PETER WALTERS

A mixing property for non-singular actions

Astérisque, tome 98-99 (1982), p. 163-183
<http://www.numdam.org/item?id=AST_1983__98-99__163_0>

A MIXING PROPERTY FOR NON-SINGULAR ACTIONS

par

Peter WALTERS

§0. INTRODUCTION

In 1972 I published a paper about recurrent sets for measure-preserving transformations and introduced a mixing concept which lies between weak mixing and strong mixing [12]. In 1978 H. Furstenberg and B. Weiss gave an equivalent form of this mixing property in terms of product transformations [3]. Both formulations became interesting to people who work on the non-singular actions of locally compact groups on measure spaces so Klaus Schmidt and myself set about proving the equivalence of the two formulations in this set up. Our proof also gives a more elementary proof of the theorem of Fursternberg and Weiss. Our method of proof uses an action, on a space of functions, and this action (and also similar ones on other function sapces) is connected with the problem of existence of an equivalent invariant measure. I shall describe mostly the case of actions of the integers: the general case can be found in [10]. The work described in this article is joint work with Klaus Schmidt.

§1. MILD MIXING OF ACTIONS OF THE INTEGERS

Let $(X,\mathcal{B},m)$ denote an atomless Lebesgue probability space. (We could have chosen an atomless standard Borel space: the results still remain true.) A non-singular automorphism of $(X,\mathcal{B},m)$ is a bijection $T:X \to X$ which is bimeasurable (i.e. $T^{-1}\mathcal{B} = \mathcal{B}$) and $m(B) = 0$ iff $m(T^{-1}B) = 0$. If X is a compact smooth manifold and m is a smooth measure then any C^1 diffeomorphism of X is a non-singular automorphism of $(X,\mathcal{B},m)$, where $\mathcal{B}$ is the completion under m of the σ-algebra of Borel subsets of X. The measure-preserving transformations constitute the best studied subset of the non-singular automorphisms. The theorem of this section can be considered as an analogue of the following result: if T is a measure-preserving transformation then T is weak mixing iff whenever $S:Y \to Y$ is an ergodic measure-preserving transformation of a Lebesgue space $(Y,\mathcal{D},p)$ the product $T \times S$ is also ergodic.

Definition 1.1.

A non-singular automorphism $T:X \to X$ is mildly mixing if whenever $S:Y \to Y$ is an ergodic non-singular automorphism of a Lebesgue space $(Y,\mathcal{D},p)$ then $T \times S$ is ergodic.

It is clear that a mildly mixing automorphism is ergodic. The following theorem connects mild mixing with the action of T on the members of $\mathcal{B}$. It was proved by Furstenberg and Weiss for the case of measure-preserving T, and is also true for non-singular actions of a locally compact group (see §4).

Theorem 1.1.

If T is a non-singular automorphism of a Lebesgue space $(X,\mathcal{B},m)$ then T is mildly mixing iff whenever $B \in \mathcal{B}$ and $0 < m(B) < 1$ then $\liminf_{|n| \to \infty} m(T^nB \,\Delta\, B) > 0$.

Remarks.

1. The condition stated in the theorem means that the map that T induces on the metric space $(\overline{\mathcal{B}},d_m)$ of equivalence classes of sets in $\mathcal{B}$, with $d_m(A,B) = m(A \,\Delta\, B)$, has no recurrent points except for the two fixed points corresponding to sets of measure 0 and sets of measure 1 . The space $(\overline{\mathcal{B}},d_m)$ is a complete separable metric space.

2. In the proof of Theorem 1.1 we shall use the following space of functions associated with $(X,\mathcal{B},m)$. Let $U(X,m,S^1)$ denote the collection of all measurable functions $\phi:X \to S^1$, where two functions are identified if they are equal almost everywhere. Let $U(X,m,S^1)$ be equipped with the metric d_m which is the restriction of the $L^1(m)$ metric i.e. $d_m(\phi,\psi) = \int |\phi-\psi|dm$. Then $U(X,m,S^1)$ is a complete, separable, metric group. The non-singular automorphism $T:X \to X$ induces a map $\tilde{T}$ of $U(X,m,S^1)$ defined by $\tilde{T}\phi = \phi \circ T^{-1}$. The map $\tilde{T}$ is a homeomorphism and each constant function in $U(X,m,S^1)$ is a fixed point of $\tilde{T}$.

3. Let S be a homeomorphism of a complete separable metric space (Y,d) . A point y of Y is a genuine recurrent point for S if y is not a periodic point but $\liminf_{|n| \to \infty} d(S^ny,y) = 0$. There is a genuine recurrent point for S iff there is a non-atomic, ergodic,

non-singular probability p for S. (See [7], [2], [5].) An analogous result holds for continuous actions of locally compact groups. (See §4.)

4. The condition stated in the theorem implies T is ergodic.

Proof of Theorem 1.1.

Suppose firstly that T is not mildly mixing. Let $S:Y \to Y$ be an ergodic, non-singular automorphism of a Lebesgue space $(Y,\mathcal{D},p)$ with $T \times S$ not ergodic. Choose $Q \in \mathcal{B} \times \mathcal{D}$ with $(T \times S)(Q) = Q$ and $0 < (m \times p)(Q) < 1$. For $y \in Y$ let $Q_y = \{x \in X | (x,y) \in Q\}$. The condition $(T \times S)^n(Q) = Q$ gives $m(T^n Q_y \Delta Q_{S^n y}) = 0$ a.e.(p) for all $n \in Z$. The conditions $0 < (m \times p)(Q) < 1$ give $p(\{y \in Y | m(Q_y) > 0\}) > 0$ and $p(\{y \in Y | m(Q_y) < 1\}) > 0$. Since S is ergodic we have $p(\{y \in Y | 0 < m(Q_y) < 1\}) > 0$.

Let $(\overline{\mathcal{B}},d_m)$ denote the complete metric space of equivalence classes of elements of $\mathcal{B}$ (see Remark 1). We have a measurable map $y \to Q_y$ of Y into $\overline{\mathcal{B}}$. Since $(Y,\mathcal{D},p)$ is a Lebesgue space we can put a metric on Y which makes Y a compact metric space and so that $\mathcal{D}$ is the completion under p of the σ-algebra of Borel sets for this metric. By Lusin's theorem there is a compact subset C of Y with $p(C \cap \{y | 0 < m(Q_y) < 1\}) > 0$ such that $y \to Q_y$ is continuous on C. Since S is ergodic we can choose $y_0 \in C \cap \{y \in Y | 0 < m(Q_y) < 1\}$ and integers $n_i \to \infty$ such that $S^{n_i} y_0 \in C$ and $y_0 \neq S^{n_i} y_0 \to y_0$. Then $0 < m(Q_{y_0}) < 1$ and, using the continuity of $y \to Q_y$ on C, $m(T^{n_i} Q_{y_0} \Delta Q_{y_0}) \to 0$. Therefore the condition stated in the theorem is not satisfied.

To prove the converse we use the space of functions described in Remark 2. Suppose the condition stated in the theorem is violated so there is a set $B \in \mathcal{B}$ and integers n_i with $0 < m(B) < 1$ and $m(T^{n_i}B \,\Delta\, B) \to 0$. Put $\phi_0 = 2\chi_B - 1 \in U(X,m,S)$. Then $d_m(\tilde{T}^n\phi_0,\phi_0) = 2m(T^nB \,\Delta\, B)$ so $d_m(\tilde{T}^{n_i}\phi_0,\phi_0) \to 0$. We now consider two cases. Suppose firstly that $m(T^nB \,\Delta\, B) > 0$ whenever $n \neq 0$. Then ϕ_0 is not a periodic point of $\tilde{T}$ and hence is a genuine recurrent point. By the result stated in Remark 3 there is a non-atomic, ergodic, non-singular probability p for $\tilde{T}$ on $U(X,m,S^1)$. Since constant functions in $U(X,m,S^1)$ are fixed points of $\tilde{T}$ the collection of constant functions has p-measure zero. Define $F:X \times U(X,m,S^1) \to S^1$ by $F(X,\phi) = \phi(x)$. This appears to be not well-defined but a measurable version of it can be obtained from the theorem on selection of representatives. ([11] p.8.) Clearly $F \circ (T \times \tilde{T}) = F$ and since F is not almost everywhere constant we conclude that $T \times \tilde{T}$ is not ergodic. Hence T is not mildly mixing.

We now consider the case when $\{n \mid m(T^nB \,\Delta\, B) = 0\} = kZ$ for $k \neq 0$. Let $R:Y_1 \to Y_1$ be a Bernoulli shift on a Lebesgue space $(Y_1,\mathcal{D}_1,p_1)$ and define $S:Y \to Y$ from it as follows. Let $Y_2 = \{0,1,\ldots,k-1\}$ and put $(Y,\mathcal{D},p) = (Y_1,\mathcal{D}_1,p_1) \times (Y_2,2^{Y_2},\lambda)$, where λ is normalised counting measure on Y_2. Define $S:Y \to Y$ by $S(y_1,y_2) = (y_1,y_2+1)$ if $y_2 \neq k-1$ and $S(y_1,k-1) = (Ry,0)$. Then S is an ergodic measure preserving transformation. However $T \times S$ is not ergodic because the set $\{(x,y_1,y_2) \mid y_1 \in Y_1,\ x \in T^{y_2}B\}$ is $T \times S$ invariant and has measure different from 0 and 1. (We used the Bernoulli shift R so that $(Y,\mathcal{D},p)$ would be a non-atomic space. If we had not wanted $(Y,\mathcal{D},p)$ non-atomic we could have taken $(Y,\mathcal{D},p)$ the same as $(Y_2,2^{Y_2},\lambda)$ and chosen S to be $S(y_2) = y_2+1$, mod k.)

The above proof also shows that T is mildly mixing iff whenever $S:Y \to Y$ is an ergodic measure-preserving transformation of a σ-finite Lebesgue space $(Y,\mathcal{D},p)$ then $T \times S$ is also ergodic. This is because whenever there exists a non-atomic, non-singular ergodic measure then there is a non-atomic, ergodic, σ-finite invariant measure. ([13])

When T is a measure-preserving transformation of a Lebesgue space the condition stated in the theorem is equivalent to $A(T) = N$ in the notation of [12]. In this case mild mixing lies between weak mixing and strong mixing. A. Saleski has shown that a measure-preserving transformation is mildly mixing iff for every non-trivial finite partition ξ of $(X,\mathcal{B},m)$ and for every subsequence N^1 of the natural numbers we have $\sup \{h_{N''}(T,\xi) \mid N''$ is a subsequence of $N^1\} > 0$. Here $h_{N''}(T,\xi)$ denotes sequence entropy.

§2. CONNECTIONS WITH THE EXISTENCE OF AN EQUIVALENT INVARIANT PROBABILITY

Given a non-singular automorphism T os a Lebesgue space $(X,\mathcal{B},m)$ it is natural to ask if there is a probability measure m' on $(X,\mathcal{B})$ which is equivalent to m (i.e. m and m' have the same null sets) and preserved by T. In this section we describe how mild mixing and certain actions, obtained from T, on function spaces are connected with the above problem.

Theorem 2.1.

Let T be a non-singular automorphism of the Lebesgue space $(X,\mathcal{B},m)$. There is an equivalent T-invariant probability iff the map $\tilde{T}$ of $U(X,m,S^1)$ is uniformly equicontinuous (i.e. $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that $d_m(\phi,\psi) < \delta$ implies $d_m(\tilde{T}^k\phi,\tilde{T}^k\psi) < \varepsilon\ \ \forall\ k \in Z)$.

Proof.

If m' is equivalent to m then $U(X,m',S^1) = U(X,m,S^1)$ and the metrics $d_{m'}$ and d_m are equivalent. Therefore if m' is also preserved by T then $\tilde{T}$ is an isometry for $d_{m'}$ and hence uniformly equicontinuous. To prove the converse we use the well-known result that if no equivalent T-invariant probability exists then there is a weakly wandering set i.e. there is a set B with $0 < m(B) < 1$ and a sequence $n_i \to \infty$ with $m(T^{n_i}B \cap T^{n_j}B) = 0$ if $i \neq j$ ([4]). Put $\phi = 1-2\chi_B \in U(X,m,S^1)$. Then $\inf_n d_m(\tilde{T}^n\phi,\tilde{T}^n 1) = \inf_n 2m(T^nB) = 0$ but $d_m(\phi,1) = 2m(B)$ showing that $\tilde{T}$ cannot be uniformly equicontinuous. □

For a non-singular automorphism T of a Lebesgue space $(X,\mathcal{B},m)$ let $R(\tilde{T})$ denote the set $\{\phi \in U(X,m,S^1) : \liminf_{|n| \to \infty} d_m(\tilde{T}^n\phi,\phi)=0\}$. This set consists of all recurrent points for $\tilde{T}$ and includes all the periodic points of $\tilde{T}$. Since constant functions are fixed points for $\tilde{T}$ we know $R(\tilde{T})$ contains all constant functions in $U(X,m,S^1)$. It is not difficult to show, using Theorem 1.1, that T is mildly mixing iff $R(\tilde{T})$ consists only of the constant functions in $U(X,m,S^1)$. The following result shows how $R(\tilde{T})$ depends on the existence of an equivalent T-invariant probability.

Theorem 2.2.

Let T be a non-singular automorphism of $(X,\mathcal{B},m)$.

(i) If no equivalent T-invariant probability exists on $(X,\mathcal{B})$ then $R(\tilde{T})$ is a dense G_δ subset of $U(X,m,S^1)$.

(ii) If there is an equivalent T-invariant probability on $(X,\mathcal{B})$ then $R(\tilde{T})$ is a closed subset of $U(X,m,S^1)$.

Proof.

(i) We have $R(\tilde{T}) = \bigcap_{\ell=1}^{\infty} \bigcup_{k\neq 0} \{\phi \in U(X,m,S^1) \mid d_m(\tilde{T}^k\phi,\phi) < \frac{1}{\ell}\}$.

This is clearly a G_δ subset and it remains to show that for each fixed ℓ the set $\bigcup_{k\neq 0} \{\phi \mid d_m(\tilde{T}^k\phi,\phi) < \frac{1}{\ell}\}$ is dense. Fix ℓ and let $\varepsilon > 0$ and $\phi \in U(X,m,S^1)$. Choose a weakly wandering set B with $m(X\backslash B) < \min(\varepsilon/2,\ 1/2\ell)$. ([4].) Let $\{n_i\}_{i=1}^{\infty}$ be a sequence with $n_1 = 0$, $n_i \to \infty$ and $m(T^{n_i}B \cap T^{n_j}B) = 0$ if $i \neq j$.

Let $\psi(x) = \chi_{X\setminus \bigcup_{i=1}^{\infty} T^{n_i}B}(x) + \sum_{i=1}^{\infty} \phi(T^{-n_i}x)\cdot\chi_{T^{n_i}B}(x)$. Then $\psi \in U(X,m,S^1)$. Also $d_m(\phi,\psi) = \int |\phi-\psi|dm \le \int_B |\phi-\psi|dm + 2m(X\setminus B) = 2m(X\setminus B) < \varepsilon$, and $d_m(\psi,\tilde{T}^{-n_i}\psi) \le \int_B |\psi-\tilde{T}^{-n_i}\psi|\, dm + 2m(X\setminus B) = 2m(X\setminus B) < \frac{1}{\ell}$. Therefore $\bigcup_{k\neq 0} \{\phi \mid d_m(\tilde{T}^k\phi,\phi)\mid\ < \frac{1}{\ell}\}$ is dense.

(ii) Let m' be an equivalent T-invariant probability. We use the metric $d_{m'}$ on $U(X,m,S^1) = U(X,m',S^1)$. Since $R(\tilde{T}) = \bigcap_{\ell} \{\phi \mid \inf_{k\neq 0} d_{m'}(\tilde{T}^k\phi,\phi) \le \frac{1}{\ell}\}$ we need only show $\{\phi \mid \inf_{k\neq 0} d_{m'}(\tilde{T}^k\phi,\phi) \le \frac{1}{\ell}\}$ is closed. Since $\tilde{T}$ is an isometry for $d_{m'}$ one readily shows $\phi \to \inf_{k\neq 0} d_{m'}(\tilde{T}^k\phi,\phi)$ is continuous, and the result follows. □

From (i) we easily deduce the following result of Aaronson, Lin and Weiss ([1]).

Corollary 2.3.

If the non-singular automorphism T of $(X,\mathcal{B},m)$ is mildly mixing there is an equivalent T-invariant probability.

Mildly mixing automorphisms are those for which $R(\tilde{T})$ is as small as possible: automorphisms for which $R(\tilde{T})$ is as large as possible are called rigid.

Definition 2.1.

A non-singular automorphism T of $(X,\mathcal{B},m)$ is called rigid if $R(\tilde{T}) = U(X,m,S^1)$.

There are many examples of rigid measure-preserving transformations ([12]), e.g. a rotation of the unit circle. There are also examples which do not have an equivalent T-invariant probability. In fact the following is an example of a rigid, non-singular automorphism with no equivalent, T-invariant σ-finite measure. Let $T:S^1 \to S^1$ be one of Katznelson's C^∞ orientation preserving diffeomorphisms of the unit circle which is ergodic with respect to Lebesgue measure λ but has no σ-finite equivalent invariant measure ([6]). Let α be the rotation member of T and let p_j/n_j be the j-th convergent in the continued fraction expansion of the irrational number α. We have $|\alpha - p_j/n_j| \le 1/n_j^2$. Then T is topologically conjugate to the rotation $z \to e^{2\pi i\alpha}z$, and since $e^{2\pi i n_j\alpha} \to 1$ we have $f \circ T^{n_j}$ converges uniformly to f for each continuous $f:S^1 \to \mathbb{C}$. One then shows $d_\lambda(\tilde{T}^{-n_j}\phi,\phi) \to 0 \quad \forall\, \phi \in U(S^1,\lambda,S^1)$ by approximating ϕ by a continuous function and using the following inequality which follows from the Denjoy-Koksma inequality: $\exists\, c > 0$ such that $e^{-c} \le DT^{n_j}(z) \le e^{c} \quad \forall\, z \in S^1$, where $DT^{n_i}(z)$ is the differential coefficient at z of T^{n_i}.

Notice in the above example that there is a sequence n_i such that $d_\lambda(\tilde{T}^{n_i}\phi,\phi) \to 0 \quad \forall\, \phi \in U(S^1,\lambda,S^1)$. Rigid measure-preserving transformations always have this stronger property:

Theorem 2.4.

Let T be a measure preserving transformation of $(X,\mathcal{B},m)$. The following are equivalent:

(i) T is rigid.

(ii) There is a sequence $n_i \to \infty$ of natural numbers with

$$d_m(\tilde{T}^{n_i}\phi,\phi) \to 0 \quad \forall \quad \phi \in U(X,m,S^1) \ .$$

(iii) The maximal spectral type σ of T satisfies

$$\limsup_{n\to\infty} \mathcal{R}(\hat{\sigma}(n)) = 1 \ , \quad \text{where } \mathcal{R} \text{ denotes "real part"}$$

and $\hat{\sigma}(n) = \int_{S^1} \lambda^n \, d\sigma(\lambda)$.

For the proof see [10] and [12]. Rigid transformations have singular spectrum and hence zero entropy [12]. However the direct product of two rigid transformations need not be rigid and can even have non-singular spectrum [10]. Hence rigidity is not the same as the condition $\mathcal{A}(T) = \mathcal{B}$ of [12].

§3. A DIFFERENT INDUCED ACTION ON A FUNCTION SPACE

We describe, in this section, a necessary and sufficient condition for the existence of an equivalent T-invariant probability. The condition involves another action on a space of functions.

Let T be a non-singular automorphism of $(X,\mathcal{B},m)$. Let $U(S^1 \times X, \lambda \times m, S^1)$ be the space of measurable functions $\phi : S^1 \times X \to S^1$ where two functions are identified if they are equal $(\lambda \times m)$-almost everywhere (λ denotes Lebesgue measure on S^1). Let $d_{\lambda \times m}$ denote the restriction of the $L^1(\lambda\times m)$ metric to $U(S^1 \times X, \lambda \times m, S^1)$. Let $\tilde{\tilde{T}}$ be the homeomorphism of $U(S^1 \times X, \lambda \times m, S^1)$ defined by $(\tilde{\tilde{T}}\phi)(z,x) = z\phi(z,T^{-1}x)$.

Theorem 3.1.

Let T be a non-singular ergodic automorphism of $(X,\mathcal{B},m)$. The following are equivalent:

(i) there exists an equivalent T-invariant probability on $(X,\mathcal{B})$;

(ii) whenever S is a non-singular ergodic automorphism of $(Y,\mathcal{D},p)$ then $T \times S$ is not of type 1(i.e. there is no set $E \in \mathcal{B} \times \mathcal{D}$ with $\bigcup_{n=\infty}^{\infty} (T \times S)^n E = X \times Y$ and $(m \times p)((T \times S)^n E \cap E) = 0$ for all $n \neq 0$.)

(iii) $\tilde{\tilde{T}}$ has no recurrent points (i.e. there is no $\phi \in U(S^1 \times X, \lambda \times m, S^1)$ with $\liminf_{|n| \to \infty} d_{\lambda\times m}(\tilde{\tilde{T}}^n\phi,\phi) = 0$.)

Proof

(ii) => (iii) . Suppose $\phi_0 \in U(S^1 \times X, \lambda \times m, S^1)$ satisfies $\liminf_{|n| \to \infty} d_{\lambda \times m}(\tilde{\tilde{T}}^n \phi_0, \phi_0) = 0$. If $\tilde{\tilde{T}}^n \phi_0 = \phi_0$ for some $n \neq 0$ then $\phi(z, T^n x) = z^n \phi(z,x)$ a.e. $(\lambda \times m)$, so for λ-almost every z we have that z^n is an L^∞-eigenvalue of T^n , and this cannot happen when T is ergodic on a non-atomic space ([8]). Therefore ϕ_0 is a genuinely recurrent point for $\tilde{\tilde{T}}$ and hence there is a non-atomic, non-singular, ergodic probability p for $\tilde{\tilde{T}}$ on $U(S^1 \times X, \lambda \times m, S^1)$. Define $F: S^1 \times X \times U(S^1 \times X, \lambda \times m, S^1) \to S^1$ by $F(z,x,\phi) = \phi(z,x)$ (using the selection of representatives theorem [11] p.8). For λ-almost every z in S^1 $F(z,\cdot,\cdot)$ is an L^∞-eigenfunction for $T \times \tilde{\tilde{T}}$ with eigenvalue z , and therefore $T \times \tilde{\tilde{T}}$ is of type 1. ([8])

(iii) => (i) . This resembles the proof of Theorem 2.2(i). Suppose there is no equivalent T-invariant probability. The recurrent points of $\tilde{\tilde{T}}$ constitute the set

$$R(\tilde{\tilde{T}}) = \bigcap_{\ell=1}^{\infty} \bigcup_{k \neq 0} \{\phi \in U(S^1 \times X, \lambda \times m, S^1) \,|\, d_{\lambda \times m}(\tilde{\tilde{T}}^k \phi, \phi) < \tfrac{1}{\ell}\} .$$

It will follow that $R(\tilde{\tilde{T}})$ is a dense G_δ if we show that for each ℓ the set $\bigcup_{k \neq 0} \{\phi \,|\, d_{\lambda \times m}(\tilde{\tilde{T}}^k \phi, \phi) < \tfrac{1}{\ell}\}$ is dense. Let $\varepsilon > 0$ be given and let $\phi \in U(S^1 \times X, \lambda \times m, S^1)$. Choose B with $m(X \backslash B) < \min(\varepsilon/2, 1/2\ell)$ and a sequence $\{n_i\}$ with $n_1 = 0$, $n_i \to \infty$ and $m(T^{n_i} B \cap T^{n_j} B) = 0$ if $i \neq j$. Let

$$\psi(z,x) = \chi_{X \backslash \bigcup_{i=1}^{\infty} T^{n_i} B}(x) + \sum_{i=1}^{\infty} z^{n_i} \phi(T^{-n_i} x) \chi_{T^{n_i} B}(x) \quad . \text{ Then}$$

$\psi \in U(S^1 \times X, \lambda \times m, S^1)$. We have $d_{\lambda\times m}(\phi,\psi) \leq \int_{S^1\times B} |\phi-\psi| d\lambda\times m + 2m(X\backslash B)$ $< \varepsilon/2$ and $d_{\lambda\times m}(\psi, \tilde{\tilde{T}}^{-n_i}\psi) \leq \int_{S^1\times B} |\psi - \tilde{\tilde{T}}^{n_i}\psi| d\lambda\times m + 2m(X\backslash B)$ $= 2m(X\backslash B) < \frac{1}{\ell}$. Therefore $\bigcup_{k\neq 0}\{\phi | d_{\lambda\times m}(\tilde{\tilde{T}}^k\phi,\phi) < \frac{1}{\ell}\}$ is dense.

(i) => (ii).

Let m' be an equivalent T-invariant probability. Suppose S is ergodic and non-singular on $(Y,\mathcal{D},p)$ and $T \times S$ is of type 1. Let $E \in \mathcal{B} \times \mathcal{D}$ be so that $\bigcup_{n=\infty}^{\infty} (T \times S)^n E = X \times Y$ and $(m'\times p')((T \times S)^n E \cap E) = 0$ when $n \neq 0$.

Define $\phi: S^1 \times X \times Y \to S^1$ by $\phi(z,x,y) = \sum_{-\infty}^{\infty} z^n \chi_{(T\times S)^n E}(x,y)$.

We have a measurable map $Y \to U(S^1 \times X, \lambda \times m', S^1)$ given by $y \to \phi(\cdot,\cdot,y)$. By choosing a metric on Y , using Lusin's theorem and then the recurrence of S (as in the proof of Theorem 1.1) we can choose some $y_0 \in Y$ with $d_{\lambda\times m'}(\phi(z,x,S^{n_i}y_0) , \phi(z,x,y_0)) \to 0$ for some $n_i \to \infty$. Since $\phi(z,x,S^{n_i}y_0) = z^{n_i}\phi(z,T^{-n_i}x,y_0)$ we get $d_{\lambda\times m'}(z^{n_i}\phi(z,T^{-n_i}x,y_0), \phi(z,x,y_0)) \to 0$, and choosing a further subsequence if necessary we have

$$z^{n_i}\phi(z,T^{-n_i}x,y_0) \to \phi(z,x,y_0) \quad \text{a.e.} \quad (\lambda \times m') .$$

For each $k \in Z$ $z^{k+n_i}\phi(z,T^{-n_i}x,y_0) \to z^k\phi(z,x,y_0)$ a.e. $(\lambda \times m')$.

Let $c_n(x) = \int_{S^1} z^n\phi(z,x,y_0)\, d\lambda(z)$ be the Fourier coefficients of $z \to \phi(z,x,y_0)$. Then $\sum_{-\infty}^{\infty} |c_n(x)|^2 = 1$ since ϕ takes values in S^1 and hence $\int_X |c_n(x)|^2\, dm'(x) \to 0$ as $n \to \infty$.

From the above we have $c_{k+n_i}(T^{-n_i}x) \to c_k(x)$ a.e. (m') so that

$$\int |c_{k+n_i}(x)|^2 \, dm'(x) \to \int |c_k(x)|^2 dm'(x)$$

and hence $\int |c_k(x)|^2 dm'(x) = 0$ for all k. Therefore $c_k(x) = 0$ a.e. so that $\phi(\cdot,\cdot,y_0) = 0$ ae $(\lambda \times m')$. This contradicts the fact that ϕ takes values in S^1. Therefore $T \times S$ is not of type 1.

§4. ACTIONS OF GENERAL GROUPS

The results of the previous sections can be proved for the case of a non-singular action of a locally compact group. We shall now describe these results; details can be found in [10].

Let $(X,\mathcal{B},m)$ be a non-atomic, standard Borel probability space and let G be a locally compact, second-countable group which is not compact. A non-singular action of G on X is a measurable action $(g,x) \to gx$ of G on X such that $m(E) = 0$ implies $m(gE) = 0 \; \forall \, g \in G$. An action is properly ergodic if it is ergodic and m is not concentrated on a single orbit. Let $U(X,m,S^1)$ be the same space as in §1 with the metric d_m. A non-singular action $(g,x) \to gx$ of G on X induces a continuous action of G on $U(X,m,S^1)$ defined by $(g,\phi) \to \tilde{T}_g\phi = \phi \circ g^{-1}$.

Definition 4.1.

The non-singular, properly ergodic action $(g,x) \to gx$ is mildly mixing if for every non-singular properly ergodic action $(g,y) \to gy$ of G on a standard Borel probability space $(Y,\mathcal{D},p)$ the product action $(g,(x,y)) \to (gx,gy)$ of G on $X \times Y$ is ergodic.

There is the following generalisation of Theorem 1.1.

Theorem 4.1.

Let $(g,x) \to gx$ be a non-singular, properly ergodic action of G on the standard Borel probability space $(X,\mathcal{B},m)$. The action is mildly mixing iff whenever $B \in \mathcal{B}$ and $0 < m(B) < 1$ then $\liminf\limits_{g \to \infty} m(gB \,\Delta\, B) > 0$.

The proof is similar to that of Theorem 1.1 and uses the following result of Glimm and Effros ([2], [5]). Suppose the locally compact, second countable group G acts continuously on a complete separable metric space Y. For $y \in Y$ let $G_y = \{g \in G | gy = y\}$ denote the stability subgroup of y. A point $y \in Y$ is recurrent for the action if the natural map $gG_{y_o} \to gy_o$ of G/G_{y_o} onto the orbit, Gy_o, of y_o is not a homeomorphism. Then the action has a recurrent point iff the action has a non-singular, properly ergodic probability measure defined on the Borel subsets of Y.

We now describe the generalisations of the results in §2. Let G act on $(X, \mathcal{B}, m)$ as above and let $(g, \phi) \to \tilde{T}_g \phi$ be the induced action on $U(X, m, S^1)$.

Theorem 4.2.

There is an equivalent G-invariant probability on $(X, \mathcal{B})$ iff the group of homeomorphisms, $\{\tilde{T}_g | g \in G\}$, of $U(X, m, S^1)$ is uniformly equicontinuous.

Let $R(G) = \{\phi \in U(X, m, S^1) | \liminf_{g \to \infty} d_m(\tilde{T}_g \phi, \phi) = 0\}$ be the collection of recurrent points for the action $(g, \phi) \to \tilde{T}_g \phi$. The constant functions in $U(X, m, S^1)$ belong to $R(G)$ and one can use Theorem 4.1 to show that the action of G is mildly mixing iff $R(G)$ consists only of constant functions. The size of $R(G)$ is connected with the existence of an equivalent G-invariant probability as follows.

Theorem 4.3.

If there is no equivalent G-invariant probability on $(X,\mathcal{B})$ then $R(G)$ is a dense G_δ subset of $U(X,m,S^1)$. If there is an equivalent G-invariant probability on $(X,\mathcal{B})$ then $R(G)$ is a closed subset of $U(X,m,S^1)$.

When G is abelian we can generalise the results of §3. Suppose G is an abelian, locally compact, second countable, non-compact group and let $(g,x) \to gx$ be a properly ergodic, non-singular action of G on a standard Borel space $(X,\mathcal{B},m)$. Let Γ be the character group of G and let λ denote Haar measure on Γ . Let $U(\Gamma \times X, \lambda \times m,S^1)$ be the space of all measurable maps $\phi:\Gamma \times X \to S^1$ where two such maps are identified if they are equal almost everywhere. The action of G on X induces the following action of G on $U(\Gamma \times X, \lambda \times m,S^1)$:

$$(g,\phi) \to g\phi \quad \text{where} \quad (g\phi)(\gamma,x) = \gamma(g)\phi(\gamma,g^{-1}x) \ .$$

Theorem 4.4.

Let G be an abelian, locally compact, non-compact, second countable group and let $(g,x) \to gx$ be a properly ergodic non-singular action of G on a standard Borel space $(X,\mathcal{B},m)$. The following three statements are mutually equivalent.

(i) there is an equivalent G-invariant probability on $(X,\mathcal{B})$

(ii) whenever $(g,y) \to gy$ is a properly ergodic, non-singular action of G on a standard Borel space $(Y,\mathcal{D},p)$ then the product action $(g,(x,y)) \to (gx,gy)$ is not of type 1

(iii) the induced action of G on $U(\Gamma \times X, \lambda \times m, S^1)$ has no recurrence i.e. there is no $\phi \in U(\Gamma \times X, \lambda \times m, S^1)$ with $\liminf_{g \to \infty} d_{\lambda \times m}(g\phi, \phi) = 0$.

Klaus Schmidt has considered the problem of which groups G are such that all their properly ergodic actions are mildly mixing [9].

References

[1] Aaronson, J., Lin, M., and Weiss, B. : Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products. Israel J. Mathematics, 33 (1979), 198-224.

[2] Effros, E.G. : Transformation groups and C*-algebras. Ann. Math. 81 (1965), 38-55.

[3] Furstenberg, H., and Weiss, B. : The finite multipliers of infinite ergodic transformations. In: The Structure of Attractors in Dynamical Systems, pp. 127-132, Springer Lecture Notes in Mathematics, Vol. 668: Springer 1978.

[4] Friedman, N.A. : Introduction to Ergodic Theory. Van Nostrand, 1970.

[5] Glimm, J. : Locally compact transformation groups. Trans. Amer. Math. Soc. 101 (1961), 124-138.

[6] Katznelson, Y. : Sigma - finite invariant measures for smooth mappings of the circle. Journal d'Analyse Math. 31 (1977), 1-18.

[7] Katznelson, Y., and Weiss, B. : The construction of quasi-invariant measures. Israel J. Math. 12 (1972), 1-4.

[8] Losert, V., and Schmidt, K. : A class of probability measures on groups arising from some problems in ergodic theory. In: Probability measures on groups, pp. 220-238, Springer Lecture Notes in Mathematics, Vol. 706. Springer 1979.

[9] Schmidt, K. : Mildly mixing actions of locally compact groups II. Preprint 1981.

[10] Schmidt, K. and Walters, P. : Mildly mixing actions of locally compact groups I. Preprint 1981.

[11] Varadarajan, V.S. : Geometry of Quantum Theory II. Van Nostrand 1970.

[12] Walters, P. : Some invariant σ-algebras for measure preserving transformation. Trans. Amer. Math. Soc. 163 (1972) 357-368.

[13] Schmidt, K. : Infinite invariant measures on the circle. Istituto Nazionale di Alta Matematica Symposia Mathematica, XXI (1977), 37-43.

Peter WALTERS

Mathematics Institute,
University of Warwick,
Coventry CV4 7AL,

ENGLAND.