normal form (1.6). This system may be considered as a particular case of a more general system

$$(1.7)\qquad \begin{cases} \dot{x} = A(y) + F(x,y) \\ \dot{y} = G(x,y) \end{cases}$$

which is a perturbation of the completely integrable form

$$(1.7^{a})\qquad \begin{cases} \dot{x} = A(y) \\ \dot{y} = 0 \,. \end{cases}$$

So the perturbation terms F and G are assumed to be small in some sense. In (1.7) x and y are both 3-vectors and we suppose that the system is $2\pi$-periodic in $x = (x_0,x_1,x_2)$.

Such systems (1.7) have been investigated by Moser [12] in the case where A, F and G are real analytic. He proved a persistence theorem for invariant tori of such systems (cf. [12], theorem 5, also compare [$11^{a}$]). An explicit formulation in the case that (1.7) is volume preserving and real analytic will be given in § 1.6, theorem A". In this paper we need an analogue of this theorem under the assumptions that A, F and G are $C^{\infty}$.

First we state a condition for volume preservation of the normal form (1.7):

PROPOSITION 1.2:

The system (1.7) is volume preserving if and only if both

(i) $$\sum_{j=0}^{2} \left(\frac{\partial F_j}{\partial x_j} + \frac{\partial G_j}{\partial y_j}\right) = 0 \qquad \text{and}$$

(ii) $$\int G(x,y)\; dx_0 \wedge dx_1 \wedge dx_2 = 0.$$

We omit the proof.

Now we consider a 3-torus $y = c$, invariant for the system $(1.7^a)$. It contains the solutions $x(t) = \omega t + \text{cst.}$, $y(t) = c$, $t \in \mathbb{R}$, where $\omega = A(c)$. Assume that $\omega$ satisfies a strong non-resonance condition: For some $\tau$ and $\gamma$ with $3 < \tau < 4$ and $\gamma > 0$ and for all tri-indices $\nu \in \mathbb{Z}^3 \setminus \{0\}$ we have

(1.8) $$|(\nu,\omega)| \geq \gamma\, |\nu|^{-\tau} .$$

Here $(\nu,\omega) = \sum_{j=0}^{2} \nu_j\omega_j$ and $|\nu| = \sum_{j=0}^{2} |\nu_j|$.
For such $\omega$ the above solution certainly is quasi-periodic. It is well known that for $\tau > 3$ (or even $\tau > 2$) these frequencies $\omega$ form a Cantor set, the measure of which tends to full measure as $\gamma \to 0$. Also note that for $\tau$ fixed, the set of $\omega$ satisfying (1.8) for some $\gamma > 0$, has a conical structure: If $\omega$ satisfies (1.8) with $\tau$ and $\gamma$, then for every $s \neq 0$ so does $s\omega$ with $\tau$ and $|s|\gamma$. Now further assume that A is a diffeomorphism on a neighbourhood of c. Let $\Delta = \{a \in \mathbb{R}^3 \mid |a - \omega| \leq d\}$ be a disc where the inverse $A^{-1}$ is well defined and $C^\ell$ with $\ell = 363$. We then have

<u>THEOREM A</u>:

Let A and c satisfy the preceding assumptions and let K be a positive constant. Then there exists a positive constant $\delta$, depending only on $\gamma$, K and the $C^\ell$-norm of $A^{-1}$ on $\Delta$, such that for all perturbations F and G $\in C^\ell(\mathbb{R}^3 \times A^{-1}(\Delta), \mathbb{R}^3)$ which are $2\pi$-periodic in $x_0$, $x_1$ and $x_2$ and such that the corresponding system (1.7) preserves the volume while

$$(1.9)\qquad \begin{cases} |F(x,y)| + |G(x,y)| \le \delta \quad \text{and} \\ |D_x^{\zeta_1} D_y^{\zeta_2} F(x,y)| + |D_x^{\zeta_1} D_y^{\zeta_2} G(x,y)| \le K \,, \end{cases}$$

for all $(x,y) \in \mathbb{R}^3 \times A^{-1}(\Delta)$ and $|\zeta_1| + |\zeta_2| = \ell$, then the system (1.7) possesses a quasi-periodic solution with frequency $\omega$. If the $C^0$-norm $|F|_0 + |G|_0 \to 0$, then the distance of this solution to $y = c$ tends to zero.

Theorem A will be proven in § 2.5. One may apply theorem A to systems without a component $y_2$, as for example (1.6), in the following way. Consider

$$(1.10)\qquad \begin{cases} \dot{x} = b(y) + f(x,y) \\ \dot{y} = g(x,y) \,, \end{cases}$$

where $x = (x_0,x_1,x_2)$, $y = (y_0,y_1)$ and where b, f and g satisfy conditions similar to those for A, F and G above, except that of course now b is not a diffeomorphism. If we rescale the time t to $(1+y_2)t$ , so introducing an artificial coordinate $y_2$, we obtain a special form of (1.7) where

$$(1.11)\qquad \begin{cases} A(y) = (1+y_2)\, b(y_0,y_1),\ F(x,y) = (1+y_2)\, f(x,y_0,y_1), \\ G_j(x,y) = (1+y_2)\, g_j(x,y) \text{ for } j = 0,1 \text{ and where } G_2 = 0. \end{cases}$$

Note that now $y = (y_0,y_1,y_2)$.

We assume that $(c_0,c_1)$ is such that $\omega = b(c_0,c_1)$ satisfies (1.8) and also we assume the following non-degeneracy condition

$$(1.12)\qquad \det \begin{bmatrix} \frac{\partial b_0}{\partial y_0} & \frac{\partial b_0}{\partial y_1} & b_0 \\ \frac{\partial b_1}{\partial y_0} & \frac{\partial b_1}{\partial y_1} & b_1 \\ \frac{\partial b_2}{\partial y_0} & \frac{\partial b_2}{\partial y_1} & b_2 \end{bmatrix} \ne 0 \qquad \text{for } y_0 = c_0,\ y_1 = c_1.$$

Then, from theorem A we immediately conclude

THEOREM B:

Suppose that $V \subseteq \mathbb{R}^2$ is a neighbourhood of $(c_0,c_1)$ and that $b \in C^\ell(V,\mathbb{R}^3)$ satisfies (1.12), while $\omega = b(c_0,c_1)$ satisfies (1.8). Let K be a positive constant. Then there exists a positive constant $\delta$ such that for all perturbation terms f and $g \in C^\ell(\mathbb{R}^3 \times V)$ which are $2\pi$-periodic in $x_0,x_1,x_2$ and such that (1.10) is volume preserving and satisfies (1.9) for all $(x,y_0,y_1) \in \mathbb{R}^3 \times V$ and $|\zeta_1| + |\zeta_2| = \ell$, the system (1.10) possesses a quasi-periodic solution with frequency $s\omega$. Here s is a scalar and $s \to 1$ as $|f|_0 + |g|_0 \to 0$.

Note that for the conditions of volume preservation and (1.9) one has to use the translation (1.11). Cf. proposition 1.2.Also observe that the frequency vector $\omega$ in perturbing may have slightly changed to $s\omega$, but that in this way the frequency ratios have been kept constant. Recent work of Pöschel [14] strongly suggests that, also in our volume preserving context, the surviving invariant tori fill up a set of positive measure, tending to full measure as $|f|_0 + |y|_0 \to 0$.

The special case (1.6) of (1.10) is obtained by taking

$$(1.13) \qquad b_0(y) = y_1^2 a_0(y_0,y_1^2),\ b_j(y) = a_j(y_1^4\, y_0,y_1^2),\ g_j \equiv 0 \quad \text{for } j = 1,\ 2\ .$$

Consider (1.6) for $y_1$ positive and small, since we investigate $\tilde{X} = \tilde{X}^\mu + p^\mu$ for small, positive $\mu$ and $y_1 = \sqrt{\mu}$. For the determinant in (1.12) we have

$$2y_1^3(\alpha_2\beta_1 - \alpha_1\beta_2)\,\frac{\partial a_0}{\partial y_0}(y_0,0) + O(y_1^4)$$

as $y_1 \to 0$, uniformly, because of (1.4) and (1.13) . This explains the generic condition

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$$

made in § 1.3. Moreover, $c = (c_0, c_1)$ should be chosen in such a way that $c_1 > 0$ is small, $\frac{\partial a_0}{\partial y_0}(c_0, 0) \neq 0$ and that

$$\omega = (c_1^2 a_0(c_0, c_1^2),\ a_1(c_1^4 c_0, c_1^2),\ a_2(c_1^4 c_0, c_1^2)) \tag{1.14}$$

satisfies (1.8), in order that theorem B may be applied.

These conditions, however, give rise to some problems. Firstly we have to establish the existence of $c = (c_0, c_1)$, with $c_1 > 0$ and small and $c_0$ near a prescribed value, such that $\omega$, defined by (1.14), satisfies (1.8). Observe that if $\omega$ satisfies (1.8) and $c_1 \to 0$, then necessarily $\gamma \to 0$. Which brings us to a second aspect of the problem: We shall see that in the theorems A and B we have $\delta \to 0$ as $\gamma \to 0$ and a priori it is not clear that (1.9) can be fulfilled.

Here we have to take into account that the map A, defined via (1.13) and (1.11), is singular for $y_1 = 0$. Our genericity assumptions, however, guarantee that the image of A implodes in a controlable way for $y_1 \to 0$. What one needs is, in the theorems A and B, an estimate for $\delta$ in terms of $\gamma$. This problem is similar to that of the "small twist", cf. Moser [11]. In analogy with [11] we give a refinement of our theorems.

In order to obtain such a refinement, it is useful to consider theorem A as stated above, and to look at the result which serves as a basis for its proof. This result concerns a system with a parameter (cf. Moser [12], § 3)

$$(1.15)\qquad \begin{cases} \dot{x} = a + f(x,y,a) \\ \dot{y} = g(x,y,a)\ . \end{cases}$$

Here $x = (x_0,x_1,x_2)$ and $y = (y_0,y_1,y_2)$, while f and g are small perturbation terms, $2\pi$-periodic in $x_0$, $x_1$ and $x_2$. Also the system is assumed to be volume preserving, cf. proposition 1.2. Let the parameter $a \in \mathbb{R}^3$ in (1.15) vary in a neighbourhood of $\omega \in \mathbb{R}^3$, where $\omega$ satisfies (1.8) with $\gamma > 0$ and $3 < \tau < 4$. The perturbation terms f and g will be estimated by several parameters; let

$$(1.16)\qquad \begin{cases} \delta > 0,\quad \varepsilon = \delta^{\sigma},\ N = \delta^{-r},\ P = \delta^{-s},\ Q = \delta^{-q}\ , \\ \sigma = 0.93;\ r = 0.1;\ s = 2;\ q = 0.95\ . \end{cases}$$

Then we have

<u>THEOREM C</u>:

Let $\gamma^* > 0$ and $\ell = 363$. Then there exists a positive number $\delta^*$, only depending on $\gamma^*$, with the following properties: Suppose that $\omega$ satisfies (1.8) for some $\gamma$ with $0 < \gamma \le \gamma^*$. Let $0 < \delta \le \delta^*$ and let $f, g \in C^{\ell}(S, \mathbb{R}^3)$, where

$$S = \mathbb{R}^3 \times \{y \in \mathbb{R}^3 \mid |y| \le P^{-1}\} \times \{a \in \mathbb{R}^3 \mid |a-\omega| \le \gamma\,\varepsilon\}$$

cf. (1.16)

and where f and g are $2\pi$-periodic in $x_0,x_1,x_2$, constituting a system (1.15) which is volume preserving. Finally assume

$$(1.17)\qquad \begin{cases} |Nf(x,y,a)| + |Pg(x,y,a)| \le \gamma\,\delta N \qquad \text{and} \\ |D_x^{\zeta_1} D_y^{\zeta_2} D_a^{\zeta_3} Nf(x,y,a)| + |D_x^{\zeta_1} D_y^{\zeta_2} D_a^{\zeta_3} Pg(x,y,a)| \le \\ \le \gamma^{1-|\zeta_3|} N^{|\zeta_1|+1} P^{|\zeta_2|} Q^{|\zeta_3|}\ , \end{cases}$$

for all $(x,y,a) \in S$ and $|\zeta_1| + |\zeta_2| + |\zeta_3| = \ell$

Then there exists a vector a with $|a - \omega| \leq \gamma\,\varepsilon$, such that the corresponding system (1.15) possesses a quasi-periodic solution with frequency $\omega$.

We shall prove theorem C in § 2.4.

From theorem C we shall derive theorem A in § 2.5, in (1.7) replacing y by $A^{-1}(a) + y$ for $a \in \Delta$ . Then (1.7) is transformed into (1.15) with $a \in \Delta$. Now we may apply theorem C to obtain theorem A. It is possible to prove refinements of the theorems A and B, which are applicable to the system (1.6). In stead of this we shall give in § 2.6 a direct application of theorem C to (1.6). The result is the original aim of our paper:

<u>THEOREM D</u>:

Let $X^{\mu} = \tilde{X}^{\mu} + p^{\mu}$ be a generic arc of divergence free vector fields on $\mathbb{R}^4$, as specified above. Then there exists a $\mu_0 > 0$, such that the family $\{\tilde{X}^{\mu}\}_{0<\mu<\mu_0}$ possesses uncountably many invariant 3-tori with a quasi-periodic flow having 3 sufficiently independent frequencies. For any of these $\tilde{X}^{\mu}$-invariant tori, there exits an appropriate $\mu'$, close to $\mu$, such that $X^{\mu'}$ possesses a slightly deformed invariant torus, again with quasi-periodic flow. Both quasi-periodic motions, occurring in $\tilde{X}^{\mu}$ and in $X^{\mu'}$, have the same frequency ratios.

<u>REMARKS</u>

i. If the results from [14], mutatis mutandis, also hold here, then the surviving tori fill up a Cantor set of positive measure (relatively tending to 1 as $\mu \to 0$). Then by Fubini's theorem, we find a set

$C_{\mu_0} \subseteq (0,\mu_0)$, having positive measure, such that for each $\mu \in C_{\mu_0}$ the vector field $X^\mu$ possesses a number of invariant tori, the union of which has positive measure.

ii. Compare the three dimensional analogue [4], where the bifurcation parameter plays a different rôle. In that case, for *each* value of the parameter, fixed sufficiently small, uncountably many invariant 2-tori survive the perturbation.

## § 1.5 A generalization to higher dimensions and some remarks on a non-divergence free analogue

a. It is straight forward to generalize the theorems A,B and C to higher dimensions. Here we present the form which is most in accordance with theorem D. For this purpose on $\mathbb{R}^n$ $(n \geq 4)$ consider an $(n-3)$-parameter family of divergence free vector fields, which forms an almost integrable system in the following way:

$$
* \quad \left\{
\begin{array}{ll}
\dot{x}_0 & = b_0(y_0,\mu) + f_0(x,y_0,\mu) \\
\hline
\dot{x}_{n-2} & = b_{n-2}(y_0,\mu) + f_{n-2}(x,y_0,\mu) \\
\dot{y}_0 & = g_0(x,y_0,\mu) \quad ,
\end{array}
\right.
$$

where $\mu = (\mu_1,\ldots,\mu_{n-3})$ is the vector of parameters and where $x = (x_0,x_1,\ldots,x_{n-2})$ are the angles. The perturbations $f_j$ and $g_0$ so have period $2\pi$ in the $x_j$, compare (1.10). Analogous to proposition 1.2 the preservation of volume translates to

$$
\frac{\partial g_0}{\partial y_0} + \sum_{j=0}^{n-2} \frac{\partial f_j}{\partial x_j} \equiv 0 \quad \text{and} \quad \int g_0(x,y_0,\mu)\, dx_0 \wedge \ldots \wedge dx_{n-2} \equiv 0.
$$

Note that if the perturbation is zero, the vector field possesses an invariant foliation of (n-1)-dimensional tori, shrinking towards an (n-2)-dimensional torus. This is what the term "almost integrable" expresses.

Analogous to (1.12) we require the following non-degeneracy condition:

$$\det \left( \frac{\partial b_i}{\partial y_j} , b_i \right)_{i=0}^{n-2} {}_{j=0}^{n-3} \neq 0 \;,$$

where $y_j = \mu_j$ for $j = 1, 2, \ldots, n-3$. We abbreviate $y = (y_0, y_1, \ldots, y_{n-3})$.

As before we single out tori $y = c$, such that the frequency $\omega = b(c)$ satisfies a strong non-resonance condition like (1.8), which for $\tau > n-1$ is easy to satisfy. We now state

THEOREM 1.3:

For sufficiently small perturbations $f_j$ and $g_0$ and each (n-1)-torus $y = c$ as above, the system $*$ possesses an invariant (n-1)-torus which is a slight deformation of $y = c$. The motion in this new torus is quasi-periodic and the frequency is a scalar multiple of $\omega$, close to $\omega$.

REMARKS :

i. Theorem 1.3 is a straight forward generalization of §1.4.

ii. It is very probable that also for $n > 4$ the almost integrable form $*$ occurs in bifurcation problems.

b. The rest of this section will be devoted to the same 4-dimensional bifurcation as before, but now without the restriction to divergence zero. So again we unfold a singularity with eigenvalues $\pm i\alpha_1$ and $\pm i\alpha_2$, for $\alpha_1$ and $\alpha_2$ positive. In this case generalically the singularity has codimension two, and we consider a generic 2-parameter family $X^{\mu_1,\mu_2}$, which unfolds our singularity that $X^{0,0}$ possesses in the origin of $\mathbb{R}^4$. Here $\mu_1$ and $\mu_2$ are real parameters.

This situation was studied before by e.g. Takens [15] and Dumortier & Roussarie [7]. Our present concern, however, is with a result formulated by Guckenheimer [8]. This result claims the existence of quasi-periodic motions with three independent frequencies, associated with a line of Hopf-bifurcations, which for an open set of unfoldings occurs in the $(\mu_1,\mu_2)$-plane. Below we shall be more precise. It is our aim to illustrate our methods - using Moser [12] - in providing Guckenheimer's statement with a sketch of a proof. For the moment we completely restrict to the "small divisor side" of the problem, foregoing the difficulties related to the "small twist". See §1.4. Also compare §2.6.

Becoming more explicit: we assume that $\alpha_1$ and $\alpha_2$ satisfy some finite non resonance condition as in § 1.2. This yields a normal form decomposition $X^{\mu_1,\mu_2} = \tilde{X}^{\mu_1,\mu_2} + p^{\mu_1,\mu_2}$ (see e.g. [15]) where $\tilde{X}$ possesses toroidal symmetry and p consists of higher order terms. The "unperturbed" family $\tilde{X}$ has the form

$$(1.1') \quad \begin{cases} \dot{\varphi}_i = \alpha_i + f_i(r_1^2,r_2^2,\mu_1,\mu_2) \\ \dot{r}_i = r_i\{\mu_i + g_i(r_1^2,r_2^2,\mu_1,\mu_2)\} \end{cases},$$

with $f_i(0,0,0,0) = g_i(0,0,0,0) = \frac{\partial g_i}{\partial \mu_j}(0,0,0,0) = 0$ for $i,j = 1,2$. See above.

We number the formulae using the following convention: if a formula in a similar form already occurred in a previous section with the number (1,i), then in this section it will be indicated by the number (1,i'). Again, in neglecting the angular components, (1.1') is reduced to the $(r_1,r_2)$-plane. Firstly we truncate at the order 3 and write for $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22} \in \mathbb{R}$:

$$\dot{r}_i = r_i(\mu_i + a_{i1}\, r_1^2 + a_{i2}\, r_2^2) \ , \quad i = 1,2 \ . \tag{1.2'}$$

Secondly, for simplicity, we impose the following conditions:

$$a_{11}a_{22} - a_{12}a_{21} > 0,\ a_{22} - a_{12} > 0,\ a_{11} \neq 0 \text{ and } \frac{a_{22}}{a_{11}} < 0 \quad .$$

Compare [7, 8, 14]. In the $(\mu_1,\mu_2)$-plane then consider the half-line $\mu_2 = \chi\mu_1$, $\mu_1 \geq 0$, where $\chi = \dfrac{a_{22}(a_{11}-a_{21})}{a_{11}(a_{12}-a_{22})}$ .

One may blow up (1.2') in the direction of this line, using techniques similar to the ones showed in § 1.2. Also compare e.g. [7]. In this way, using the implicit function theorem, one finds a $C^\infty$-curve $\tilde{L}$ in the $(\mu_1,\mu_2)$-plane, which for $\mu_1 \geq 0$ and small, is smoothly parametrized by its $\mu_1$-coordinate. Moreover:

i. For $\mu_1 = 0$ the curve $\tilde{L}$ is tangent to $\mu_2 = \chi\mu_1$,

ii. For $(\mu_1,\mu_2) \in \tilde{L} \setminus \{(0,0)\}$ the reduced system possesses an isolated singularity at an asymptotic distance $\sqrt{\mu_1}$ from $(r_1,r_2) = (0,0)$. The eigenvalues in this singularity are purely imaginary and non-zero.

Furthermore, if one moves transversally to $\tilde{L}$ the eigenvalues cross the imaginary axis with a positive velocity. In [8] it is stated that generic conditions have to be imposed on the fifth order terms, to ensure that we are dealing with a "stable" Hopf-bifurcation. (Cf. Marsden & McCracken [10].)

So the reduced system possesses a 2-parameter family of hyperbolic closed orbits. Let $\nu_1 \geq 0$ parametrise the curve $\tilde{L}$, while $\nu_2$ is a Hopf-parameter, moving transversally to $\tilde{L}$. Assume that $\nu_2|_{\tilde{L}} \equiv 0$ and that the closed orbits come into existence for $\nu_2 > 0$. For $\nu_1$ and $\nu_2$ positive, and both small, we find suitable coordinates $\varphi_0$ and $y_0$, defined near the closed orbit of the reduced system. See fig. 2.

Here $\varphi_0$ is angular, defined mod $2\pi$, and $y_0$ is a normal coordinate. The reduced system then can be written as

$$\begin{cases} \dot{\varphi}_0 = b_0(y_0,\nu_1,\nu_2) \\ \dot{y}_0 = y_0\, n(y_0,\nu_1,\nu_2) \end{cases},$$

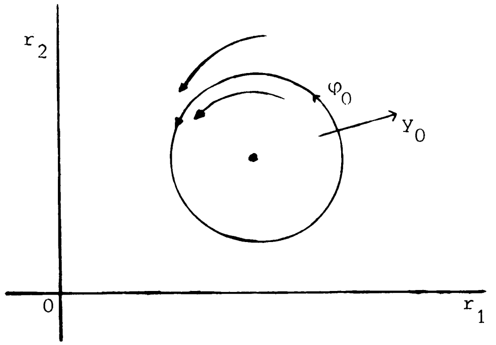


fig. 2

where the hyperbolicity means that $n(0,\nu_1,\nu_2) \neq 0$.

The symmetric vector field $\tilde{X}$ possesses a corresponding family of invariant 3-tori, constituting a normally hyperbolic invariant $C^\infty$-manifold $\tilde{M}$ of dimension 5.

In the 6-dimensional space coordinatised by $(\varphi_0,\varphi_1,\varphi_2, y_0,\nu_1,\nu_2)$ we have $\tilde{M} = \{y_0 = 0\}$.

Near $\tilde{M}$ the vector field $\tilde{X}$ has the form

$$\begin{cases} \dot{\varphi}_0 = b_0(y_0,\nu_1,\nu_2) \\ \dot{\varphi}_1 = b_1(y_0,\nu_1,\nu_2) \\ \dot{\varphi}_2 = b_2(y_0,\nu_1,\nu_2) \\ \dot{y}_0 = y_0\; n(y_0,\nu_1,\nu_2) \;. \end{cases}$$

Note that we replaced $\mu_1$ and $\mu_2$ by the pair $(\nu_1,\nu_2)$. We recall that $\nu_1$ and $\nu_2$ are positive but small, and that $y_0$ varies in a neighbourhood of 0.

We now follow the same strategy as before and perturb to $X^{\nu_1,\nu_2} = \tilde{X}^{\nu_1,\nu_2} + p^{\nu_1,\nu_2}$, so obtaining

$$(1.6') \qquad \begin{cases} \dot{\varphi} = b(y_0,\nu_1,\nu_2) + f(\varphi,y_0,\nu_1,\nu_2) \\ \dot{y}_0 = y_0\; n(y_0,\nu_1,\nu_2) + g_0(\varphi,y_0,\nu_1,\nu_2) \;, \end{cases}$$

where $\varphi = (\varphi_0,\varphi_1,\varphi_2)$ etc., and the perturbations f and $g_0$ are $2\pi$-periodic in $\varphi$.

According to Hirsch, Pugh & Shub [9] and to Palis & Takens [13] the vector field X, as a consequence of the normal hyperbolicity, possesses and invariant manifold M, close to $\tilde{M}$. The dynamics of X near M is given by (1.6') and one may well ask whether M contains quasi-periodic motions As is said before, an affirmative answer is given in [8], and we shall proceed in indicating a proof of this.

First we restrict ourselves to a region where $\nu_2$ is bounded away from 0. Here we reparametrize the vector field by the factor $^1/n(0,\nu_1,\nu_2)$. So without loss of generality we may assume that in (1.6') we have $n(y_0,\nu_1,\nu_2) \equiv 1$. Similar to (1.12) we now require

$$\det\left(\frac{\partial b_i}{\partial \nu_j}, b_i\right)_{i=0}^{2}\,{}_{j=0}^{2} \neq 0 \quad \text{for } y_0 = 0.$$

As in § 1.4 we introduce an artificial variable $\nu_3$ which rescales the time t to $(1+\nu_3)t$. Then our system translates to

$$(1.7')\qquad \begin{cases} \dot{\varphi} = A(y_0,\nu) + F(\varphi,y_0,\nu) \\ \dot{y}_0 = y_0 + G_0(\varphi,y_0,\nu) \\ \dot{\nu} = 0\ , \end{cases}$$

where $\nu = (\nu_1,\nu_2,\nu_3)$, $A(y_0,\nu) = (1+\nu_3)\ a(y_0,\nu_1,\nu_2)$,
$F(\varphi,y_0,\nu) = (1+\nu_3)\ f(\varphi,y_0,\nu_1,\nu_2)$ and $G_0(\varphi,y_0,\nu) = (1+\nu_3)\ g_0(\varphi,y_0,\nu_1,\nu_2)$.

Compare (1.11). The non-degeneracy condition (1.12') now obviously rewrites to

$$\det\left(\frac{\partial A_i}{\partial \nu_j}(0,\nu)\right)^2_{i,j-1=0} \neq 0\ .$$

This implies that the map $(\nu_1,\nu_2,\nu_3) \to \bar{A}(\nu_1,\nu_2,\nu_3) = A(0,\nu_1,\nu_2,\nu_3)$ is a local diffeomorphism. It is easy now to find $\omega = \bar{A}(\nu)$ such that for some $\gamma > 0$ and $\tau > 0$ we have

$$(1.8')\qquad |k + i(j,\omega)| \geq \gamma |j|^{-\tau}$$

for $k = -1,0,1$ and all $j \in \mathbb{Z}^3 \setminus \{0\}$. Let $\Delta = \{a \in \mathbb{R}^3\,|\,|a-\omega| \leq d\}$ be a disc where $\bar{A}^{-1}$ is well defined and for $a \in \Delta$ write

$$\nu = \bar{A}^{-1}(a) + (y_1,y_2,y_3)$$

(Compare the construction following theorem C, § 1.4). So now $y = (y_0,y_1,y_2,y_3)$ is close to zero, while (1.7') transforms to

$$\begin{cases} \dot{x} = a + \bar{f}(x,y,a) \\ \dot{y}_0 = y_0 + \bar{g}_0(x,y,a) \\ \dot{y}_1 = \dot{y}_2 = \dot{y}_3 = 0\ , \end{cases}$$

where $\bar{f}$ and $\bar{g}_0$ have an obvious relation to F and $G_0$. Once more consider a rescaling of the time by a scalar B, i.e. in stead of (1.15') write

$$\begin{cases} \dot{x} = Ba + B\bar{f}(x,y,a) \\ \dot{y}_0 = By_0 + B\bar{g}_0(x,y,a) \\ \dot{y}_1 = \dot{y}_2 = \dot{y}_3 = 0. \end{cases}$$

Here B is close to 1. Recall that a is close to $\omega$. In the last system replace Ba by c and so obtain

$$\begin{cases} \dot{x} = c + \bar{\bar{f}}\,(x,y,B,c) \\ \dot{y}_0 = By_0 + \bar{\bar{g}}(x,y,B,c) \\ \dot{y}_1 = \dot{y}_2 = \dot{y}_3 = 0 \end{cases}$$

Now B is close to 1 and c is close to $\omega$. This form is suitable for our purposes, for we can make the following considerations:

Analogous to theorem C one can also prove a $C^\infty$ variant of (12], theorem 3 stating that, modulo smallness conditions - compare (1.9) or (1.17) - , there exist $c^*$ close to $\omega$ and $B^*$ close to 1 such that this system, with $c = c^*$ and $B = B^*$, possesses a quasi-periodic solution with the characteristic numbers $\omega_0, \omega_1, \omega_2$ and 1.

Then, writing $a^* = \frac{1}{B^*} c^*$, one obtains a quasi-periodic solution with frequency $\frac{1}{B^*}\,\omega$ of the system (1.15') and hence of (1.7').

Similar methods are used in Moser [12, §6], also see Moser [$11^a$].

Recapitulating we conclude the following: Consider the $\tilde{X}$-invariant mani-

fold $\tilde{M} = \{y_0 = 0\}$ which is a 2-parameter family of 3-tori, parametrised by $\nu_1$ and $\nu_2$. Condition (1.12') yields a Cantor set of frequencies $\omega = b(0,\nu_1,\nu_2)$ satisfying (1.8') for some sufficiently large $\tau$ and some $\gamma > 0$. To each of these frequencies we apply the above procedure and so find a deformed X-invariant 3-torus where the flow is quasi-periodic with frequency $s\omega$, for some s close to 1. Note that, similar to the volume preserving case, the parameters $\nu_1$ and $\nu_2$ may have shifted in perturbing, but that the frequency ratios have been kept constant.
Also compare remark i. following theorem D, §1.4.

## §1.6 A real analytic analogue

In this section we present a real analytic form of the theorems A and C. This theory is essentially contained in Moser [12].
It is included here because it provides us with several tools for our treatment of the $C^\infty$-case.
We number the formulae using a convention similar to the one in §1.5: if a formula in a similar form already occurred in §1.4 with number (1,i), then in this section it will be indicated by the number (1,i").

a. Similar to (1.7) we consider a system

$$(1.7")\qquad \begin{cases} \dot{x} = A(y) + \varepsilon F(x,y,\varepsilon) \\ \dot{y} = \varepsilon G(x,y,\varepsilon), \end{cases}$$

where $\varepsilon > 0$ is a perturbation parameter.
As above $x = (x_0,x_1,x_2)$ and $y = (y_0,y_1,y_2)$, while the functions $F = (F_0,F_1,F_2)$ and $G = (G_0,G_1,G_2)$ are real analytic in all arguments and $2\pi$-periodic in $x = (x_0,x_1,x_2)$. Also we assume volume preservation in the sense of proposition 1.2.

Consider a 2-torus $y = c$, which is invariant for (1.7") with $\varepsilon = 0$, such that the frequency vector $\omega = A(c)$ satisfies

$$(1.8") \qquad |(\nu,\omega)| \geq \gamma|\nu|^{-\tau} \qquad \text{for all } \nu \in \mathbb{Z}^3-\{0\}.$$

Also assume the non-degeneracy condition

$$(1.18) \qquad \det\left(\frac{\partial A_i}{\partial y_j}(c)\right)^2_{i,j=0} \neq 0.$$

We formulate, in analogy with theorem A:

<u>Theorem A"</u>:

For sufficiently small $\varepsilon > 0$ the system (1.7"), as specified above, possesses an invariant 3-torus with a quasi-periodic flow of frequency $\omega$. This torus is an analytic deformation of the torus $y = c$, the deformation being parametrized by $\sqrt{\varepsilon}$.

One easily may restrict to relevant special cases of theorem A". For example one may immediately conclude a real analytic analogue of theorem B, cf. §1.4. In that case the variable $y_2$ plays a special rôle: the time t is rescaled to $(1 + y_2)t$ and we have that $G_2 = 0$. Compare (1.10) and (1.11).

If one imposes as an extra condition that also $G_1 = 0$, then (1.7") becomes a 1-parameter family of differential equations like (1.10), parametrized by $y_1$. This second special case would be more or less in accordance with theorem D.

Presently it will appear that the language of Lie algebras is convenient for these situations which are volume preserving and which have some extra verticalness conditions (such as e.g. $G_2 = 0$ or $G_1 = G_2 = 0$). Cf. [12].

In order to formulate a real analytic analogue of theorem C we have to introduce some concepts.

Consider a real analytic system

$$(1.19)\quad \begin{cases} \dot{x} = \omega + \varepsilon f\,(x,y,\varepsilon,c) \\ \dot{y} = \varepsilon g\,(x,y,\varepsilon,c)\,, \end{cases}$$

as above $2\pi$-periodic in $x = (x_0, x_1, x_2)$. The letter c denotes a 3-dimensional parameter. As in [12] we consider a modification of (1.19):

$$(1.15'')\quad \begin{cases} \dot{x} = \omega + \lambda + \varepsilon f\,(x,y,\varepsilon,c) \\ \dot{y} = \varepsilon g\,(x,y,\varepsilon,c)\,, \end{cases}$$

where $\lambda = (\lambda_1, \lambda_1, \lambda_2)$ is independant of x and y.

This system is similar to (1.15) if one puts $a = \omega + \lambda$.

The main result of this section will be that for given f and g there exists a modifying term $\lambda$, analytic in $\varepsilon$ and c and which vanishes for $\varepsilon = 0$, such that the corresponding system (1.15") has quasi-periodic solutions depending analytically on $\varepsilon$ and c and possessing the same frequency $\omega = (\omega_0, \omega_1, \omega_2)$ for all $\varepsilon$ and c.

Here we use that $\omega$ is as before, i.e. obeys the diophantic condition (1.8"). This result is a direct application of [12], theorem 5.

We now shall become more precise:

First consider a normal form

$$(1.20)\quad \begin{cases} \dot{\xi} = \omega + O(\eta) \\ \dot{\eta} = O(\eta) \end{cases}, \qquad \xi = (\xi_0, \xi_1, \xi_2)\,,\ \eta = (\eta_0, \eta_1, \eta_2)\,.$$

The system (1.15") will be put into this normal form by a change of coordinates

$$(1.21)\quad \begin{cases} x = \xi + \varepsilon u(\xi,\varepsilon,c) \\ y = \eta + \varepsilon v(\xi,\eta,\varepsilon,c), \end{cases}$$

where u and v have period $2\pi$ in $\xi_0,\xi_1,\xi_2$ and are real analytic in all their arguments.

Moreover this conjugacy may be chosen volume preserving and linear in $\eta$. We formulate analogous to theorem C, partly recapitulating this introduction:

Theorem C":

There exists a unique real analytic $\lambda = \lambda(\varepsilon,c)$ and there exist real analytic $u = u(\xi,\varepsilon,c)$ and $v = v(\xi,\eta,\varepsilon,c)$, as above, such that the transformation (1.21) conjugates (1.15") with the normal form (1.20).
In particular

$$(1.22)\quad \begin{cases} x(t) = \omega t + \varepsilon u\ (\omega t,\varepsilon,c) \\ y(t) = \varepsilon v\ (\omega t,0,\varepsilon,c) \qquad , \ t \in \mathbb{R} \end{cases}$$

is a quasi-periodic solution of (1.15") with frequency $\omega$.

Remark:

If the system (1.19) and so (1.15") is vertical in the sense that $g_2 = 0$ or $g_1 = g_2 = 0$, then the conjugacy (1.21) may be chosen in such a way that $v_2 = 0$ or $v_1 = v_2 = 0$ respectively.
Observe that in that case the normal form (1.20) has the same verticalness property.

b. Before we prove theorem C" from [12], we shall derive theorem A" from it. So consider a 3-torus $y = \overset{o}{c}$ such that $\omega = A(\overset{o}{c})$ satisfies (1.8"). Firstly we stretch the normal variables near this torus by introducing $Y = (Y_0, Y_1, Y_2)$ via $y - \overset{o}{c} = \sqrt{\varepsilon}\, Y$.
This transforms (1.7") to

$$(1.23) \quad \begin{cases} \dot{x} = \omega + \sqrt{\varepsilon}\ \bar{F}(x, Y, \sqrt{\varepsilon}, \overset{o}{c}) \\ \dot{Y} = \sqrt{\varepsilon}\ \bar{G}(x, Y, \sqrt{\varepsilon}, \overset{o}{c}). \end{cases}$$

We now apply theorem C" to the system

$$(1.24) \quad \begin{cases} \dot{x} = \omega + \sqrt{\varepsilon}\ \bar{F}(x, Y, \sqrt{\varepsilon}, c) \\ \dot{Y} = \sqrt{\varepsilon}\ \bar{G}(x, Y, \sqrt{\varepsilon}, c), \end{cases}$$

where $\omega = A(\overset{o}{c})$, but where $c$ ranges over a full neighbourhood of $\overset{o}{c}$. This yields $\lambda = \lambda(\sqrt{\varepsilon}, c)$, such that the modified version of (1.24) possesses quasi-periodic solution of frequency $\omega$. If we can solve $c$ from the equation

$$(1.25) \qquad A(c) = \omega + \lambda(\sqrt{\varepsilon}, c)$$

then we clearly have put the obtained quasi-periodic solution into our original system (1.7"). In this real analytic case we solve (1.25) using the implicit function theorem: for $\varepsilon = 0$ the equation has solution $c = \overset{o}{c}$ and the non-degeneracy condition (1.18) then assures the existence of a (local) analytic curve $c = c(\sqrt{\varepsilon})$ with $c(0) = \overset{o}{c}$.
This proves theorem A".

c. This section is concluded by giving a proof of theorem C", applying [12], theorem 5. For this purpose we introduce the Lie algebra $L$ consisting of all vector fields

$$Z = \Sigma_{j=0}^{2} \{F_j(x,y) \frac{\partial}{\partial x_j} + G_j(x,y) \frac{\partial}{\partial y_j}\},$$

where $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$, all functions have period $2\pi$ in $x$ and are analytic in all their arguments in a neighbourhood of the 3-torus $y = 0$.

Also we consider the Lie subalgebra $\dot{L}$ of $L$ which contains vector fields $Z$ as above which preserve the volume in the sense of proposition 1.2.

Moreover $\ddot{L} \subseteq \dot{L}$ shall denote the Lie subalgebra of vector fields which have the extra verticalness property that $G_1 = G_2 = 0$.

Observe that the perturbation in (1.19) and (1.15"), modulo an translation of $y = (y_0, y_1, y_2)$ over the vector $c$, is an element of $\dot{L}$.

Also compare (1.23) and (1.24).

To $Z \in L$ we associate its linearised version

$$(Z)_1 = F(x,0) \frac{\partial}{\partial x} + \{G(x,0) + G_y(x,0)y\} \frac{\partial}{\partial y},$$

where $G_y$ denotes the derivative of $G$ in the y-directions.

Thus we obtain another Lie subalgebra $L_1$ of $L$.

Define $\dot{L}_1 = L_1 \cap \dot{L}$ and $\ddot{L}_1 = L_1 \cap \ddot{L}$.

Note that $\dot{L}_1$ contains the infinitesimal generators of the transformations of type (1.21). The corresponding Lie group of transformations is denoted by $\dot{G}_1$. Similarly we have $\ddot{G}_1$ associated with $\ddot{L}_1$. (Cf. the remark made after theorem C".)

In [12] a general theory is developed for "arbitrary" Lie subalgebras of $L$. We will restrict to the volume preserving case $\dot{L}$ and also consider one of the relevant vertical subcases $\bar{L}$. The following considerations are made for $\dot{L}$ only, but hold for $\bar{L}$ as well.

Now consider $D = \omega\frac{\partial}{\partial x}$, belonging to $\dot{L}_1$, and the commutator $\Theta : Z \mapsto [D,Z]$, the Lie bracket of D and Z. This commutator defines a linear map $L_1 \to L_1$, and its restriction to $\dot{L}_1$ is a linear map $\dot{L}_1 \to \dot{L}_1$. Let $N$ and $\dot{N}$ denote the nullspaces of these respective maps: $\dot{N} = N \cap \dot{L}_1$. The fact that $\omega$ satisfies the condition (1.8") implies that $N$ has finite dimension.

A typical element of $N$ has the form

$$\lambda \frac{\partial}{\partial x} + (\mu + My) \frac{\partial}{\partial y} ,$$

$\lambda$ and $\mu$ being constants in $\mathbb{R}^3$, and M a constant 3 x 3 matrix. Moreover, if the range $\Theta(L_1)$ is denoted by $R_1$ and if $\Theta(\dot{L}_1) = \dot{R}_1$, then

$$N \oplus R_1 = L_1 \quad \text{and} \quad \dot{N} \oplus \dot{R}_1 = \dot{L}_1 ,$$

where $\oplus$ denotes the direct sum of vector spaces.

According to [12], theorem 5, under these circumstances we have that for each vector field

$$\{\omega + \varepsilon f(x,y,\varepsilon)\} \frac{\partial}{\partial x} + \varepsilon g(x,y,\varepsilon) \frac{\partial}{\partial y}$$

of type (1.19), belonging to $\dot{L}$, a unique modifying

$$\lambda(\varepsilon) \frac{\partial}{\partial x} + \{\mu(\varepsilon) + M(\varepsilon)y\} \frac{\partial}{\partial y}$$

exists in $\dot{N}$, such that the sum of these two is conjugate to the normal form (1.20). The conjugacy is a transformation in $\dot{G}_1$, so of type (1.21).

Our proof of theorem C" now is completed by the observation, that modifying terms in $\dot{N}$ have $\mu(\varepsilon) \equiv 0$ and $M(\varepsilon) \equiv 0$. This is easily seen from proposition 1.2. Note that we suppressed the parameter c.

This formal argument will be illustrated by a brief digression into the method of Moser's proof.

We have to find u and $v_0$ from (1.21) and $\lambda$ from (1.15"), such that (1.21) conjugates (1.15") and the normal form (1.20).

Expanding formally as a power series in $\varepsilon$ we write

$$(1.26) \qquad \begin{cases} u(\xi,\varepsilon) & = u^1(\xi) + O(\varepsilon) \ , \\ v(\xi,\eta,\varepsilon) & = v^1(\xi) + V^1(\xi)\eta + O(\varepsilon) \quad \text{and} \\ \lambda(\varepsilon) & = \varepsilon\lambda^1 + O(\varepsilon^2) \ . \end{cases}$$

On the level of first order terms the conjugation property now can be expressed as

$$(1.27) \qquad \begin{cases} u^1_\xi\omega = f(\xi,0,0) + \lambda^1 \ , \\ v^1_\xi\,\omega = g(\xi,0,0) \ , \\ V^1_\xi\omega = g_y(\xi,0,0) \ . \end{cases}$$

If one writes for the perturbation, the conjugation and the modifying term - up to higher order terms -

$$(1.28) \qquad \begin{cases} F = f(\xi,0,0)\,\dfrac{\partial}{\partial\xi} + \{g(\xi,0,0) + g_y(\xi,0,0)\eta\}\,\dfrac{\partial}{\partial\eta} \ , \\ U = u^1(\xi)\,\dfrac{\partial}{\partial\xi} + \{v^1(\xi) + V^1(\xi)\eta\}\,\dfrac{\partial}{\partial\eta} \ , \\ N = \lambda^1\,\dfrac{\partial}{\partial\xi} \ , \end{cases}$$

then the equations (1.27) take the form

$$(1.29) \qquad [D,U] = F + N,$$

which is suitable for the Lie algebra talk we held before.
Note that indeed $[D,N] = 0$, meaning that $N \in \dot{N}$. To discuss the above equations we expand in Fourier series

$$f(\xi,0,0) = \sum_{\nu \in \mathbb{Z}^3} f_\nu \, e^{i(\nu,\xi)}, \text{ etc.}$$

and write formally

$$u^1(\xi) = \sum_{\nu \in \mathbb{Z}^3} u^1_\nu \, e^{i(\nu,\xi)}, \text{ etc.}$$

Now (1.27) takes the form

$$(1.30) \quad \begin{cases} i(\nu,\omega)\, u^1_\nu = f_\nu,\ i(\nu,\omega) v^1_\nu = g_\nu,\ \ i(\nu,\omega) V^1_\nu = g_{y\nu} \text{ for } \nu \neq 0 \text{ and} \\ 0 = f_0 + \lambda^1. \end{cases}$$

Note that by proposition 1.2 we have $g_0 = 0$ and $g_{y0} = 0$. Condition (1.8') yields unique real analytic solutions

$$v^1(\xi) = \sum_{\nu \neq 0} \frac{g_\nu}{i(\nu,\omega)} e^{i(\nu,\xi)} ,$$

$$v^1(\xi) = \sum_{\nu \neq 0} \frac{g_{y\nu}}{i(\nu,\omega)} e^{i(\nu,\xi)} ,$$

and a real analytic

$$u^1(\xi) = u^1_0 + \sum_{\nu \neq 0} \frac{f_\nu}{i(\nu,\omega)} e^{i(\nu,\xi)} ,$$

where $u^1_0$ remains arbitrary.
Form proposition 1.2 it is evident that for $F \in \dot{L}$ also $U \in \dot{L}$.
Similarly we have that $U \in \ddot{L}$ as soon as $F \in \ddot{L}$.

Finally we conclude from (1.30) that

$$\lambda^1 = -f_0.$$

Note that from (1.30) it follows that the operator $\Theta$ has eigenvalues $i(\nu,\omega)$, $\nu \in \mathbb{Z}^3$.

Also note that if we choose $u_0^1 = 0$, then the solution $u^1(\xi)$ also becomes unique and the whole problem has a unique solution $N \in \dot{N}$, $U \in \dot{R}$. The desired transformation (1.21) in $\dot{G}_1$ now is obtained by taking $\exp(\varepsilon U)$. Cf. [12], §5c.

If one proceeds in this way, determining coefficients in a power series expansion in $\varepsilon$, a formal solution is obtained for the modifying term $\lambda$ and the conjugation (1.21). A convergent construction can be performed as in [12], but in chapter 2 we shall present such a construction with less assumptions, in order to prove theorem C.

# CHAPTER 2 - PROOFS

## § 2.1 Introduction

The main problem is the proof of theorem C concerning system (1.15). We recall that $x = (x_0,x_1,x_2)$, $y = (y_0,y_1,y_2)$ and that the righthand side of (1.15) is $2\pi$-periodic in $x_0$, $x_1$ and $x_2$. We will consider two cases for this system:

Case I: the system is volume-preserving, i.e. (cf. proposition 1.2)

$$\sum_{j=0}^{2} \left( \frac{\partial f_j}{\partial x_j} + \frac{\partial g_j}{\partial y_j} \right) = 0, \quad \int_0^{2\pi} dx_0 \int_0^{2\pi} dx_1 \int_0^{2\pi} dx_2 g(x,y) = 0$$

Case II: the system is volume-preserving and $g_1 = g_2 = 0$ (verticalness condition).

In the language of Lie algebras (cf. § 1.4c)) it is easy to verify that the vector fields corresponding to systems in case I and in case II form a Lie algebra $\dot{L}$ and $\ddot{L}$ respectively, where $\ddot{L} \subset \dot{L} \subset L$. Note that our differentiability assumptions are less than those in §1.4.

Theorem C will be proven by the construction of a coordinate transformation $U$ : $x = u(\xi,\eta)$, $y = v(\xi,\eta)$ which transforms system (1.15) into a form similar to (1.20):

$$(2.1) \qquad \dot{\xi} = \omega + \varphi(\xi,\eta), \quad \dot{\eta} = \psi(\xi,\eta) \ ,$$

where $\varphi(\xi,0) = \psi(\xi,0) = 0$. This yields a quasi-periodic solution on an invariant torus of (2.1), and therefore of (1.15), corresponding to $\eta = 0$. The conjugacy $U$ will be volume-preserving, $u - \xi$ and $v$ will be

$2\pi$-periodic in $\xi_0$, $\xi_1$, $\xi_2$, whereas in case II also $v_1 = \eta_1$, $v_2 = \eta_2$, Note that now (2.1) will also be volume preserving and in case II it will be vertical as well.

Compare §1.4c. We do not express this in Lie-canguage, because U is only $C^1$ and (2.1) only $C^o$. Moreover we may choose the conjugacy to be of the special form:

(2.2) $\quad x = \xi + u(\xi), \quad y = \eta + \bar{v}(\xi) + V(\xi)\eta \quad ,$

where u and $\bar{v}$ are $2\pi$-periodic 3-vectors and V is a $2\pi$-periodic 3 x 3-matrix (compare §1.4). Note that the condition that U is volume preserving implies that the averages of $\bar{v}$ and V over the set $0 \leq \xi_j \leq 2\pi$, $j = 0,1,2$ are zero.

The construction of this coordinate transformation proceeds via an iterative procedure. In each step a result of the following type will be applied: given a differential equation which is a perturbation of $\dot{x} = \omega$, $\dot{y} = 0$ with certain bounds on the perturbation terms, one can construct a coordinate transformation of the form (2.2) which transforms the given differential equation into another which belongs to $L$ or $\bar{L}$ and is a perturbation of $\dot{\xi} = \omega$, $\dot{\eta} = 0$ with certain smaller bounds on the perturbation terms. This result should be such that it is possible to set up a sequence of coordinate transformations $U_1$, $U_2$, $U_3$, ... such that $U_1 \circ U_2 \circ \ldots \circ U_n$ converges to a transformation $U_\infty$ of the form (2.2) which conjugates (1.15) with (2.1).

The transformations $U_\nu$ reduce in particular the main terms in the righthand side of (1.15) and its iterates, i.e. the terms corresponding to $a = \omega + \lambda$, and these transformations will depend on the value of a: Therefore the coordinate transformation will transform x, y and the parameter a to coordinates $\xi$, $\eta$ and a parameter $\alpha$. The limit transformation $U_\infty$ will reduce $a = \omega + \lambda$ to $\alpha = \omega$. Compare Moser [12].

Because of the presence of small divisors these requirements restrict severely the type of result mentioned above and they are the reason that it is of a very complicated nature. The iteration step will be formulated as theorem 2.4 in §2.3. In §2.2 some auxiliary results needed for the proof of theorem 2.4 are give.

In §2.4 the iteration procedure is carried out in order to prove theorem C. In §2.5 and §2.6 we apply theorem C to prove the theorems A and D respectively.

## § 2.2 Preliminary lemmas

In this section we discuss three auxiliary results, one on a periodic differential equation, one on a smooting operator and one on the exponential map.

a. Let $k, n \in \mathbb{N}$ and let $g \in C^k(\mathbb{R}^n, \mathbb{R})$ be $2\pi$-periodic in $\xi_0, \xi_1, \ldots, \xi_{n-1}$. Let $[g]$ denote the mean of $g$, i.e.

$$[g] = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\xi_0 \cdot \ldots \int_0^{2\pi} d\xi_{n-1} g(\xi) \ ;$$

moreover $|g|_k$ will denote the $C^k$-norm of $g$, cf. § 1.4. Consider the differential equation

$$u_\xi \omega = g(\xi) \quad , \tag{2.3}$$

where $\omega$ is an $n$-vector satisfying the small divisor condition: there exist positive constants $\gamma$ and $\tau$ such that

$$|(j,\omega)| \geq \gamma |j|^{-\tau} \quad \text{for all } j \in \mathbb{Z}^n \setminus \{0\} \ . \tag{2.4}$$

Compare (1.8). Here $u_\xi \omega$ abbreviates $\sum_{i=0}^{n-1} \omega_i \frac{\partial u}{\partial \xi_i}$. We require that $u \in C^1(\mathbb{R}^n, \mathbb{R})$. Now we have

LEMMA 2.1:

Let the above assumptions be satisfied with $k > n + \tau$ and $[g] = 0$. Then there exists a unique $2\pi$-periodic solution u of (2.3) with $[u] = 0$. Let $Lg$ denote this solution. There exists a positive constant K, independent of g and $\gamma$ such that

$$|Lg|_0 \leq \frac{K}{\gamma} |g|_k .$$

PROOF:

We have $g(\xi) = \sum_{j \in \mathbb{Z}^n} g_j e^{i(j,\xi)}$, where

$$g_j = [g(\xi) e^{-i(j,\xi)}] .$$

Since $g \in C^k$ we may integrate by parts k times to obtain

$$|g_j| \leq n^k |j|^{-k} |g|_k .$$

If we write $u(\xi) = \sum u_j e^{i(j,\xi)}$ then (2.3) implies that for all j: $i(j,\omega)u_j = g_j$. Since $g_0 = [g] = 0$, all $u_j$ are uniquely determined if we require $[u] = 0$. Now

$$|u_j| = \left|\frac{g_j}{(j,\omega)}\right| \leq \frac{1}{\gamma} n^k |j|^{\tau-k} |g|_k \qquad \text{if } j \neq 0.$$

Since $\sum_{j\neq 0} |j|^{\tau-k}$ is convergent for $k > n + \tau$ we see that

$$(2.5) \qquad (Lg)(\xi) = \sum_{j\neq 0} \frac{g_j}{i(j,\omega)} e^{i(j,\xi)}$$

is the unique solution of (2.3) with $[u] = 0$. Moreover

$$|Lg|_0 \leq \frac{K}{\gamma} |g|_k \quad ,$$

where $K = n^k \sum_{j \neq 0} |j|^{\tau - k}$.

QED

Compare § 1.4, formula (1.30) and what follows.

Also compare e.g. [11].

b. Following Moser [11], § 3b, we use a smoothing operator.

Let $n, m \in \mathbb{N}$, $\ell > 0$, $N > 1$ and $b_j < c_j$, $M_j > \max(1, 2/(c_j - b_j))$

for $j = 0, 1, \ldots, m-1$.

Let $A = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid b_j \leq y_j \leq c_j,\ j = 0, 1, \ldots, m-1\}$

and $A_1 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid b_j + M_j^{-1} \leq y_j \leq c_j - M_j^{-1},\ 0 \leq j \leq m-1\}$.

Then we define the smooting operator

$$T : C(A, \mathbb{R}^h) \to C^\infty(A_1, \mathbb{R}^h) \quad ,$$

where $h \in \mathbb{N}$, by means of a convolution as follows:

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi(x) = 0$ if $|x| \geq 1$, and

$$\int_{-\infty}^{+\infty} x^p \chi(x)\, dx = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } 0 < p < \ell. \end{cases}$$

Then define

$$\tilde{\chi}(x,y) = \{\prod_{j=0}^{n-1} N\chi(Nx_j)\}\{\prod_{j=0}^{m-1} M_j\, \chi(M_j y_j)\} \quad ,$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$. Finally, if $f \in C(A, \mathbb{R}^h)$, we define

$$(Tf)(x,y) = \iint_A \tilde{\chi}(x-\xi, y-\eta)\, f(\xi,\eta)\, d\xi d\eta \quad ,$$

for $(x,y) \in A_1$.

Thus T leaves polynomials in $x_0, x_1, \ldots, x_{n-1}, y_0, y_1, y_{m-1}$, of degree less than $\ell$ in each variable, unchanged.

If f is 2 -periodic in $x_0, x_1, \ldots, x_{n-1}$, then the same holds for Tf. One now may prove

<u>LEMMA 2.2</u>:

If $f \in C(A, \mathbb{R}^h)$, then

$$\left| \frac{\partial^{|\zeta_1|+|\zeta_1|}}{\partial x^{\zeta_1} \partial y^{\zeta_2}} Tf(x,y) \right| \leq c_\zeta N^{|\zeta_1|} M^{|\zeta_2|} |f|_0 ,$$

for all multi indices $\zeta_1$ and $\zeta_2$ and $(x,y) \in A_1$.

Here $c_\zeta$ is a constant depending on $\chi$, n, m and $\zeta = (\zeta_1, \zeta_2)$ only.

If $f \in C(A, \mathbb{R}^h)$, then

$$|(f - Tf)(x,y)| \leq c \sup_{|\zeta_1|+|\zeta_2|=\ell} N^{-|\zeta_1|} M^{-|\zeta_2|} \left| \frac{\partial^{|\zeta_1|+|\zeta_2|}}{\partial x^{\zeta_1} \partial y^{\zeta_2}} f \right|_0$$

if $(x,y) \in A_1$.

Here c is a constant independent of f, M and N. The norms $|.|_0$ are those in $C(A, \mathbb{R}^h)$. Compare § 1.4.

For $M = (M_0, M_1, \ldots, M_{m-1})$ and $\alpha \in \mathbb{Z}^m$ we write $M^\alpha = M_0^{\alpha_0} \cdot \ldots M_{m-1}^{\alpha_{m-1}}$.

For a proof see [11], §3b. For use later on, we note the following corollaries of both lemmas:

Assume n = 3, m = 6, $f \in C(A, \mathbb{R}^3)$, f is periodic in the first three scalar variables with mean $[f] = 0$.

Let $k > \tau + 3$ and let L be defined as in lemma 2.1 with n = 3. Then there exists a constant $K_0 > 0$, independent of f and $\gamma$ such that

$$(2.6) \qquad |(L\,T\,f)(x,y)| \le \gamma^{-1}\, K_0 N^k |f|_0 \;, \text{ for } (x,y) \in A_1 \;.$$

If, moreover, $\zeta = (\zeta_1,\zeta_2)$, then there exists a constant $K(\zeta) > 0$, independent of f and $\gamma$, such that

$$(2.7) \qquad \left| \frac{\partial^{|\zeta_1|+|\zeta_2|}}{\partial x^{\zeta_1} \partial y^{\zeta_2}} (L\,T\,f)(x,y) \right| \le \gamma^{-1}\, K(\zeta)\; N^{|\zeta_1|+k}\, M^{\zeta_2} |f|_0 \;,$$

for $(x,y) \in A_1$.

Formula (2.6) immediately follows from the lemmas 2.1 and 2.2, whereas for (2.7) we use that that the differential operator $\dfrac{\partial^{|\zeta_1|+|\zeta_1|}}{\partial x^{\zeta_1}\, y^{\zeta_2}}$ and L commute.

(The latter easily follows from 2.5.)

c. We need one more technical result in proving theorem C.

As we indicated in §2.1 in this chapter the Lie algebras $\ddot{L} \subseteq \dot{L} \subseteq L$ consist of $C^\ell$ vector fields, where $\ell$ will be specified later on. Compare § 1.4 where everything is real analytic. We recall the notion of the exponential map exp, which associates to a vector field its time-1 flow. Observe that exp is well-defined in $L_1$. We have $\exp: L_1 \to G_1$, while $\exp(\dot{L}_1) \subseteq \dot{G}_1$ and $\exp(\dot{L}_1) \subseteq \dot{G}_1$.

Here $G_1, \dot{G}_1$ and $\ddot{G}_1$ are the Lie groups, corresponging to $L_1, \dot{L}_1$ and $\ddot{L}_1$, consisting of $C^\ell$-diffeomorphisms of type (2.2). Compare §1.4c.

Our last lemma reads

<u>LEMMA 2.3:</u>

There exists a positive constant $K_\ell$ such that for all $X \in L_1$ and all $1 \le p \le \ell$ :

$$|\exp X - \mathrm{id}|_p \le K_\ell\; |X|_p \quad .$$

This result can be proven using Gronwall's lemma.

A straight forward extension of this lemma holds if the vector fields depend smoothly on some extra parameters.

## § 2.3 A transformation theorem for the iteration

In this section we consider a transformation theorem that will be used to generate the Newtonian iteration procedure for the construction of the conjugacy in theorem C. Compare e.g. [11].

Let F be the system

$$F \quad \begin{cases} \dot{x} = a + f(x,y,a) \\ \dot{y} = g(x,y,a) \ , \end{cases}$$

where f and g are of class $C^\ell$ ($\ell \in \mathbb{N}$) with range $\mathbb{R}^3$ and domain $S = \mathbb{R}^3 \times \{ y \in \mathbb{R}^3 \mid |y| \leq \frac{1}{p}\} \times \{a \in \mathbb{R}^3 \mid |a - \omega| \leq \gamma\, \varepsilon\}$, and $2\pi$-periodic in $x_0, x_1, x_2$. Note that F has the form (1.15).
Here P, $\gamma$ and $\varepsilon$ are positive numbers, while $\omega \in \mathbb{R}^3$ satisfies the small divisor condition (1.8). The number $\tau$ in (1.8) lies between 3 and 4.
We recall the fact that F is supposed to belong to $\dot{L}$ (or $\bar{L}$).

We shall transform F by means of a change of coordinates $U$

$$U \quad \begin{cases} x = \xi + u(\xi,\alpha) \\ y = \eta + v(\xi,\eta,\alpha) \\ a = \alpha + \omega(\alpha) \ , \end{cases}$$

where $v(\xi,\eta,\alpha) = \bar{v}(\xi,\alpha) + V(\xi,\alpha)\eta$, $u$, $\bar{v}$ and $w$ are 3-vectors and $V$ is a $3 \times 3$-matrix of class $C^{\infty}$ in $(\xi,\alpha)$ if $\xi \in \mathbb{R}^3$, $\alpha \in \mathbb{R}^3$, $|\alpha - \omega| \leq \gamma\, \varepsilon_+$, which are $2\pi$-periodic in $\xi_0$, $\xi_1$, $\xi_2$. Here $\varepsilon_+$ is some positive number, to be specified later on.
Again recall that $U$ should respect the nature of F.

The system F is transformed by $U$ into the system $\Phi$:

$$\Phi \qquad \begin{cases} \dot{\xi} = \alpha + \varphi(\xi,\eta,\alpha) \\ \dot{\eta} = \psi(\xi,\eta,\alpha) \quad , \end{cases}$$

defined on a set $S_+ = \mathbb{R}^3 \times \{\eta \in \mathbb{R}^3 \mid |\eta| \leq P_+^{-1}\} \times \{\alpha \in \mathbb{R}^3 \mid |\alpha - \omega| \leq \gamma\, \varepsilon\}$, where $P_+ > 0$ is to be specified later on and where $\varphi$ and $\psi$ are of class $C^{\ell}$ on $S_+$, $2\pi$-periodic in $\xi_0$, $\xi_1$ and $\xi_2$. We write $\Phi = U^*F$
Note that $F \in \dot{L}\ (\ddot{L})$ and $U \in \dot{G}_1(\ddot{G}_1)$ imply that $\Phi \in \dot{L}\ (\ddot{L})$.

The perturbation terms f, g, $\varphi$ and $\psi$ will be estimated by several parameters which we recall form §1.4 (see(1.16)):

$$(2.8) \qquad \begin{cases} \text{Let } \delta > 0,\ \varepsilon = \delta^{\sigma},\ N = \delta^{-r},\ P = \delta^{-s},\ Q = \delta^{-q} \text{ and } M = \delta^{-t}\ , \\ \text{where } \sigma = 0.93;\ r = 0.1;\ s = 2;\ q = 0.95;\ t = 2.01. \\ \text{Let } \delta_+ = \delta^{\kappa} \text{ where } \kappa = 1.03. \\ \text{Also we set } \varepsilon_+ = \delta_+^{\sigma},\ N_+ = \delta_+^{-r} \text{ etc.} \end{cases}$$

We then have

<u>THEOREM 2.4</u>

Let $\gamma^* > 0$ and $\ell = 363$. There exists a positive number $\delta^*$, only depending on $\gamma^*$, with the following properties:
If $0 < \delta \leq \delta^*$ , $0 < \gamma \leq \gamma^*$ and $F \in \dot{L}\ (\ddot{L})$ satisfy the above assumptions with

$$(2.9^a)\qquad |Nf(x,y,a)| + |Pg(x,y,a)| \le \gamma N \delta \ ,$$

$$(2.9^b)\qquad |D_x^{\zeta_1} D_y^{\zeta_2} D_a^{\zeta_3}\, Nf(x,y,a)| + |D_x^{\zeta_1} D_y^{\zeta_2} D_a^{\zeta_3}\, Pg(x,y,a)| \le \le \gamma^{1-|\zeta_3|} N^{|\zeta_1|+1} P^{|\zeta_2|} Q^{|\zeta_3|} \ ,$$

for all $(x,y,a) \in S$, and tri-indices $\zeta_1$, $\zeta_2$, $\zeta_3$ with $|\zeta_1| + |\zeta_2| + |\zeta_3| = \ell$, then a $C^\infty$ transformation $U \in \dot{G}_1 (\ddot{G}_1)$ can be chosen such that $\Phi \in \dot{L}\ (\bar{L})$ satisfies

$$(2.10^a)\qquad |N_+\, \varphi(\xi,\eta,\alpha)| + |P_+\, \psi(\xi,\eta,\alpha)| \le \gamma N_+ \delta_+ \ ,$$

$$(2.10^b)\qquad |D_\xi^{\zeta_1} D_\eta^{\zeta_2} D_\alpha^{\zeta_3}\, N_+\varphi(\xi,\eta,\alpha)| + |D_\xi^{\zeta_1} D_\eta^{\zeta_2} D_\alpha^{\zeta_3}\, P_+\psi(\xi,\eta,\alpha)| \le \le \gamma^{1-|\zeta_3|} N_+^{|\zeta_1|+1} P_+^{|\zeta_2|} Q_+^{|\zeta_3|} \ ,$$

for all $(\xi,\eta,\alpha) \in S_+$ and tri-indices $\zeta_1$, $\zeta_2$, $\zeta_3$ with $|\zeta_1| + |\zeta_2| + |\zeta_3| = \ell$, while U satisfies

$$(2.11)\quad \left\{ \begin{array}{l} |w(\alpha) + \alpha - \omega| \le \gamma(\varepsilon - Q_+^{-1}) \ , \\ |D_\xi^{\zeta_1} D_\alpha^{\zeta_2}\, u(\xi,\alpha)| \le K_\zeta\, N_+^{7+|\zeta_1|} (\gamma^{-1} Q_+)^{|\zeta_2|}\, \delta \ , \\ |D_\xi^{\zeta_1} D_\alpha^{\zeta_2}\, \bar{v}(\xi,\alpha)| \le K_\zeta N_+^{7+|\zeta_1|} \frac{N}{P} (\gamma^{-1} Q_+)^{|\zeta_2|} \delta \ , \\ |D_\zeta^{\zeta_1} D_\alpha^{\zeta_2}\, V(\xi,\alpha)| \le K_\zeta N_+^{7+|\zeta_1|}\, M\, \frac{N}{P} (\gamma^{-1} Q_+)^{|\zeta_2|}\, \delta \ , \end{array} \right.$$

for all $\xi \in \mathbb{R}^3$, $\alpha \in \mathbb{R}^3$ with $|\alpha - \omega| \le \gamma\, \varepsilon_+$, and tri-indices $\zeta_1$, $\zeta_2$. The constant $K_\zeta$ only depends on $\gamma^*$ and $\zeta = (\zeta_1,\zeta_2)$.

PROOF

The transformation $U$ conjugates F and $\Phi$, which can be expressed as

$$(2.12)\qquad \begin{cases} (\mathrm{id} + D_\xi u)\varphi + D_\xi u\,\alpha = w(\alpha) + f(\xi + u,\ \eta + v,\ \alpha + w) \\ (\mathrm{id} + V)\psi + (D_\xi \bar{v} + D_\xi V\eta)(\alpha + \varphi) = g(\xi + u,\ \eta + v,\ \alpha + w). \end{cases}$$

We have to find u, v and w such that $\varphi$ and $\psi$ are small compared with the bounds on f and g. In stead of determining $U \in \dot{G}_1(\bar{G}_1)$ we shall construct an infinitesimal generator

$$\bar{U} \in \dot{L}_1(\bar{L}_1), \text{ and then we define } U = \exp(\bar{U}) \quad .$$

We apply the same procedure as in §1.4. We determine $\bar{U}$ from (1.28) and (1.29) with the following substitutions: U, N and F from (1.28) are replaced by

$$\begin{cases} \bar{U} = \bar{u}(\xi)\dfrac{\partial}{\partial\xi} + \{\bar{\bar{v}}(\xi) + \bar{V}(\xi)\eta\}\dfrac{\partial}{\partial\eta} \quad , w(\alpha)\dfrac{\partial}{\partial\xi} \quad \text{and} \\ (Tf)(\xi,0,\alpha)\dfrac{\partial}{\partial\xi} + \{(Tg)(\xi,0,\alpha) + D_\eta(Tg)(\xi,0,\alpha)\eta\}\dfrac{\partial}{\partial\eta} \end{cases}$$

respectively.

Here T is the smoothing operator of § 2.2, operating on the variables $x = (x_0,x_1,x_2) \in \mathbb{R}^3$ and $y = (y_0,y_1,y_2,a_0,a_1,a_2) \in \mathbb{R}^6$ (so $y_3 = a_0$, $y_4 = a_1$, $y_5 = a_2$) with the choice

$$(2.13)\qquad \begin{cases} \ell = 363,\ N = N_+,\ c_j = -b_j = P^{-1},\ M_j = M = \delta^{-t} \\ \text{if } 0 \leq j \leq 2 \text{ , and} \\ c_j = \omega_j + \gamma\varepsilon,\ b_j = \omega_j - \gamma\varepsilon,\ M_j = \gamma^{-1}Q_+ \quad \text{if } 3 \leq j \leq 5 \ . \end{cases}$$

So we determine $\bar{u}$ and $\bar{v}$ such that

$$(2.14)\qquad \begin{cases} D_\xi\bar{u}(\xi,\alpha)\omega = (Tf)(\xi,0,\alpha) + \omega(\alpha) \\ D_\xi\bar{\bar{v}}(\xi,\alpha)\omega = (Tg)(\xi,0,\alpha) \\ D_\xi\bar{V}(\xi,\alpha)\omega = D_\eta(Tg)(\xi,0,\alpha) \quad . \end{cases}$$

Observe that, according to lemma 2.1, the equations (2.14) are solvable if the right hand sides have mean $[\cdot]= 0$. Since $F \in \dot{L}$, we have that $[g] = 0$ and therefore $[Tg] = 0$ and $[D_\eta(Tg)] = 0$. So the equations for $\bar{\bar{v}}$ and $\bar{V}$ are not problematic.
The equation for $\bar{u}$ is somewhat more subtle: the "parameter" $w(\alpha)$ is necessary to balance $[(Tf)(.,0,a)]$. Compare § 1.4, formula (1.27).

So first we set

$$(2.15) \qquad w(\alpha) = -[(Tf)(.,0,a)], \text{ i.e. } \alpha = a + [(Tf)(.,0,a)].$$

We consider this equation for $|\alpha - \omega| \leq \gamma\varepsilon_+$ and look for solutions in $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$.
If $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$, then, according to lemma 2.2 and (2.9[a]) we have $|[(Tf)(.,0,a)]| \leq \gamma\delta c_0$. Now if $\delta^*$ is sufficiently small then

$$(2.16) \qquad c_0\delta < \varepsilon - Q_+^{-1} - \varepsilon_+ \quad \text{for } 0 < \delta \leq \delta^* ,$$

in view of (2.8). Hence

$$|[(Tf)(.,0,a)]| < |a - \alpha| \text{ if } |a - \omega| = \gamma(\varepsilon - Q_+^{-1}) \text{ and}$$

$$|\alpha - \omega| \leq \gamma\varepsilon_+.$$

By an extension of Rouché's theorem (cf. Deimling [6[a], p.45]) the function $a \mapsto a - \alpha + [(Tf)(.,0,a)]$ now has at least one zero in the disc $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ for any $\alpha$ with $|\alpha - \omega| \leq \gamma\varepsilon_+$. So (2.15) has a solution in $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ as soon as $|\alpha - \omega| \leq \gamma\varepsilon_+$ and the first line of (2.11) follows.

Lemma 2.2 and (2.8) imply that for $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ also

$$|D_a\, Tf(x,0,a)| \leq \tilde{c}\, Q_+\delta .$$

Here and in the following numbers $c$, $\tilde{c}$, $c_0$, $K_0$, $K_1$ etc. always denote constants independent of $f$, $g$, $\delta$ and $\gamma$.

From (2.8) it follows that $|D_a Tf| < \frac{1}{2}$ if $0 < \delta \leq \delta^*$, with $\delta^*$ sufficiently small and positive. Hence (2.15) has at most one solution in $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$.

Combining this with the previous result we see that (2.15) has a unique solution in the disc $|q - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ if $|\alpha - \omega| \leq \gamma\varepsilon_+$. This solution is $C^\infty$ by the implicit function theorem, because Tf is $C^\infty$.

We now solve (2.14) using lemma 2.1 and (2.15), (2.16). It so follows that for $|\alpha - \omega| \leq \gamma\varepsilon_+$ there exists a unique solution $\bar{U}$ of (2.14) with $[\bar{u}] = [\bar{\bar{v}}] = 0$ and $[\bar{V}] = 0$, which is $2\pi$-periodic in $\xi_0$, $\xi_1$, $\xi_2$ and of class $C^\infty$.
As in § 1.4 one shows that $\bar{U} \in \dot{L}_1(\bar{L}_1)$.
From (2.8), (2.15) and the corollaries of lemmas 2.1 and 2.2 we may derive the estimates (2.11) where u, $\bar{v}$ and V are replaced by $\bar{u}$, $\bar{\bar{v}}$ and $\bar{V}$ respectively.
In view of lemma 2.3 this yields (2.11) in its original form.

Next we prove (2.10$^a$) for $\Phi = U^*F$ with help of (2.12), (2.11) and (2.9). From (2.12) and (2.14) we deduce

$$(2.17) \qquad \varphi = (\mathrm{id} + D_\xi u)^{-1}\{-D_\xi u\alpha + D_\xi \bar{u}(\xi,\alpha)\omega + f(\xi + v, \eta + v, a) - (Tf)(\xi,0,\alpha)\},$$

where $\varphi = \varphi(\xi,\eta,\alpha)$, $u = u(\xi,\alpha)$, $v = v(\xi,\eta,\alpha)$, a is determined by (2.15) and where $(\xi,\eta,\alpha) \in S_+$.

If $(\xi,\eta,\alpha) \in S_+$ then we have by (2.11) and (2.8)

$$(2.18)\qquad |v(\xi,\eta,\alpha)| \le K_0 N_+^7 \frac{N}{P}\,\delta(1+\frac{M}{P_+}) \le \frac{1}{P_+}$$

if $\delta^*$ is sufficiently small positive and $0 < \delta \le \delta^*$.

Hence

$$(2.19)\qquad |\eta + v(\xi,\eta,\alpha)| \le P^{-1} - M^{-1}, \quad \text{if } (\xi,\eta,\alpha) \in S_+$$

and $\delta^*$ is sufficiently small. So the function $f(\xi + u, \eta + v, a)$ in (2.17) makes sense if $(\xi,\eta,\alpha) \in S_+$.

From (2.8) and (2.11) we also deduce that

$$(2.20)\qquad |(id + D_\xi u)^{-1}| \le 2 \qquad \text{if } |\alpha - \omega| \le \gamma\varepsilon_+$$

and $\delta^*$ is sufficiently small positive.

Now we consider

$$(2.21)\qquad -D_\xi u\alpha + D_\xi \bar{u}\omega = -D_\xi u(\alpha - \omega) + D_\xi(\bar{u} - u)\omega .$$

The first term can be easily estimated using (2.11) and (2.8):

$$(2.22)\qquad |D_\xi u(\omega - \alpha)| \le K_1 N^8\, \gamma\delta\varepsilon_+ \le \frac{1}{10}\gamma\delta_+ \text{ if } |\omega - \alpha| \le \gamma\varepsilon_+$$

and if $\delta^*$ is sufficiently small.

In order to estimate the second term we consider $\exp(t\bar{U})$ for $0 \le t \le 1$ and therefore introduce the following notation, which suppresses $\alpha$:

$$(\exp(t\bar{U}))\binom{\xi}{\eta} = \begin{pmatrix} x(t,\xi) \\ y(t,\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi + u(t,\xi) \\ \eta + v(t,\xi,\eta) \end{pmatrix},$$

so $u(\xi) = u(1,\xi)$ and $v(\xi,\eta) = v(1,\xi,\eta)$.

We then have that $x(t,\xi)$ and $y(t,\xi,\eta)$ satisfy the following differential equations

$$(2.23)\quad \begin{cases} \dfrac{d}{dt}\, x(t,\xi) = \bar{u}(x(t,\xi))\ , \quad x(0,\xi) = \xi \quad \text{and} \\[2ex] \dfrac{d}{dt}\, y(t,\xi,\eta) = \bar{\bar{v}}(x(t,\xi)) + \bar{V}(x(t,\xi),\ y(t,\xi,\eta)),\ y(0,\xi,\eta) = \eta\ . \end{cases}$$

Since $u(t,\xi) = \int_0^t \bar{u}(x(s,\xi))ds$, it easily follows

$$(2.24)\quad \begin{cases} |u(t,\xi)| \le t|\bar{u}|_0 \quad \text{for } t \ge 0 \quad \text{and} \\[2ex] |u(\xi) - \bar{u}(\xi)| \le \frac{1}{2}\, |\bar{u}|_0 |\bar{u}|_1\ . \end{cases}$$

From (2.23) we derive

$$(2.25)\quad \frac{d}{dt}\, D_\xi\, u(t,\xi) = D_x\bar{u}(x(t,\xi))\ (\text{id} + D_\xi\, u(t,\xi))\ ,$$

which by Gronwall's lemma yields

$$|D_\xi u(t,\xi)| \le c\ |\bar{u}|_1 \qquad \text{if } 0 \le t \le 1\ .$$

Now multiply (2.25) on the right by $\omega$ and use the fact that

$$|D_x\bar{u}(x)\omega| \le c\gamma\delta$$

(a consequence of (2.14), (2.15) and (2.9)).

This yields by Gronwall:

$$(2.26)\quad |D_\xi u(t,\xi)\omega| \le c\gamma\delta \qquad \text{if } 0 \le t \le 1\ .$$

Now by (2.25):

$$D_\xi(u-\bar{u})(\xi)\omega = \int_0^1 \{D_x\bar{u}(x(t,\xi)) - D_\xi\bar{u}(\xi)\}\,\omega\, dt\ +$$

$$+ \int_0^1 D_x\bar{u}(x(t,\xi))D_\xi u(t,\xi)\,\omega\, dt\ .$$

The second integral, with help of (2.26) is easily estimated by $c|\bar{u}|_1\,\gamma\delta$.

For the first integral we use (2.14), (2.15) and the mean value theorem. This yields as an upper bound

$$|(Tf)_x| \int_0^1 |u(t,\xi)|\,dt \le N|f|_0|\bar{u}|_0$$

by (2.24) and lemma 2.2.

We now conclude

$$(2.27) \qquad |D_\xi(u - \bar{u})(\xi)\omega| \le \frac{1}{10}\,\gamma\delta_+ ,$$

using (2.8), (2.9) and (2.11) and the fact that $\delta^*$ is small.

Now consider the last part of (2.17).

We have

$$(2.28) \quad \begin{cases} |f(\xi + u,\ \eta + v,\ a) - (Tf)(\xi,0,a)| \le \\ < |f(\xi + u,\ \eta + v,\ a) - (Tf)(\xi + u,\eta + v,\ a)| + \\ + |(Tf)(\xi + u,\eta + v,\ a) - (Tf)(\xi,0,a)| . \end{cases}$$

Using lemma 2.2 and (2.9) we deduce

$$(2.29) \qquad |f(\xi + u,\ \eta + v,\ a) - (Tf)(\xi + u,\ \eta + v,\ a) \le$$

$$\le c\gamma\ \sup\{(\frac{N}{N_+})^{|\zeta_1|}(\frac{D}{M})^{|\zeta_2|}(\frac{Q}{Q_+})^{|\zeta_3|}\ \Big|\ |\zeta_1|+|\zeta_2|+|\zeta_3| = \ell\} = c\gamma(\frac{N}{N_+})^{\ell}$$

if $|\eta + v| \le P^{-1} - M^{-1}$, $|a - \omega| \le \gamma(\varepsilon - Q_+^{-1})$. In the last part of (2.29) we used $\frac{D}{M} \le \frac{N}{N_+}$ and $\frac{Q}{Q_+} \le \frac{N}{N_+}$ which is a consequence of (2.8). Because of (2.19) we may apply (2.29) with $u = u(\xi,\alpha)$, $v = v(\xi,\eta,\alpha)$ if $(\xi,\eta,\alpha) \in S_+$. From (2.8) it follows that the right hand side of

(2.29) is $o(\gamma\delta_+)$ as $\delta \to 0$.

To the last part of (2.28) we apply the mean value theorem and then use lemma 2.2. Thus we get

$$(2.30)\quad \begin{cases} |(Tf)(\xi + u,\eta + v,\ a) - (Tf)(\xi,0,a)| \leq \\ \leq |D_x(Tf)|\ |u| + |D_y(Tf)|\ |\eta + v| \leq \\ \leq c_1\gamma\delta\ (N_+|u| + M|\eta + v|)\ , \end{cases}$$

if $|\eta + v| \leq P^{-1} - M^{-1}$, $|a - \omega| \leq (\varepsilon - Q_+^{-1})$. We apply (2.30) with $u = u(\xi,\alpha)$, $v = v(\xi,\eta,\alpha)$ and $a = \alpha + w(\alpha)$ for $(\xi,\eta,\alpha) \in S_+$. Cf. (2.19) and (2.11).

Using (2.8), (2.11) and (2.19) we deduce

$$(Tf)(\xi + u,\ \eta + v,\ a) - (Tf)(\xi,0,a) = o(\gamma\delta_+) \quad \text{as } \delta \to 0.$$

Combination of this result with the estimate on (2.29), (2.28), (2.27), (2.22) and (2.17) shows that

$$(2.31)\qquad |\varphi(\xi,\eta,\alpha)| \leq \frac{1}{2}\gamma\delta_+ \qquad \text{if } (\xi,\eta,\alpha) \in S_+\ ,$$

provided that $0 < \delta \leq \delta^*$ with $\delta^*$ sufficiently small.

We now indicate an analogous estimate on $\psi$, which together with (2.31) will imply ($2.10^a$).

From (2.12) and (2.14) we deduce

$$(2.32)\qquad \psi = (\mathrm{id} + V)^{-1}\ \{-(D_\xi\bar{v} + D_\xi V_\eta)(\alpha + \varphi) + (D_\xi\bar{\bar{v}} + D_\xi\bar{V}_\eta)\omega\ +$$
$$+\ g(\xi + u,\ \eta + v,\ a) - (Tg)(\xi,0,a) - D_\eta(Tg)(\xi,0,a)\eta\}\ ,$$

where $\psi = \psi(\xi,\eta,\alpha)$, $\varphi = \varphi(\xi,\eta,\alpha)$, $u = u(\xi,\alpha)$, $v = v(\xi,\eta,\alpha)$,

$\bar{v} = \bar{v}(\xi,\alpha)$, $V = V(\xi,\alpha)$, $\bar{\bar{v}} = \bar{\bar{v}}(\xi,\alpha)$ and $\bar{V} = \bar{V}(\xi,\alpha)$.

As before we have from (2.11)

$$(2.33) \qquad |(\mathrm{id} + V)^{-1}| \leq 2 \qquad \text{if} \quad |\alpha - \omega| \leq \gamma\varepsilon_+$$

and $\delta^*$ is small.

From (2.11) and (2.31) we deduce

$$(2.34) \qquad |(D_\xi \bar{v} + D_\xi V\eta)(\alpha + \varphi + \omega)| \leq K_1 N_+^8 \frac{N}{P} \delta (1 + \frac{M}{P_+})\gamma \, (\varepsilon_+ + \frac{1}{2}\delta_+) \ ,$$

if $(\xi,\eta,\alpha) \in S_+$. Using (2.8) we may show that this bound is $\gamma N_+ P_+^{-1} \delta_+ o(1)$ as $\delta \to 0$.

We now have to estimate

$$D_\xi(\bar{\bar{v}} - \bar{v})\omega + D_\xi(\bar{V} - V)\eta\omega \ .$$

This can be achieved analogous to the estimate (2.27). We now use the second part of (2.23) which falls apart into

$$\begin{cases} \dfrac{d}{dt}\bar{v}(x(t,\xi)) = \bar{\bar{v}}(x(t,\xi)) & \text{and} \\[2ex] \dfrac{d}{dt}V(x(t,\xi)) = \bar{V}(x(t,\xi)) \ . \end{cases}$$

We so obtain

$$(2.35) \qquad D_\xi(\bar{\bar{v}} - \bar{v})\omega + D_\xi(\bar{V} - V)\eta\,\omega| \leq \frac{1}{10}\gamma\, N_+ P_+^{-1} \delta_+ \ ,$$

if $(\xi,\eta,\alpha) \in S_+$ and $\delta^*$ is small.

In order to estimate the last part of (2.32) we consider separately several differences. From lemma 2.2 and (2.9) we derive similarly to (2.29):

$$(2.36) \qquad |g(\xi + u, \eta + v, a) - (Tg)(\xi + u, \eta + v, a)| \leq c\gamma \left(\frac{N}{N_+}\right)^{\ell} \frac{N}{P},$$

if $|\eta + v| \leq P^{-1} - M^{-1}$, $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$. Because of (2.8) the righthand side is $\gamma\, N_+P_+^{-1}\delta_+ o(1)$ as $\delta \to 0$.

An application of the mean value theorem, lemma 2.2 and (2.9) may be used to show that

$$(2.37) \quad \begin{cases} |(Tg)(\xi + u, \eta + v, a) - (Tg)(\xi,\eta, a)| \leq \\ |D_x(Tg)|\,|u| + |D_y(Tg)|\ |v| \leq c_1\ |g|_0(N_+|u| + M|v|) \\ \leq c_1 \frac{N}{P} \gamma\delta(N_+|u| + M|v|) \end{cases}$$

if $|\eta|$ and $|\eta + v| \leq P^{-1} - M^{-1}$, $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ .

Finally we have by lemma 2.2 and (2.9)

$$(2.38) \quad \begin{cases} |(Tg)(\xi,\eta,a) - (Tg)(\xi,0,a) - D_\eta(Tg)(\xi,0,a)\eta| \leq \\ \leq 5|\eta|^2 \sup\{ \frac{\partial^{|\zeta|}}{\partial\eta^\zeta} (Tg)(\xi,\eta,a)| \ \Big|\ |\zeta|= 2,\ |\eta| \leq P_+^{-1}\} \\ \leq c|\eta|^2\ M^2|g|_0 \leq c|\eta|^2\ M^2 \frac{N}{P} \gamma\delta \quad , \end{cases}$$

if $|\eta| \leq P_+^{-1}, |a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$ .

Now let $u = u(\xi,\alpha)$, $v = v(\xi,\eta,\alpha)$ where $|\eta| \leq P^{-1}$, $|\omega - \alpha| \leq \gamma\varepsilon_+$. Then $|\eta| \leq P^{-1} - M^{-1}$, $|\eta + v| \leq P^{-1} - M^{-1}$ (cf. (2.19), $|a - \omega| \leq \gamma(\varepsilon - Q_+^{-1})$, and u and v satisfy (2.11). Now combine (2.32) with (2.33) - (2.38), and take into account (2.11) and (2.8). Then we may derive

$$(2.39) \qquad |\psi(\xi,\eta,\alpha)| \leq \frac{1}{2} N_+P_+^{-1}\delta_+, \text{ if } (\xi,\eta,\alpha) \in S_+ \quad ,$$

if $\delta^*$ is sufficiently small positive. With (2.39) thus ($2.10^a$) follows.

We next consider $(2.10^b)$. As in Moser [11] we blow up $F$, $U$ and $\Phi$ by $\hat{x} = N_+x$, $\hat{\xi} = N_+\xi$, $\hat{y} = P_+y$, $\hat{\eta} = P_+\eta$, $\hat{a} = \gamma^{-1}Q_+a$, $\hat{\alpha} = \gamma^{-1}Q_+\alpha$. Let

$$\hat{f}(\hat{x},\hat{y},\hat{a}) = N_+f\left(\frac{\hat{x}}{N_+}, \frac{\hat{y}}{P_+}, \gamma\frac{\hat{a}}{Q_+}\right),$$

$$\hat{g}(\hat{x},\hat{y},\hat{a}) = P_+g\left(\frac{\hat{x}}{N_+}, \frac{\hat{y}}{P_+}, \gamma\frac{\hat{a}}{Q_+}\right), \text{ etc.}$$

Then we get

$$\begin{cases} \hat{F} : \dot{\hat{x}} = \gamma\dfrac{N_+}{Q_+}\hat{a} + \hat{f}(\hat{x},\hat{y},\hat{a}), \quad \dot{\hat{y}} = \hat{g}(\hat{x},\hat{y},\hat{a}) \\[2ex] \hat{U} : \hat{x} = \hat{\xi} + \hat{u}(\hat{\xi},\hat{\alpha}), \quad \hat{y} = \hat{\eta} + \hat{v}(\hat{\xi},\hat{\eta},\hat{\alpha}), \quad \hat{a} = \hat{\alpha} + \hat{w}(\hat{\alpha}) \\[2ex] \hat{\Phi} : \dot{\hat{\xi}} = \gamma\dfrac{N_+}{Q_+}\hat{\alpha} + \hat{\varphi}(\hat{\xi},\hat{\eta},\hat{\alpha}), \quad \dot{\hat{\eta}} = \hat{\psi}(\hat{\xi},\hat{\eta},\hat{\alpha}) . \end{cases}$$

From (2.8) and (2.9) we deduce

$$(2.40)\qquad \begin{cases} |\hat{f}| + |\hat{g}| \le \dfrac{P_+}{P}(|Nf| + |Pg|) \le \dfrac{P_+}{P}N\gamma\delta \le \gamma, \\[2ex] |D_{\hat{x}}^{\zeta_1}D_{\hat{y}}^{\zeta_2}D_{\hat{a}}^{\zeta_3}\hat{f}| + |D_{\hat{x}}^{\zeta_1}D_{\hat{y}}^{\zeta_2}D_{\hat{a}}^{\zeta_3}\hat{g}| \le \gamma\dfrac{P_+}{P}N\left(\dfrac{N}{N_+}\right)^{\ell} \le \gamma \\[2ex] \text{if } |\zeta_1|+|\zeta_2|+|\zeta_3| = \ell, \ |\hat{y}| \le 1, \ \left|\hat{a} - \dfrac{Q_+\omega}{\gamma}\right| \le Q_+\varepsilon . \end{cases}$$

Here $Q_+\varepsilon \ge 1$. From this we may conclude that

$$|\hat{f}(\hat{x},\hat{y},\hat{a})|_{\ell} + |\hat{g}(\hat{x},\hat{y},\hat{a})|_{\ell} \le K_{\ell}\gamma ,$$

where $|\cdot|_{\ell}$ denotes the $C^{\ell}$-norm on $|\hat{y}| \le 1$, $\left|\hat{a} - \frac{Q_+\omega}{\gamma}\right| \le Q_+\varepsilon$, $K_{\ell}$ is a constant depending on $\ell$, but independent of $\gamma$ and $\delta$(cf. Moser [11]; we use a Taylor approximation of order $\ell$ and obtain bounds for the Taylor coefficients from this and (2.40)). Moreover, we may deduce from lemma (2.9) and (2.8)

$$|N_+(Tf)(\frac{\hat{x}}{N_+}, \frac{\hat{y}}{P_+}, \gamma\frac{\hat{a}}{Q_+})|_\ell + |P_+(Tg)(\frac{\hat{x}}{N_+}, \frac{\hat{y}}{P_+}, \frac{\hat{a}}{Q_+})|_\ell$$

$$+ |(Tg)_y (\frac{\hat{x}}{N_+}, \frac{\hat{y}}{P_+}, \gamma\frac{\hat{a}}{Q_+}|_\ell \leq K\gamma(N_+|f|_0 + P_+|g|_0) \leq K_1\gamma \; .$$

Note that since $\Phi = U^*F$ we have to consider the $C^{\ell+1}$-norm of the transformation $U$. But $|\hat{u}|_{\ell+1}$ and $|\hat{v}|_{\ell+1}$ are bounded because of (2.11) and (2.8); the bounds are independent of $\delta$ and $\gamma$.

Now $\hat{\varphi}$ and $\hat{\psi}$ are determined by (2.17) and (2.32) so

$$\left\{\begin{aligned} &\hat{\varphi} = (\mathrm{id} + D_{\hat{\xi}}\hat{u})^{-1} \{D_{\hat{\xi}}\hat{u}\, N_+(\omega - \alpha) + \hat{f}(\hat{\xi} + \hat{u}, \hat{\eta} + \hat{v}, \hat{a}) \\ &- N_+(Tf)(\frac{\hat{\xi}}{N_+}, 0, \gamma\frac{\hat{a}}{Q_+})\} \text{ and} \\ &\hat{\psi} = (\mathrm{id} + V)^{-1} \{- D_{\hat{\xi}}\hat{v}(P_+(\alpha - \omega) + \frac{P_+}{N_+}\hat{\varphi}) \quad + \\ &+ \hat{g}(\hat{\xi} + \hat{u}, \hat{\eta} + \hat{v}, \hat{a}) - P_+(Tg)(\frac{\hat{\xi}}{N_+}, 0, \gamma\frac{\hat{a}}{Q_+}) \\ &- (Tg)_y(\frac{\hat{\xi}}{N_+}, 0, \gamma\frac{\hat{a}}{Q_+})\hat{\eta}\} \; . \end{aligned}\right.$$

Using the estimates above we see that $\gamma^{-1}\hat{\varphi}$ and $\gamma^{-1}\hat{\psi}$ have $C^\ell$-norms which are bounded by a constant independent of $\gamma$ and $\delta$. Hence

$$|D_\xi^{\zeta_1} D_\eta^{\zeta_2} D_\alpha^{\zeta_3} N_+\varphi(\xi,\eta,\alpha)| + |D_\xi^{\zeta_1} D_\eta^{\zeta_2} D_\alpha^{\zeta_3} P_+\psi(\upsilon,\eta,\alpha)$$
$$\leq C\, N_+^{|\zeta_1|} P_+^{|\zeta_2|} Q_+^{|\zeta_3|} \gamma^{1-|\zeta_3|} .$$

If $\delta^*$ is sufficiently small positive, then $N_+ > C$ and we obtain $(2.10^b)$.

QED

§ 2.4 A proof of theorem C by iteration

In this section we shall prove theorem C, applying theorem 2.4 repeatedly to a system $F \in \dot{L}$ (or $\bar{L}$) of the form (1.15), in order to set up an iteration process which reduces the error terms. For a global description see § 2.1.

Note the equivalence of (1.17) and (2.9).

Proof of theorem C:

We start the iteration process with $F_0 = F$.

In stead of $\delta$, $\varepsilon$, P, S etc. we write $\delta_0$, $\varepsilon_0$, $P_0$, $S_0$ etc. We apply theorem 2.4 and denote $S_+$, $U$ and $\Phi$ by $S_1$, $U_0$ and $F_1$ respectively.

We may apply theorem 2.4 again to $F_1$, since $F_1$ satisfies the assumptions on S with $\delta = \delta_1 = (\delta_0)_+ = \delta_0^{\kappa}$. Thus we obtain a transforamtion $U_2$ and a conjugate system $F_2$ defined on $S_2$. Proceeding in this manner we obtain at the $n^{th}$ step

$$U_n \begin{cases} x = \xi + u_n(\xi,\alpha) \\ y = \eta + \bar{v}_n(\xi,\alpha) + V_n(\xi,\alpha)\eta \\ a = \alpha + w_n(\alpha) \end{cases}$$

and

$$F_n \begin{cases} \dot{\xi} = \alpha + \varphi_n(\xi,\eta,\alpha) \\ \dot{\eta} = \psi_n(\xi,\eta,\alpha) \; . \end{cases}$$

$U_n$ and $F_n$ are defined on

$$S_n = \mathbb{R}^3 \times \{\eta \in \mathbb{R}^3 \,|\, |\eta| \leq P_n^{-1}\} \times \{\alpha \in \mathbb{R}^3 \,|\, |\alpha - \omega| \leq \gamma\, \varepsilon_n\} \; ,$$

where $P_n = P^{\kappa^n}$ and $\varepsilon_n = \varepsilon^{\kappa^n}$ .

Observe that $F_n \in \dot{L}$ (or $\ddot{L}$) and that $U_n \in \dot{G}_1$ (or $\ddot{G}_1$) for all n. From (2.10) we deduce that

$$\varphi_n(\xi,0,\omega) \to 0 \text{ and } \psi_n(\xi,0,\omega) \to 0 \text{ as } n \to \infty . \tag{2.41}$$

Now consider $\widetilde{U}_n = U_1 0 U_2 0 \dots 0\, U_n$, which maps $S_n$ into $S_0 = S$. We may write

$$\widetilde{U}_n \begin{cases} x = \xi + \tilde{u}_n(\xi,\alpha) \\ y = \eta + \tilde{\tilde{v}}_n(\xi,\alpha) + \tilde{V}_n(\xi,\alpha)\eta \\ a = \alpha + \tilde{w}_n(\alpha) . \end{cases}$$

Here

$$\begin{cases} \tilde{u}_n = \sum_{j=1}^{n} u_j,\ \tilde{\tilde{v}}_n = \sum_{j=1}^{n} (\mathrm{id} + V_1) \dots (\mathrm{id} + V_{j-1})\, \bar{v}_j , \\ \tilde{V}_n = (\mathrm{id} + V_1) \dots (\mathrm{id} + V_n) - \mathrm{id} \quad \text{and} \\ \tilde{w}_n = w_1 + w_2 + \dots + w_n , \end{cases}$$

where the arguments in the functions have to be chosen properly: if $\tilde{u}_n = \tilde{u}_n(\xi,\alpha)$ then $u_j = u_j(\xi_j,\alpha_j)$ with $(\xi_j,\alpha_j) = \widetilde{U}_j^{-1}\widetilde{U}_n(\xi,\alpha)$. Hence

$$\begin{cases} |\tilde{u}_n| \le \sum_{j=1}^{n} |u_j|,\ |\tilde{\tilde{v}}_n| \le \sum_{j=1}^{n} \left( \prod_{h=1}^{j-1} |\mathrm{id} + V_h| \right) |\bar{v}_j| , \\ |\tilde{V}_n| \le \sum_{j=1}^{n} \ln(1 + |V_j|) \text{ and } |\tilde{w}_n| \le \sum_{j=1}^{n} |w_j| . \end{cases} \tag{2.42}$$

Now apply (2.11):

$$\begin{cases} |w_n| \le \gamma\, \varepsilon_n = \gamma\, \varepsilon^{\kappa^n} , \\ |D_\xi^\zeta\, u_n(\xi,\alpha)| \le K_\zeta (N^{7+|\zeta|}\delta)^{\kappa^n} , \text{ etc.} \end{cases} \tag{2.43}$$

Hence $\tilde{w}_n(\omega)$, $\tilde{u}_n(\xi,\omega)$, $\tilde{\tilde{v}}_n(\xi,\omega)$ and $\tilde{V}_n(\xi,\omega)$ converge as $n \to \infty$, uniformly in $\xi$, to $w_\infty$, $u_\infty(\xi)$, $\bar{v}_\infty(\xi)$ and $V_\infty(\xi)$ respectively.

We now prove that also the derivative of $\tilde{U}_n$, with respect to $\xi$ and $\eta$, converges on $S_n$, where $S_n$ shrinks to $\mathbb{R}^3 \times \{0\} \times \{\omega\}$. We have

$$D\,\tilde{U}_n = D\,U_1 \,.\, D\,U_2 . \,\ldots\, . D\,U_n \;,$$

where

$$D\,U_n = \begin{pmatrix} id + D_\xi u_n & 0 \\ D_\xi(\bar{v}_n + V_n\eta) & id + V_n \end{pmatrix} .$$

Note that here we suppress the parameter $\alpha$. So for the convergence of $D\,\tilde{U}_n$ it is sufficient to prove the convergence of

$$\prod_{n=1}^{\infty} (id + \rho_n J), \text{ where } J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and}$$

$$\rho_n = \sup_{S_n} \{|D_\xi u_n|,\ |D_\xi \bar{v}_n|,\ |D_\xi V_n\eta|\ ,\ |V_n|\} \ .$$

But this convergence follows from (2.43).

Hence $\tilde{U}_n \to U_\infty$ as $n \to \infty$ for $\eta = 0$, $\alpha = \omega$ uniformly in $\xi$, while $D\,\tilde{U}_n$ converges in the same sense. Since $\tilde{U}_n$ is linear in $\eta$ we may extend $U_\infty$ to a map

$$x = \xi + u_\infty(\xi)\,,\ y = \eta + \bar{v}_\infty(\xi) + V_\infty(\xi)\eta\ ,\ a = \omega + w_\infty(\omega)\ .$$

Here $\tilde{u}_n(\xi,\omega) \to u_\infty(\xi)$, $\tilde{\bar{v}}_n(\xi,\omega) \to \bar{v}_\infty(\xi)$ and $V_n(\xi,\omega) \to V_\infty(\xi)$ in $C^1$-norm. Therefore the original system F with $a = \omega + w_\infty(\omega)$, by means of $U_\infty$, is transformed into

$$F_\infty \begin{cases} \dot{\xi} = \omega + \varphi_\infty(\xi,\eta) \\ \dot{\eta} = \psi_\infty(\xi,\eta)\,, \end{cases}$$

where $\varphi_\infty(\xi,\eta)$ and $\psi_\infty(\xi,\eta)$ are defined and continuous on $S_\infty =$ $= \mathbb{R}^3 \times \{\eta \in \mathbb{R}^3 \mid |\eta| \leq d_\infty\}$ for some $d_\infty > 0$.

Because of (2.43) we have $\varphi_\infty(\xi,0) = \psi_\infty(\xi,0) = 0$.

QED

## § 2.5 A proof of theorem A

In this section we give a proof of theorem A, using theorem C. In $\dot{L}$(or $\ddot{L}$) consider a system of the form (1.7). As we remarked already after the formulation of theorem C in § 1.4, for $a \in \Delta$ we replace y by $A^{-1}(a) + y$. So now y is close to zero. This transforms (1.7) into (1.15) with $a \in \Delta$, where

$$\begin{cases} f(x,y,a) = A(A^{-1}(a) + y) - a + F(x,A^{-1}(a) + y) \quad \text{and} \\ g(x,y,a) = G(x,A^{-1}(a) + y) \ . \end{cases}$$

We now apply theorem C to this system. Note that we have

$$(2.44) \qquad A(A^{-1}(a) + y) - a = O(|y|) \ .$$

The application is straight forward. We have to restrict y to a disc $|y| \leq C\gamma$ in order to fulfill ($1.17^a$), using (2.44).

## § 2.6 A proof of theorem D

Below we shall prove theorem D directly from theorem C. We have to deal with problems related to the "small twist", see [11], as indicated in § 1.4. Also we shall present a proof of proposition 1.1 concerning the action variable $y_0$.

a. Consider a 1-parameter family $X^{\mu}$ as specified in §§ 1.2 and 1.3, which has the system form (1.6) being a special case of (1.10), compare (1.13). Recall that $y_1 = \sqrt{\mu}$ , which has to be chosen sufficiently close to zero, in order to control the size of the perturbation. In order to solve the problems, mentioned in § 1.4 on behalf of system (1.6), we first state

<u>LEMMA 2.5:</u>

For $\gamma > 0$ consider the set $K_\gamma = \{\omega \in \mathbb{R}^3 \,|\, \exists\, j \in \mathbb{Z}^3 : |(j,\omega)| < \gamma |j|^{-\tau}\}$ and for $\zeta > 0$ let $D_\zeta$ be any disc in $\mathbb{R}^3$ with radius $\zeta$. Then, for $\tau > 2$ and $\gamma < \zeta$ we have

$$\text{measure } (D_\zeta \cap K_\gamma) \leq C \, \gamma \, \zeta^2 ,$$

where C depends only on $\tau$.

We omit the proof. Recall that in our case $3 < \tau < 4$.

Subsequently we introduce the variable $y_2$, which rescales the time t to $(1 + y_2)t$. We restrict $y_2$ to the interval $[\frac{1}{2},2]$. So the system obtains the form (1.7), transforming (1.10) via (1.11). Let us consider the map $y \to A(y)$, $y = (y_0,y_1,y_2)$, more closely. For $y_2 = 0$ the determinant in (1.12) exactly is the Jacobian determinant of A and, as we saw in § 1.4, its value is

$$2y_1^3(\alpha_2\beta_1 - \alpha_1\beta_2) \frac{\partial a_0}{\partial y_0} (y_0,0) + O(y_1^4) ,$$

as $y_1 \to 0$, uniformly. Now consider any interval $[p_1,p_2]$, with $0 < p_1 < p_2$, where $\frac{\partial a_0}{\partial y_0}(\cdot,0)$ is definite. According to proposition 1.1 there are many such intervals. We are then given a positive number $\eta_0$, depending on $p_1$ and $p_2$, such that the map A, restricted

to $[p_1,p_2] \times [0,\eta_0] \times [\frac{1}{2},2]$, is a diffeomorphism onto its image. Now choose $q_1$ and $q_2$ with $p_1 < q_1 < q_2 < p_2$. For sufficiently small positive $\eta$ the image $A(W)$ of the set $W = [q_1,q_2] \times [\frac{1}{2}\eta,\frac{3}{2}\eta] \times [\frac{1}{2},2]$ contains a 3-disc with a radius of order $\eta^3$. This can be seen using (1.4) and (1.13). Then we apply lemma 2.5 with $\gamma = \eta^{3+\alpha}$ for some $\alpha > 0$, to be specified lateron. If $\eta$ is sufficiently small, we so find many frequencies $\omega$ in $A(W)$, which satisfy (1.8) with this $\gamma$. Let $c \in W$ be such that $\omega = A(c)$ is such a frequency.
We now also have that $A^{-1}$ is a diffeomorphism of a disc $\Delta =$ $= \{d \in \mathbb{R}^3 \,|\, |d - \omega| \leq \zeta\eta^3\}$ where $A^{-1}(\Delta) \subseteq W$. Here we require that $0 < \eta < \eta_1$. We may choose suitable $\eta_1$ and $\zeta > 0$ independent of the choice of c.

b. Our proof of theorem D now proceeds in the same manner as in e.g. § 2.5. For $d \in \Delta$ we replace y by $A^{-1}(d) + y$. As in § 2.5 this transforms our system to a system of type (1.15):

$$\begin{cases} \dot{x} = d + f(x,y,d) \\ \dot{y} = g(x,y,d) \,, \end{cases}$$

where $f(x,y,d) = A(A^{-1}(d) + y) - d + F(x, A^{-1}(d) + y)$ and

$$g(x,y,d) = G(x,A^{-1}(d) + y).$$

Once more observe that y is close to zero.
Recall that the perturbations F and G are of order $\eta^m$, uniformly in the whole region. Eventually we shall fix m and choose $\eta_1$ sufficiently small. From now on we use the notations from §§ 2.3 and 2.4. We choose $\gamma^* = 1$ and determine the associated $\delta^*$ as in theorem 2.4. So first we set $\eta_1 \leq 1$, for then $\gamma \leq \gamma^*$.
In applying theorem C to our situation we have to choose

$\delta \in (0, \min\{\delta^*, \frac{1}{2}\})$.

We restrict ourselves to the domain

$$S = \mathbb{R}^3 \times \{y \in \mathbb{R}^3 \mid |y| \leq P^{-1}\} \times \{d \in \mathbb{R}^3 \mid |d - \omega| \leq \gamma\varepsilon\} .$$

See (2.8). We need that

$$\gamma\varepsilon \leq \zeta\eta^3 \tag{2.45}$$

Also we have to satisfy the smallness condition (2.9). For these purposes we shall make appropriate choices for $\eta_1, m, \delta$ and $\alpha$.

First consider (2.9$^a$) which is fulfilled if

$$|f| \leq \frac{1}{2}\gamma\delta \qquad \text{and } |g| \leq \frac{1}{2}\gamma \frac{N}{P}\delta$$

for $(x,y,d) \in S$.

We write $A(A^{-1}(d) + y) - d = DA(A^{-1}(d))y + \text{HOT}$.

Since $|DA(y)|$ is of order 1 in our region we have

$$|A(A^{-1}(d) + y) - d| \leq C_1|y| ,$$

where $C_1$ only depends on $q_1$ and $q_2$. So we require $C_1P^{-1} \leq \frac{1}{4}\gamma\delta$ or equivalently (cf. (2.8))

$$\delta \leq \frac{1}{4}\frac{1}{C_1}\gamma = \frac{1}{4C_1}\eta^{3+\alpha} . \tag{2.46}$$

Next use $|F(x,A^{-1}(d) + y)| \leq C_2\eta^m$ and require that $C_2\eta^m \leq \frac{1}{4}\gamma\delta$ or equivalently

$$\eta^{m-3-\alpha} \leq \frac{1}{4C_2}\delta . \tag{2.47}$$

Similarly for the term G we obtain $C_2\eta^m \leq \frac{1}{2}\gamma\frac{N}{P}\delta$ or

$$(2.48) \qquad \eta^{m-3-\alpha} \leq \frac{1}{2C_2}\, \delta^{2.9} \quad .$$

If we choose $\delta = C_3\, \eta^{3+\alpha}$, according to (2.46), then (2.47) is easy to satisfy for $m > 6 + 2\alpha$ and $\eta$ small. Also (2.48) is easy to fulfill if $m > 11.7 + 3.9 \times \alpha$ and $\eta$ small. So choose $m = 12$ and $\alpha = \frac{1}{39}$ .
Then ($2.9^a$) is satisfied for sufficiently small $\eta$.
Note that indeed $\gamma\varepsilon < \zeta\eta^3$ for sufficiently small $\eta$, see (2.8), proving (2.45).

We proceed in considering ($2.9^b$). The derivatives in the x- and y-directions are no problem by our choice of $\eta$ in part a. of this section. Remains the d-direction. Note the argument $A^{-1}(d)$ in f and g. Also note that d is close to $\omega$, not to 0. However, we may derive from (1.4) with $\mu = y_1^2$

$$|D_a^{\zeta_3}\, A^{-1}(a)| \leq C_4\, \eta^{1-2|\zeta_3|} .$$

So we require for $k = 1,2, \ldots, \ell$ :

$$C_4\, \eta^{1-2k} \leq \gamma^{1-k}\, \varrho^k$$

which is fulfilled for sufficiently small $\eta$. Application of theorem C is allowed and it yields a quasi-periodic solution of (2.50) with frequency $\omega$. So our vector field X possesses a quasi-periodic orbit with the same frequency ratio's.
Since we can apply this result to each appropriate c, theorem D is proven.

c. Our considerations will be concluded in giving a proof of proposition 1.1. First we carry out some manipulations, in order to give the action integral $y_0$ a more suitable form. Modulo reparametrisation

and an appropriate change of coordinates of type $\xi = \alpha^2 r_1^2$, $\eta = \beta^2 r_2^2$.
We may write

$$H^0(\xi,\eta) = \xi\eta(\xi + \eta - 1), \quad \xi > 0, \quad \eta > 0,$$

hamiltonian to $Z^0$ via the standard form $d\xi \wedge d\eta$.
The elliptic singularity of $Z^0$ is at $(\xi,\eta) = (\frac{1}{3}, \frac{1}{3})$. Then write $u = \xi - \frac{1}{3}$, $v = \eta - \frac{1}{3}$, and again modulo a reparametrisation

$$H^0(u,v) = u^2 + v^2 + uv + 3u^2v + 3uv^2, \quad u > -\frac{1}{3}, \quad v > -\frac{1}{3}$$

is the hamiltonian of $Z^0(u,v)$ via $du \wedge dv$. Of course now the elliptic singularity is at $(u,v) = (0,0)$. The orbit cylinder of $Z^0$ is $\{H^0(u,v) = h\}_{0<h<\frac{1}{9}}$. Finally we substitute $p = u + \frac{1}{2}v$, $q = v$ and obtain

$$H^0(p,q) = p^2 + \frac{3}{4}q^2 + 3q(p^2 - \frac{1}{4}q^2)$$

which belongs to a reparametrised $Z^0$ via $dp \wedge dq$. Note that

$$H^0(p,q) = h \Longleftrightarrow p^2 = \frac{1}{4}\,\frac{4h-3q^2(1-q)}{1+3q}$$

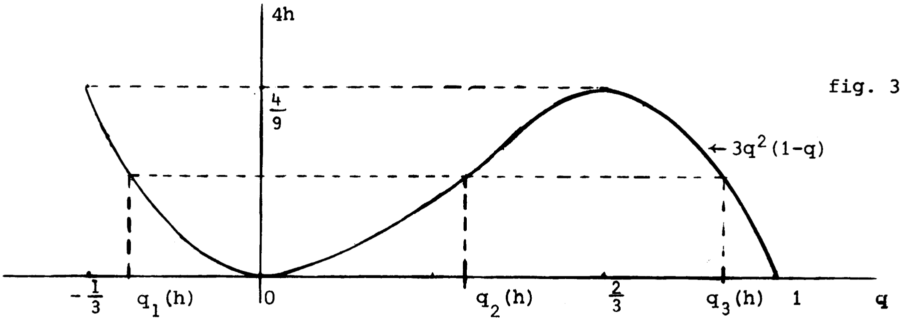


fig. 3

Let $q_1(h)$, $q_2(h)$ respectively $q_3(h)$ denote the zero's of the polynomial $4h - 3q^2(1-q)$ in climbing order, see fig. 3, then

$$y_0(h,0) = \frac{1}{2\pi} \int_{q_1(h)}^{q_2(h)} \sqrt{\frac{4h-3q^2(1-q)}{1+3q}} \cdot dq \qquad \text{and}$$

$$\frac{\partial y_0}{\partial h}(h,0) = \frac{1}{\pi} \int_{q_1(h)}^{q_2(h)} \frac{dq}{\sqrt{(1+3q)(4h-3q^2(1-q))}} \; .$$

We have that $y_0(0,0) = 0$ and $y_0(\frac{1}{9},0) = \frac{1}{2}$ .

Concerning the smoothness of $y_0(h,\mu)$ we only have to worry about the variable h near h = 0. So consider the Birkhoff normal form

$$H^{\mu}(y_0,\varphi_0) = \beta_1(\mu)\, y_0 + \beta_2(\mu)\, y_0^2 + O(y_0^3) \; .$$

We shall show that $\beta_1(0) \neq 0$ which yields

$$y_0(h,\mu) = \frac{1}{\beta_1(\mu)} h - \frac{\beta_2(\mu_1)}{(\beta_1(\mu))^3} h^2 + O(h^3) \; .$$

As a matter of fact we compute $1/\beta_1(0)$ directly from the second elliptic integral, by substituting a new variable $\tau$ by

$$q = q_1(h) + \tau\,\{q_2(h) - q_1(h)\}$$

and expanding as a power series in $\sqrt{h}$:

$$\begin{aligned} q_1(h) &= -\tfrac{2}{3}\sqrt{3}\,\sqrt{h} + \tfrac{2}{3}h + O(h^{3/2}) \; , \\ q_2(h) &= \tfrac{2}{3}\sqrt{3}\,\sqrt{h} + \tfrac{2}{3}h + O(h^{3/2}) \; , \\ q_3(h) &= 1 - \tfrac{4}{3}h + O(h^2) \; . \end{aligned}$$

Note that $q_1(h) + q_2(h) + q_3(h) \equiv 1$ .

$$\frac{\partial y_0}{\partial h}(h,0) = \frac{1}{\pi\sqrt{3}} \int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)\{1+3q_1+3\tau(q_2-q_1)\}\{q_3-q_1-\tau(q_2-q_1)\}}} =$$

$$= \frac{1}{\pi\sqrt{3}} \int_0^1 \frac{1-\frac{2}{3}\sqrt{3}(2\tau-1)\sqrt{h}+4(2\tau-1)^2 h}{\sqrt{\tau(1-\tau)}} d\tau + O(h^{3/2})$$

$$= \frac{1}{\pi\sqrt{3}} B(\tfrac{1}{2},\tfrac{1}{2}) - \frac{2}{3\pi}\sqrt{h} \int_0^1 \frac{2\tau-1}{\sqrt{\tau(1-\tau)}} dt \; +$$

$$+ \frac{2}{\pi} h \; \{4\, B(\tfrac{5}{2},\tfrac{1}{2}) - 4\, B(\tfrac{3}{2},\tfrac{1}{2}) + B(\tfrac{1}{2},\tfrac{1}{2})\} + O(h^{3/2})$$

$$= \frac{1}{\sqrt{3}} + h + O(h^2) \; .$$

So we find $\beta_1(0) = \sqrt{3}$ and $\beta_2(0) = -3\sqrt{3}$ .

This proves (i).

In order to see (ii) one may have a quick glance ar the second elliptic integral, but also a more elementary investigation of $H^0$ will suffice.

A proof of (iii) can be given as follows: First note that on the interval $0 < y_0 < \frac{1}{2}$ the function $a_0(y_0,0)$ is real analytic. Then, near $y_0 = 0$, we obtain from the Birkhoff normal form:

$$a_0(y_0,\mu) = \beta_1(\mu) + 2\beta_2(\mu)\, y_0 + O(y_0^2) \; ,$$

such that $\frac{\partial a_0}{\partial y_0}(0,0) = 2\beta_2(0) < 0$.

We now are finished as soon as we have shown that near the boundary point $y_0 = \frac{1}{2}$ the derivative $\frac{\partial a_0}{\partial y_0}$ also has a constant sign.

For this purpose we realize that in general

$$\frac{\partial a_0}{y_0} = - \frac{1}{\left(\frac{\partial y_0}{\partial h}\right)^3} \frac{\partial^2 y_0}{\partial h^2} ,$$

where

$$\frac{\partial^2 y_0}{\partial h^2}(h,0) =$$

$$= - \frac{1}{2\pi\sqrt{3}} \int_0^1 \frac{d\tau}{\sqrt{\ }}\left\{ \frac{3q_1'+3\tau(q_2'-q_1')}{1+3q_1+3\tau(q_2-q_1)} + \frac{q_3'-q_1'-\tau(q_2'-q_1')}{q_3-q_1-\tau(q_2-q_1)} \right\} ,$$

Here $q_i' = \frac{dq_i}{dh}$ and $\sqrt{\ }$ abreviates

$$\sqrt{\tau(1-\tau)\{1+3q_1+3\tau(q_2-q_1)\}\{q_3-q_1-\tau(q_2-q_1)\}}.$$

We now expand as a power series in $g = \sqrt{\frac{1}{9} - h}$, and obtain in analogy to previous calculations:

$$\begin{aligned}
q_1 &= -\frac{1}{3} + \frac{4}{3} g^2 + O(g^3) , \\
q_2 &= \frac{2}{3} - \frac{2}{\sqrt{3}} g - \frac{2}{3} g^2 + O(g^3) , \\
q_3 &= \frac{2}{3} + \frac{2}{\sqrt{3}} g - \frac{2}{3} g^2 + O(g^3) \quad \text{and so} \\
q_1' &= O(1) ,\ q_2' = \frac{1}{g\sqrt{3}} + O(1) ,\ q_3' = -\frac{1}{g\sqrt{3}} + O(1) .
\end{aligned}$$

Now the expression in accolades can be easily estimated. It appears that the coefficient of $g^{-1}$ is negative. This shows that $\frac{\partial a_0}{\partial y_0}$ also is negative near $y_0 = \frac{1}{2}$ .

QED

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