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PARTIAL RESOLUTIONS OF NILPOTENT VARIETIES

Walter Borho and Robert MacPherson*

INTRODUCTION

Springer [S3] uses the variety $\mathcal{B}$ of flags in $\mathbb{C}^n$ to define a resolution of singularities $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ for the variety $\mathcal{N}$ of nilpotent matrices in $GL(n,\mathbb{C})$. We define "partial resolutions" $\xi : \tilde{\mathcal{N}}^P \to \mathcal{N}$ of $\mathcal{N}$ by replacing $\mathcal{B}$ by a variety $\mathcal{P}$ of partial flags. We use them to extend our analysis [BM] of the topology of the singularities of $\mathcal{N}$ by means of intersection homology and by Springer's theory of Weyl-group representations. Both Springer's analysis and ours hold for a general reductive group. In <u>chapter</u> 0, for the convenience of the reader not familiar with the theory of Lie groups, we interpret the spaces and constructions involved for the special case $GL(n,\mathbb{C})$, the group of complex invertible $n \times n$ matrices.

In <u>chapter</u> 1 we develop the general theory of a composition of maps of algebraic varieties

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, "\eta"'] \arrow[rr, bend left, "\pi"] & \mathcal{N}^P \arrow[r, "\xi"'] & \mathcal{N} \end{tikzcd}$$

possessing the key property for our work: all three maps are semismall, i.e. the dimension of the inverse image of a point in a stratum is at most half the codimension of that stratum. We relate the topology of the maps to the intersection homology of the closures of the strata, using a decomposition theorem of Beilinson-Bernstein and Deligne-Gabber (1.7).

In <u>chapter</u> #2 we set up in more detail the concrete group theoretical situation studied in this paper. We recall Springer's theory of Weyl group

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representations on the homology of the fibers of π, and extend it in an appropriate way to the fibers of η and ξ. We give some new applications. For instance, we prove that the variety $\mathcal{N}$ of nilpotent elements is a rational homology manifold (2.3).

In chapter 3 we apply the analysis of the first two chapters to a new, unified study of certain fixed point subvarieties in generalized flag varieties, studied first by Steinberg (case $\mathcal{P}_x$, [St2]), resp. by Spaltenstein (case $\mathcal{P}_x^\circ$, [Sp]), resp. by Springer (case $\mathcal{P}_x^t$, [S2]), and since then in many other papers, e.g. [HS], [HSh]. We show how to deduce geometrical data about these varieties from Springer's Weyl group representations in terms of induction and restriction. In particular, we compute the homology of Steinberg's varieties (2.8, 3.7), which extends the results of Hotta-Shimomura [HSh] on GL_n, and also of Spaltenstein's varieties (3.7), and we count components in Springer's varieties $\mathcal{P}_x^t$ (3.1). Our setting gives also some geometrical understanding of the "induction" of nilpotent orbits in the sense of Lusztig-Spaltenstein (3.9).

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Contents

§0 THE CASE OF $GL(n,\mathbb{C})$

In this paper, we shall deal with a reductive complex Lie-group G, and with the variety $\mathscr{P}$ of conjugates of a parabolic subgroup P with a Levi-subgroup L. However, let us consider for the purpose of this chapter only $G = GL(n,\mathbb{C})$, the group of invertible complex $n \times n$ matrices. Then L and P are determined by the choice of a decomposition of $\mathbb{C}^n$ into a direct sum of subspaces $E_1, \dots ,E_r$, of dimension $p_1, \dots ,p_r$, say: L resp. P is the subgroup stabilizing each subspace E_i resp. each subspace $F_i = E_1+E_2 + \dots + E_i$ for $i=1, \dots ,r$. Then $F = (F_1, \dots ,F_r)$ is called a partial flag of type $p = (p_1, \dots ,p_r)$, and $\mathscr{P}$ is (isomorphic to) the variety of all partial flags of type p. One of our goals is to study the subvarieties $\mathscr{P}_x$ of all such partial flags fixed by a given nilpotent matrix x (meaning that $xF_i \subset F_i$ for all i), which were introduced by Steinberg [St2].

The partial Weyl-group W^p is the finite group of permutations of the spaces $E_1, \dots ,E_r$ preserving dimensions. We shall define a linear W^p-action on the cohomology groups $H^i(\mathscr{P}_x,\mathbb{Q})$, called partial Springer representations. In our analysis, the Steinberg varieties $\mathscr{P}_x$ shall be just the fibres of a certain map $\xi : \tilde{\mathscr{N}}^p \to \mathscr{N}$, called partial (Springer) resolution, which is the main subject of this paper. It is defined as follows: $\mathscr{N}$ is the variety of all nilpotent complex $n \times n$ matrices, $\tilde{\mathscr{N}}^p$ is the variety of pairs (x,F) consisting of a nilpotent matrix x and a flag F of type p fixed by x, and ξ is the map which forgets F.

Consider the special case $p = (1, \dots ,1)$, that is to say F is a complete flag, or equivalently, P is a Borel subgroup. Notationally, we write $\mathscr{B}$ instead of $\mathscr{P}$ in this case, and we omit the superscripts p or the word "partial" everywhere: The situation described above reduces to the (ordinary) Springer representations of the (ordinary) Weyl group W on the cohomology $H^i(\mathcal{B}_x,\mathbb{Q})$ of the fibres of Springer's resolution, which is denoted $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$. The map π is actually a resolution of the singularities of $\mathcal{N}$.

We shall study the partial resolutions as defined above by means of a factorization $\pi = \xi\eta : \tilde{\mathcal{N}} \xrightarrow{\eta} \tilde{\mathcal{N}}^P \xrightarrow{\xi} \mathcal{N}$ of Springer's resolution: the map η forgets the complete flag partially, while the map ξ forgets the partial flag completely. We can stratify these maps ξ resp. η by strata $\mathcal{O}_x \subset \tilde{\mathcal{N}}$ resp. $\mathcal{O}_y \subset \tilde{\mathcal{N}}^P$ as follows: The strata $\mathcal{O}_x$ are just the conjugacy classes (G-orbits) of nilpotent matrices. A stratum $\mathcal{O}_y$ consists of all pairs (x,F) of a nilpotent matrix x and a partial flag $F \in P$ such that x fixes F, and induces on all subquotients $F_i/F_{i-1} \cong E_i$ an endomorphism t_i of a given Jordan-type (depending on i). For these stratifications of ξ resp. η, it turns out that the fibre always has dimension less than or equal to half the codimension of the corresponding stratum. A proper algebraic map with this property is called <u>semismall</u>. For the singularities of such maps, a particularly elegant description in terms of intersection homology sheaves, is available, see 1.5. We apply it to π,η,ξ to derive our results.

Let us consider the singularity structure of the varieties $\bar{\mathcal{O}}_y$, the closures of the strata $\mathcal{O}_y$ of $\tilde{\mathcal{N}}^P$. There is a unique closed stratum, say $\mathcal{O}_{y_0}$, consisting of those pairs (x,F) such that the nilpotent matrix x acts trivially on all subquotients of the flag F. It is easy to see that $\mathcal{O}_{y_0}$ identifies with the cotangent bundle $T^*\mathcal{P}$ of $\mathcal{P}$ (by the map forgetting x). There is a topological (not algebraic) fibration of $\mathcal{N}^P$ onto T^*P which makes $\mathcal{O}_y$ into a fibre-bundle with base $T^*\mathcal{P}$, and fibre $\mathcal{O}_t$, the predescribed (L-)

conjucacy class of nilpotent endomorphisms $(t_1, \dots, t_r) = t$, say, of $E_1, \dots, E_r$. Since $T^*\mathcal{F}$ is smooth, the varieties $\overline{\mathcal{O}}_y$ have the same singularity type as the varieties $\overline{\mathcal{O}}_t$. This explains how the intersection homology sheaves of $\overline{\mathcal{O}}_y$ relate to those of $\overline{\mathcal{O}}_t$.

In each Steinberg variety $\mathcal{F}_x$ of flags F fixed by x, we may consider the subvariety $\mathcal{F}_x^{(t)}$ of flags such that the endomorphisms induced by x on the subquotients belong to $\mathcal{O}_t$. The closures $\mathcal{F}_x^t$ of these varieties generalize the Steinberg varieties (case t "in general position"), as well as the varieties studied by Spaltenstein in [Sp] (case $t = 0$), and in [BM], §7. In our setting, we can study them in a unified manner, since they all occur as fibres of our partial resolution restricted to $\overline{\mathcal{O}}_y$, the closure of the appropriate stratum. Extending results of Springer [S2], and Hotta-Shimomura [HSh], we will show how to deduce geometrical information on the varieties $\mathcal{F}_x^t$ from representation theory of Weyl groups. For the case GL_n considered here numerical data such as the Betti numbers have elegant explicit combinatorial descriptions in terms of Kostka numbers, semi-standard tableaux etc. However, we will forego the combinatorial aspects here, and refer instead to [HSh], and to [Md] for this topic.

[W], m. 39

§1 SEMISMALL MAPS, INTERSECTION HOMOLOGY, AND THE DECOMPOSITION THEOREM

1.1 Semismall maps and relevant strata

Let $\pi : Z \to X$ be a proper algebraic map of one (nonempty) irreducible complex algebraic variety onto another. Let $X = \bigcup_x \mathcal{O}_x$ be a disjoint decomposition of X into a finite number of irreducible smooth subvarieties, called <u>strata</u>. Here x denotes a distinguished base point in the stratum $\mathcal{O}_x$. We assume that this stratification makes π a <u>weakly stratified mapping</u>, i.e. that it satisfies the following condition: For each stratum $\mathcal{O}_x$, the restriction of π to its preimage $\pi^{-1}\mathcal{O}_x$ is a topological fibration with base $\mathcal{O}_x$ and fibre $\pi^{-1}x$. (Stratifications satisfying this condition always exist, see [Hd] and [T], p. 276.)

We denote by d_x the dimension of the fibre $\pi^{-1}x$ and by c_x the codimension of the stratum $\mathcal{O}_x$ i.e. $c_x = \dim X - \dim \mathcal{O}_x$. If not otherwise stated, "dimension" always means complex dimension.

<u>Definition</u>: The map π is <u>semismall</u>, if $2d_x \leq c_x$ for all x. A stratum $\mathcal{O}_x$ is <u>relevant</u> for π, if equality holds, $2d_x = c_x$. A map is <u>small</u>, if it is semismall, and the only relevant stratum is the dense one.

<u>Remark</u>: The properties of π being semismall or small do not depend on the choice of a stratification. That π is semismall can be rephrased in a stratification free way as follows:

$$\text{for all } i\ , \quad \dim\{p \in X \mid \dim \pi^{-1}p \geq i\} \leq \dim X - 2i$$

If π is semismall and is stratified in two different maps, there will be a one to one correspondence between the sets of relevant strata for the two stratifications. Corresponding strata are characterized by the property that their intersection is open (and dense) in each of them.

1.2 Monodromy-representations of local systems

The fundamental group of a stratum $\mathcal{O}_x$ acts on the highest cohomology group of the corresponding fibre $\pi^{-1}x$, denoted $V_x = H^{2d_x}(\pi^{-1}x,\mathbb{Q})$, by monodromy. Since $2d_x \le c_x$, V_x has a basis corresponding to the d_x dimensional irreducible components of $\pi^{-1}x$, and $\pi_1(\mathcal{O}_x)$ acts by permuting these. This linear representation of $\pi_1(\mathcal{O}_x)$ on V_x is denoted μ_x. It is the monodromy representation of the local system on $\mathcal{O}_x$ obtained from the sheaf $R^{2d_x}\pi_*\underline{\mathbb{Q}}(Z)$ by restriction to $\mathcal{O}_x$. Here $\underline{\mathbb{Q}}(Z)$ denotes the constant sheaf with stalk $\mathbb{Q}$ on Z. Let us write

$$(1) \qquad \mu_x = \sum_\phi m_{(x,\phi)}\phi \quad \text{or also} \quad V_x = \bigoplus_\phi V_\phi \otimes V_{(x,\phi)}$$

for the decomposition of μ_x (within the Grothendieck group) into inequivalent irreducible representations $\phi : \pi_1(\mathcal{O}_x) \to \mathrm{End}V_\phi$. Here $V_{(x,\phi)} = \mathrm{Hom}_{\pi_1(\mathcal{O}_x)}(V_\phi,V_x)$ is a $\mathbb{Q}$-vector-space of dimension $m_{(x,\phi)} = \mathrm{mtp}(\phi,\mu_x)$, the multiplicity of ϕ in μ_x. Now any linear representation ρ of $\pi_1(\mathcal{O}_x)$ on a $\mathbb{Q}$-vector-space V determines a unique local system L_ρ with monodromy representation ρ (i.e. a locally constant sheaf with stalk V_ρ over x) on $\mathcal{O}_x$. With this notation, we may write

$$(2) \qquad L_{\mu_x} = \bigoplus_\phi L_\phi \otimes V_{(x,\phi)}$$

for the decomposition of $R^{2d_x}\pi_*\underline{\mathbb{Q}}(Z)\big|_{\mathcal{O}_x}$ into indecomposable local systems derived from (1).

<u>Definition</u>: Assume π semismall. A pair (x,ϕ) as above is <u>relevant</u> for π, if the stratum $\mathcal{O}_x$ is relevant (1.1), and if the irreducible representation ϕ of its fundamental groups occurs in μ_x, i.e. if ϕ satisfies $V_{(x,\phi)} \neq 0$.

<u>Lemma</u>:

a) The multiplicity of the trivial representation $\phi = 1$ in μ_x, $m_{(x,1)} = \dim V_{(x,1)}$, coincides with the number of $\pi_1(\mathcal{O}_x)$-orbits in the set of d_x-dimensional components of $\pi^{-1}x$.

b) The following are equivalent:

(i) For at least one ϕ, (x,ϕ) is relevant for π.

(ii) $(x,1)$ is relevant for π.

(iii) $\mathcal{O}_x$ is relevant for π.

<u>Proof</u>: It is implied by the definitions, that $V_{(x,1)}$ identifies with the space of $\pi_1(\mathcal{O}_x)$-invariants in $V_x = H^{2d_x}(\pi^{-1}x,\mathbb{Q})$. The lemma is now clear from the interpretation of μ_x as a permutation representation. Q.E.D.

<u>1.3 Intersection homology</u>

Given a local system L_ϕ on a stratum $\mathcal{O}_x$ as above, let $\underline{\underline{IC}}^\bullet(L_\phi)$ denote the <u>intersection homology sheaf</u> with coefficients in L_ϕ. This is a certain complex of sheaves of $\mathbb{Q}$-modules on $\overline{\mathcal{O}}_x$ defined as in ([GM2], §2.1 or §3.1, middle perversity) up to a dimension-shift of $-2\dim\mathcal{O}_x$, such that $\underline{\underline{H}}^\circ(\underline{\underline{IC}}^\bullet(L_\phi))$ restricted to $\mathcal{O}_x$ is L_ϕ. Here and below $\underline{\underline{H}}^i(\ldots)$ denotes the i-th cohomology-sheaf of a complex, and $(\ldots)_u$ will denote the stalk at u of a sheaf. The local intersection homology groups of $\overline{\mathcal{O}}_x$ at a point $u \in \overline{\mathcal{O}}_x$ are defined by

$$IH^i_u(\overline{\mathcal{O}}_x, L_\phi) = \underline{\underline{H}}^i(\underline{\underline{IC}}^\bullet(L_\phi))_u \, .$$

1.4 Rational homology manifolds

Recall that a complex variety X is called a rational homology manifold (of dimension n), or is said to be rationally smooth, if for all points $u \in X$ we have

$$H_i(X, X - \{u\}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 2n \\ 0 & \text{otherwise} \end{cases}$$

Here H_i denotes ordinary homology. A rational homology manifold of dimension n has pure dimension n as a complex variety.

Rational homology manifolds are classical objects of topological study. They may be thought of as "nonsingular for purposes of rational homology". For example, Poincaré and Lefschetz duality hold for them in rational homology. Examples of rational homology manifolds include surfaces with Kleinian singularities, the moduli space for curves of a given genus. More generally, v-manifolds are rational homology manifolds.

PROPOSITION. <u>The following are equivalent</u>:

(i) X <u>is a rational homology manifold</u> .

(ii) <u>For all</u> $u \in X$,

$$IH_u^i(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

(iii)

$$\underline{\underline{H}}^i(\underline{\underline{IC}}^\bullet(\mathbb{Q})) = \begin{cases} \underline{\underline{\mathbb{Q}}}(X) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

<u>Proof</u>: To prove that (i) implies (iii), consider the dualizing complex $\underline{\underline{D}}_X^\bullet[-2n]$ of X with rational coefficients (see [GM2], §1.6) where $[-2n]$ denotes a dimension shift by $-2n$. The dualizing complex satisfies

$$H^i(\underline{\underline{D}}_X^\bullet[-2n])_u = H_{2n-i}(X, X - u; \mathbb{Q})$$

so by (i) the map $\underline{\underline{\mathbb{Q}}}(X) \to \underline{\underline{D}}_X^\bullet[-2n]$ iduced by capping with the fundamental class of X must be an isomorphism. Further the Verdier dual of $\underline{\underline{D}}_X^\bullet[-2n]$ is $\underline{\underline{\mathbb{Q}}}(X)$. Therefore $\underline{\underline{\mathbb{Q}}}(X)$ satisfies the axioms AX3 of [GM2] characterizing $\underline{\underline{IC}}^\bullet(\mathbb{Q})$. Statement (iii) clearly implies statement (ii), so the interesting part is to show that (ii) implies (i).

To do this, we proceed by induction on the codimension c of the stratum of a Whitney stratification of X containing u . For $c = 0$, X is smooth at u so (i) is clear. Suppose we have established (i) for all strata of codimension less than c. By standard Kunneth and coneing arguments,

$$H_i(X, X - u; \mathbb{Q}) \cong \begin{cases} 0 & \text{for } i \leq 2n-2c \\ H_{i-2n+2c-1}(\mathscr{L}, \mathbb{Q}) & \text{for } i > 2n-2c \end{cases}$$

where $\mathscr{L}$ is the link of the stratum containing u. By the induction hypotheses $\mathscr{L}$ is a rational homology manifold of dimension $2c - 1$ so it satisfies Poincaré duality over $\mathbb{Q}$. Therefore it is enough to calculate its homology in half the dimensions; that is it is enough to show

$$H_i(\mathscr{L}, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i = 2c-1 \\ 0 & \text{for } c \leq i < 2c-1 \end{cases}$$

However, for $c \leq i < 2c$, we have

$$\begin{aligned} H_i(\mathscr{L}, Q) &\cong IH^{2c-1-i}(\mathscr{L}, \mathbb{Q}) \\ &\cong IH_u^{2c-1-i}(X, \mathbb{Q}) \end{aligned}$$

so we are done.

Remark: It is not true that condition (ii) of the proposition for a single point $u \in X$ implies the rational homology manifold condition at that point. For example consider the cone over a surface obtained by identifying two points of the complex projective plane and suitably embedding it in projective n-space.

1.5 Decomposition of the cohomology of a fibre in terms of intersection homology of the strata

We now come to the central topic of this chapter. From now on, our map $\pi : Z \to X$ will always be projective and semismall, Z will be rationally smooth, and X will have the same dimension as Z.

Theorem: *Assume that our map* $\pi : Z \to X$ *is projective and semismall, that* Z *is rationally smooth and that* $\dim X = \dim Z$. *Then for any* $u \in X$, *we have*

$$(*) \qquad H^i(\pi^{-1}z, \mathbb{Q}) \cong \bigoplus_{(x,\phi)} IH_u^{i-2d_x}(\overline{\mathcal{O}}_x, L_\phi) \otimes V_{(x,\phi)},$$

where the summation is over all pairs (x,ϕ) *relevant for* π *(but the contribution is zero unless* $u \in \overline{\mathcal{O}}_x$ *).*

In other words, the cohomology groups of a fibre $\pi^{-1}u$ can be computed from the intersection homology of the closures of strata $\overline{\mathcal{O}}_x$ containing u, using only the highest cohomology groups of the corresponding fibres $\pi^{-1}x$. This theorem will be an immediate consequence of the decomposition theorem §1.7, established by Deligne, Gabber, Beilinson, and Bernstein on the more abstract level of derived categories. Formula (*) will follow from the decomposition formula §1.7 (**) by applying the functor $\underline{\underline{H}}^i(\ldots)_u$.

In the next section, we develop preliminaries necessary for the decomposition theorem.

1.6 Perverse sheaves

The idea of a perverse object in $D^b(X)$ was introduced by Deligne and Beilinson-Bernstein. This is a purely topological concept. A complex $\underline{\underline{S}}^\bullet$ on a purely n-dimensional variety is perverse if it satisfies the support condition

$$\dim(\text{support } \underline{\underline{H}}^i \underline{\underline{S}}^\bullet) \le n-i$$

and the dual support condition

$$\dim(\text{support } \underline{\underline{H}}^i \underline{\underline{V}}\underline{\underline{S}}^\bullet[-2n]) \le n-i$$

where $\underline{\underline{V}}$ is the Verdier duality map (normalized so that on a smooth variety M of dimension n, $\underline{\underline{V}}(\underline{\underline{Q}}_M) = \underline{\underline{Q}}_M[2n]$).

We shall need the following properties of perverse objects:

1. ([GM2]), §6.1) for a stratum $\mathcal{O}_x$, $j^x_* \underline{\underline{IC}}^\bullet(L_\phi)[\ell]$ is a perverse object if and only if $\ell = -2d_x$.
2. (Beilinson, Bernstein, Deligne, Gabber [D3]) The full subcategory of $D^b_c(X)$ whose objects are perverse forms an Abelian category whose simple objects are exactly those of the form $j^x_* \underline{\underline{IC}}^\bullet(L_\phi)[-2d_x]$ for some stratification $\{\mathcal{O}_x\}$ of X and some simple representation ϕ of $\pi_1(\mathcal{O}_x)$.

1.7 The decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber

Suppose now that $\pi : Z \to X$ is a proper semismall map of a rational homology manifold Z of dimension n to a purely n-dimensional variety X. Then $\underline{\underline{A}}^\bullet = R\pi_* \underline{\underline{Q}}(Z)$ is perverse. The support condition follows directly from the fact that π is semismall and proper. Since Z is a rational homology manifold of dimension n, we have $\underline{\underline{V}}\ \underline{\underline{Q}}(Z) = \underline{\underline{Q}}(Z)[2n]$. Therefore

$$\underline{\underline{V}}\ \underline{\underline{A}}^\bullet = \underline{\underline{V}}\ R\pi_*\ \underline{\underline{Q}}(Z) = R\pi_*\ \underline{\underline{V}}\ \underline{\underline{Q}}(Z)$$
$$= R\pi_*(\underline{\underline{Q}}(Z)[2n]) = (R\pi_*\ \underline{\underline{Q}}(Z))[2n] = \underline{\underline{A}}^\bullet[2n]$$

so the dual support condition holds also.

<u>Decomposition theorem</u>: <u>Assume that our map</u> $\pi : Z \to X$ <u>is projective and semismall, that</u> Z <u>is rationally smooth and that</u> $\dim X = \dim Z$. <u>Then in the category of perverse sheaves on</u> X,

a) $Rj^x_* \underline{\underline{IC}}^\bullet(L_\phi)[-2d_x]$ <u>is a simple object for each</u> (x,ϕ),

b) $\underline{\underline{A}}^\bullet = R\pi_*\underline{\underline{Q}}(Z) \cong \bigoplus_{(x,\phi)} Rj^x_*\underline{\underline{IC}}^\bullet(L_\phi)[-2d_x] \otimes V_{(x,\phi)}$, (**)

<u>where the sum extends over all pairs</u> (x,ϕ) <u>relevant for</u> π.

<u>Proof</u>: Part a) follows from property 2) of §1.6. To obtain part b), we first write the more general decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [D3], [BeBe], [GM3] which asserts that for any projective map $\pi : Z \to X$,

$$R_* \underline{\underline{IC}}^\bullet(Q) \cong \bigoplus_{(x,\phi,\ell)} Rj^x_* \underline{\underline{IC}}^\bullet(L_\phi)[\ell] \otimes \tilde{V}_{(x,\phi,\ell)}$$

for some subvarieties $\tilde{\mathcal{O}}_x$, some local systems $\tilde{L}_\phi$ on $\tilde{\mathcal{O}}_x$, some integers ℓ, and some vector spaces $\tilde{V}_{(x,\phi,\ell)}$. In our case, the left hand side of this formula is just $R\pi_* \underline{\underline{Q}}(Z) = \underline{\underline{A}}^\bullet$ since Z is a rational homology manifold. It remains to check that the data $\tilde{\mathcal{O}}_x, \tilde{L}_\phi, \tilde{\ell}, \tilde{V}_{(x,\phi,\ell)}$ on the right hand side are as claimed in the theorem.

Since $\pi : Z \to X$ is weakly stratified by the $\mathcal{O}_x$, it follows that the sheave $\underline{\underline{H}}^i \underline{\underline{A}}$ are locally constant on the $\mathcal{O}_x$, therefore the $\tilde{\mathcal{O}}_x$ may be taken to be a subset of the $\mathcal{O}_x$. Since $\underline{\underline{A}}^\bullet$ is perverse, the integers ℓ must be $-2d_x$. To determine the remaining data, we consider $\underline{\underline{H}}^{2d_{x'}} \underline{\underline{A}}^\bullet$ restricted to $\mathcal{O}_{x'}$. Only the terms of the right hand side with $x = x'$ contribute to this by the support condition for $\underline{\underline{IC}}^\bullet$ (see axioms AX1 of [GM2]). For these we see $\tilde{L}_\phi$ must be L_ϕ and $\tilde{V}_{(x,\phi,\ell)}$ must be $V_{(x,\phi)}$ for equality to hold.

1.8 Example: small maps

Assume that the only stratum relevant for π is the dense one, and that π is an isomorphism above this stratum. Or in other words, assume π is a small resolution. In this case, the sum in (**) reduces to a single simple summand. More generally, if π is small, then (**) reduces essentially to a result of Goresky and MacPherson [GM2], §6.2.

1.9 Reformulation as an Artin-Wedderburn decomposition

The decomposition theorem 1.8 says that $\underline{\underline{A}}^{\bullet} = R\pi_*\underline{\underline{Q}}(Z)$ is semisimple as an object in the category of perverse sheaves (§1.6), so it has a semisimple endomorphism-ring, which admits an Artin-Wedderburn decomposition into a direct product of matrix rings over skew fields. We may express it in terms of the decomposition formula 1.7 (**) as follows:

Corollary: $\mathrm{End}\,\underline{\underline{A}}^{\bullet} \cong \prod_{(x,\phi)} K_{(x,\phi)} \otimes \mathrm{End}_{\mathbb{Q}} V_{(x,\phi)}$, where $K_{(x,\phi)}$ is the skew-field of endomorphisms of ϕ .

1.10 Product of two semismall maps

Assume that $\pi = \xi\eta$ is a product of two proper algebraic maps $Z \xrightarrow{\eta} Y \xrightarrow{\xi} X$, that π,ξ,η are semismall, that Z is an n-dimensional rational homology manifold, and that Y and X are purely n dimensional. We choose stratifications $\{\mathcal{O}_y\}$ of Y and $\{\mathcal{O}_x\}$ of X so that π and η are weakly stratified. Maintaining all notations previously introduced for π , we have a decomposition formula

$$(1) \qquad \underline{\underline{A}}^{\bullet} = R\pi_*\underline{\underline{Q}}(Z) = \bigoplus_{(x,\phi)} Rj_*^x\, \underline{\underline{IC}}^{\bullet}(L_\phi)[-2d_x] \otimes V_{(x,\phi)}$$

for the map π , and using the strata $\mathcal{O}_y \subset Y$ for η we have with completely analogous notation another decomposition formula

$$(2) \qquad \underline{\underline{B}}^{\bullet} = R\eta_*\underline{\underline{Q}}(Z) = \bigoplus_{(y,\psi)} Rj_*^y\, \underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_y] \otimes V_{(y,\psi)}$$

for the map η. They are related by the fact that

$$\text{(3)} \qquad \underline{\underline{A}}^{\bullet} = R\xi_{*}\underline{\underline{B}}^{\bullet} .$$

Our purpose is to analyze this relation, and to draw some conclusions concerning the fibres of ξ.

Let us consider for each $\mathcal{O}_y$ relevant for η the restriction $\xi_y = \xi j^y$ of ξ to $\overline{\mathcal{O}}_y$.

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "\eta"'] & X \\ Y \arrow[ur, "\xi"] & \overline{\mathcal{O}}_y \arrow[l, hook', "j^y"] \arrow[u, "\xi_y"'] \end{tikzcd}$$

Lemma: For each pair (y,ψ) relevant for η and for each x, the sheaf $R^{2d_x}(\xi_y)_{*}\,\underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_y]$ will be a locally free sheaf when restricted to $\mathcal{O}_x$ with stalk $\mathbb{H}^{2d_x-2d_y}(\xi_y^{-1}x,\ \underline{\underline{IC}}^{\bullet}(L_\psi))$.

Here $\mathbb{H}(\ldots)$ denotes hypercohomology of the complex of sheaves on $\xi_y^{-1}x$ obtained by restricting $\underline{\underline{IC}}^{\bullet}(L_\psi)$.

Proof: This is locally free because, by the decomposition formula for η, $R^{2d_x}(\xi_y)_{*}\,\underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_x]$ will be a direct summand of $R^{2d_x}\pi_{*}\underline{\mathbb{Q}}(Z)$ which, since π is weakly stratified, is locally free on $\mathcal{O}_x$.

To identify the stalk, form the diagram

$$\begin{tikzcd} \xi_y^{-1}x \arrow[r, "b"] \arrow[d, "a"'] & \bar{\mathcal{O}}_y \arrow[d, "\xi_y"] \\ \{x\} \arrow[r, "c"'] & X \end{tikzcd}$$

Then
$$\begin{aligned} & R^{2d_x}(\xi_y)_* \underline{\underline{IC}}^\bullet (L_\psi)[-2d_y]_x \\ &= \mathbb{H}^{2d_x-2d_y} c^* R(\xi_y)_* \underline{\underline{IC}}^\bullet (L_\psi) \\ &= \mathbb{H}^{2d_x-2d_y} Ra_* b^* \underline{\underline{IC}}^\bullet (L_\psi) \\ &= \mathbb{H}^{2d_x-2d_y}(\xi_y^{-1}x, \underline{\underline{IC}}^\bullet (L_\psi)) \ . \end{aligned}$$

<u>Remark</u>: In the event that $\bar{\mathcal{O}}_y$ is rationally smooth at all points in $\xi_y^{-1}x$, and L_ψ extends to a local system $\bar{L}_\psi$ over $\xi_y^{-1}x$, then

$$\mathbb{H}^i(\xi_y^{-1}x, \underline{\underline{IC}}^\bullet(L_\psi)) = H^i(\xi_y^{-1}x, \bar{L}_\psi)$$

so the stalk of the local system is an ordinary homology group which twisted coefficients. If $\bar{\mathcal{O}}_y$ is rationally smooth at all points of $\xi_y^{-1}x$ and $\psi = 1$, then

$$H^i(\xi_y^{-1}x, \underline{\underline{IC}}^\bullet(L_\psi)) = H^i(\xi_y^{-1}x, \mathbb{Q}) \ .$$

We write the decomposition of the monodromy representation for this local system in to isotypical components as follows:

$$(4)\qquad \mathbb{H}^{2d_x-2d_y}(\xi_y^{-1}x,\ \underline{\underline{IC}}^{\bullet}(L_\psi)) = \bigoplus_{\phi} V_\phi \otimes V_{(x,\phi)}^{(y,\psi)}$$

where V_ϕ is a simple $\pi_1(\mathcal{O}_x)$ representation occurring with multiplicity $m_{(x,\phi)}^{(y,\psi)} = \dim V_{(x,\phi)}^{(y,\psi)}$. Here we chose the notation analogous to that used for the decompositions of the monodromy representations for π resp. η (cf. 1.2),

$$(5)\qquad \begin{aligned} H^{2d_x}(\pi^{-1}x,\ \mathbb{Q}) &= \bigoplus_{\phi} V_\phi \otimes V_{(x,\phi)} \quad \text{resp.} \\ H^{2d_y}(\eta^{-1}y,\ \mathbb{Q}) &= \bigoplus_{\psi} V_\psi \otimes V_{(y,\psi)} , \end{aligned}$$

by which we introduced the vector-spaces $V_{(x,\phi)}$ resp. $V_{(y,\psi)}$.

Proposition: For all pairs (y,ψ) relevant for η , we have

$$(6)\qquad R\xi_* Rj_*^y\ \underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_y] \cong \bigoplus_{(x,\phi)} Rj_*^x\ \underline{\underline{IC}}^{\bullet}(L_\phi)[-2d_x] \otimes V_{(x,\phi)}^{(y,\psi)} .$$

Proof: The perverse sheaf $Rj_*^y\ \underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_y]$ is a direct summand of $\underline{\underline{B}}^{\bullet}$, so $R\xi_* Rj_*^y\ \underline{\underline{IC}}^{\bullet}(L_\psi)[-2d_y]$ is a direct summand of $R\xi_*\underline{\underline{B}}^{\bullet} = \underline{\underline{A}}^{\bullet}$. Since $\underline{\underline{A}}^{\bullet}$ is semisimple in the category of perverse sheaves on X , any direct summand of $\underline{\underline{A}}^{\bullet}$ must be a direct sum of simple constituents of $\underline{\underline{A}}^{\bullet}$. Thus one has some decomposition of $R\xi_* Rj_*^y \underline{\underline{IC}}^\bullet(L_\psi)[-2d_y]$ into intersection homology sheaves of subvarieties of X as in the proof of theorem 1.7. The identification of the data for these as the data given above proceeds exactly as in the proof of theorem 1.7.

1.11 A double decomposition formula

From (6), (2), and (3) above, we obtain the following double decomposition formula for $\underline{\underline{A}}^\bullet = R\pi_* \underline{\mathbb{Q}}(Z)$:

$$\underline{\underline{A}}^\bullet \cong \bigoplus_{(x,\phi)} \bigoplus_{(y,\psi)} Rj_*^x \underline{\underline{IC}}^\bullet(L_\phi)[-2d_x] \otimes V_{(x,\phi)}^{(y,\psi)} \otimes V_{(y,\psi)} \tag{7}$$

Comparing this with the decomposition (1), we obtain the

(8) Corollary: $$V_{(x,\phi)} \cong \bigoplus_{(y,\psi)} V_{(x,\phi)}^{(y,\psi)} \otimes V_{(y,\psi)} .$$

1.12 Strata with rationally smooth closure

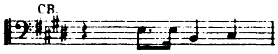
Theorem: Let $u \in X$. Assume that $\overline{\mathcal{O}}_y$ is rationally smooth (1.3) at all points which ξ maps to u . Then the cohomology of the fibre $\xi_y^{-1}u$ is given by the following formula:

$$H^{i-2d_y}(\xi_y^{-1}u, \mathbb{Q}) \cong \bigoplus_{(x,\psi)} IH_u^{i-2d_x}(\overline{\mathcal{O}}_x, L_\phi) \otimes V_{(x,\phi)}^{(y,1)} \tag{9}$$

If L_ψ extends to a local system $\overline{L}_\psi$ over $\xi_y^{-1}u$, then

$$H^{i-2d_y}(\xi_y^{-1}u, \bar{L}_\psi) \cong \bigoplus_{(x,\phi)} IH_u^{i-2d_x}(\bar{\mathcal{O}}_x, L_\phi) \otimes V_{(x,\phi)}^{(y,\psi)}$$

Proof: In view of the remark in §1.10, this results from applying the functor $\underline{\underline{H}}^i(\ldots)_u$ to formula 6 in §1.10.



[W], #85, m. 9

§2 GENERALIZATIONS OF SPRINGER'S RESOLUTION AND SPRINGER'S THEORY OF WEYL-GROUP REPRESENTATIONS

2.1 Steinberg varieties as fibres of partial resolutions

Let G be a connected reductive complex algebraic group, $\mathcal{N} = \mathcal{N}(G)$ the variety of all nilpotent elements in its Lie-algebra $\underline{g}$, $\mathcal{B} = \mathcal{B}(G)$ the variety of all Borel subgroups of G , and $\mathcal{P}$ the variety of all parabolic subgroups conjugate to a fixed one, denoted P , which is chosen once and for all. We are going to apply the general ideas of the preceding section to analyze Springer's resolution of singularities for the variety $\mathcal{N}$. For this application, let us specify now the data considered in §1 as follows: First, $X = \mathcal{N}$, and $Z = \tilde{\mathcal{N}} \subset \mathcal{N} \times \mathcal{B}$ is the variety $\tilde{\mathcal{N}}$ of pairs (x,B) with $x \in \text{Lie } B$. Moreover, $\pi : Z \to X$ is the map which forgets B . This is the Springer resolution. Alternatively, $\mathcal{N}$ may be identified with the cotangent bundle $T^*\mathcal{B}$, and then π is the moment map of $\mathcal{B}$, see [BB], §2. Second, Y is the variety $\tilde{\mathcal{N}}^P \subset \mathcal{N} \times \mathcal{P}$ of pairs (x,P') with $x \in \text{Lie}P'$, and $\xi : Y \to X$ forgets P' . This we call a "<u>partial resolution</u>". Finally, $\eta : Z \to Y$ sends (x,B) to (x,P') , where P' is the unique parabolic subgroup of G belonging to $\mathcal{P}$ and containing B . Note that our factorization $\pi = \xi\eta$ means just this: Forget B not at once, but "in two steps".

It remains to specify the stratifications $X = \bigcup_x \mathcal{O}_x$ for π and ξ , resp. $Y = \bigcup_y \mathcal{O}_y$ for η (as in 1.10). The strata $\mathcal{O}_x$ are just the orbits of nilpotent elements under the adjoint action of G in $\underline{g}$. The fibres $\pi^{-1}x$ identify with the varieties $\mathcal{B}_x$ of all $B \in \mathcal{B}$ with $x \in \text{Lie}B$. Their dimension d_x is known ([St2], Thm. 4.6) to satisfy $2d_x = c_x$, c_x being the codimension of

$\mathcal{O}_x$ in X (cf. 1.1). In other words, the Springer resolution π is <u>semismall with all $\mathcal{O}_x$ relevant</u>. It follows that the partial resolution ξ is also semismall. Its fibres $\xi^{-1}x$ identify with the varieties $\mathcal{O}_x$ of all $P' \in \mathcal{P}$ with $x \in \mathrm{Lie}P'$, which have been extensively studied, first by Steinberg [St2]. One of our goals is to reformulate and to extend the theory of <u>Steinberg varieties</u> $\mathcal{P}_x$.

Since the specification of our strata $\mathcal{O}_y$ is slightly more subtle, we postpone it until section 2.7, where we shall see that η is also semismall.

<u>Remark</u>: In [BM] §7, we considered the "generalized Springer resolution" $\tilde{\mathcal{N}}_P$ (the moment map of $\mathcal{P}$). In the present terminology, $\tilde{\mathcal{N}}_P$ is the unique closed stratum in $\hat{\mathcal{N}}^P$ (see §2.10). Moreover, the varieties denoted $\mathcal{P}_x$ in [BM], §7, are denoted $\mathcal{P}_x^0$ in the present paper (see 3.2).

2.2 Recollections on Springer's correspondence

Now the general theory of §1 applies, and we have e.g. a decomposition formula for the object $\underline{\underline{A}}^\bullet = R\pi_*\mathbb{Q}(Z)$ in the derived category $D^b(X)$, as stated in 1.7. But in the specific situation considered here, $\underline{\underline{A}}^\bullet$ carries also an interesting additional structure: An action of the Weyl-group W of G which, by the main result of [BM], gives rise to an <u>isomorphism</u>

$$\alpha : \mathbb{Q}[W] \xrightarrow{\sim} \mathrm{End}\, \underline{\underline{A}}^\bullet$$

from the group ring of W to the endomorphism ring of $\underline{\underline{A}}^\bullet$. We shall recall the definition of this action (due to Lusztig [L]) below in §2.6. Since any automorphism of $\underline{\underline{A}}^\bullet$ can only act by "permuting" isotypical simple direct summands, the W-action on $\underline{\underline{A}}^{\bullet}$ is given by a collection of linear representations $\rho_{(x,\phi)}$ on the $\mathbb{Q}$ vector spaces $V_{(x,\phi)}$ occurring in the decomposition formula 1.7 (**).

THEOREM (Springer correspondence): <u>The pairs</u> (x,ϕ) <u>relevant for the Springer resolution</u> π <u>are in bijective correspondence to the irreducible characters of the Weyl group</u> W , <u>by</u> $(x,\phi) \longleftrightarrow \rho_{(x,\phi)}$.

In fact, this is an immediate corollary of the isomorphism α above, which follows by comparing the Artin-Wedderburn decomposition for $\mathbb{Q}[W]$ on one hand, with the decomposition for $\underline{\underline{A}}^{\bullet}$ (in the form §1.10) on the other hand. (It turns out that all the $K_{(x,\phi)}$ of §1.9 are $\mathbb{Q}$).

Finally, we recall that the W action on $\underline{\underline{A}}^{\bullet}$ gives rise to a W action on $H^i(\mathcal{B}_u, \mathbb{Q}) = \underline{\underline{H}}^i(\underline{\underline{A}}^{\bullet})_u$ by functoriality for each i and $u \in X$. This turns out to coincide, after a multiplication with the sign-character, with Springer's representation defined in [S1], [S2], see Hotta [H], and [AL]. Moreover, it is clear now that formula 1.5 (*) describes exactly the decomposition of Springer's W representations on $H^i(\mathcal{B}, \mathbb{Q})$ into irreducible constituents, as conjectured and proved for $G = SL_n$ by Lusztig [L], and first proved in general in [BM].

The formula may be stated alternatively this way:

$$(*) \qquad \mathrm{Hom}_W(V_{(x,\phi)}, H^i(\mathcal{B}_u, \mathbb{Q})) \cong IH_u^{i-2d_x}(\overline{\mathcal{O}}_x, L_\phi) .$$

It has recently been used to compute explicitly all $\rho_{(x,\phi)}$ with $\phi = 1$ [AL].*)

*)N. Spaltenstein told us in December 1981 that he has completed the explicit computation of Springer's correspondence for all ϕ using this formula.

2.3 The nilpotent cone is rationally smooth

We now give an application of the above theory. The result could have been stated in 1930, but seems to be new:

THEOREM: $\mathcal{N}$ *is a rational homology manifold*.

For the definition in classical terms, as well as in terms of intersection homology, recall 1.4. We have to prove for each $u \in \mathcal{N}$

$$IH^i_u(\mathcal{N},\mathbb{Q}) = \mathbb{Q} \text{ resp. } 0 \quad \text{if} \quad i = 0 \text{ resp. } > 0 \ .$$

Let $\mathcal{O}_x$ be the dense stratum of $\mathcal{N}$, $\bar{\mathcal{O}}_x = \mathcal{N}$. Then $d_x = 0$, and $\rho_{(x,1)} = 1$ is the trivial representation of W. Hence 2.2(*) says that

$$IH^i_u(\mathcal{N},\mathbb{Q}) \cong H^i(\mathcal{B}_u, \mathbb{Q})^W \text{ (= W-invariants) } .$$

So the theorem reduces to the following lemma, which will be proved in §2.9.

LEMMA (Lusztig): *The trivial representation* 1 *occurs in the Springer representation* $H^i(\mathcal{B}_u, \mathbb{Q})$ *with multiplicity* 1 resp. 0 *if* $i=0$ resp. > 0 .

2.4 Generalization of Grothendieck's simultaneous resolution

The Springer resolution $\pi : \tilde{\mathcal{N}} = Z \to X = \mathcal{N}$ extends to a map $\pi' : \tilde{\underline{g}} \to \underline{g}$ which is well-known as the Grothendieck *simultaneous resolution* (cf. [St1], p.131), and which is defined just by omitting the restriction $x \in \mathcal{N}$ in the definition of π (cf. 2.1). Similarly, and more generally, we may extend the *partial* resolution $\xi : \tilde{\mathcal{N}}^P = Y \to X = \mathcal{N}$ to a "*partial simultaneous resolution*"

$\xi' : \tilde{\mathfrak{g}}^P \to \mathfrak{g}$. Here $\tilde{\mathfrak{g}}^P$ is the variety of pairs (x,P') with $x \in \mathfrak{g}$, $P' \in \mathcal{P}$ such that $x \in \operatorname{Lie}P'$, and ξ' forgets P' . Alternatively, the variety $\tilde{\mathfrak{g}}^P$ may be described as an associated fibre-bundle $G \times^P \operatorname{Lie}P$. Here we use the following notation: If P acts on a set M , then $G \times^P M = G \times M/\sim$ is the set of orbits under the free P-action $(g,m) \to (g\,p^{-1}, p\,m)$ in the product $G \times M$. (If M is a variety with algebraic P-action, then $G \times^P M$ is a variety with algebraic G-action.)

LEMMA: <u>The map</u> $\tilde{\mathcal{N}}^P \to \mathcal{P}$ <u>forgetting</u> x <u>is a vector bundle with fibre</u> $\operatorname{Lie}P$, <u>which identifies with</u> $G \times^P \operatorname{Lie}P \to G/P$.

<u>Remark</u>: Let us mention that the decomposition of $\mathfrak{g}$ into finitely many "decomposition classes" (Zerlegungs-klassen), as studied in [Bl], provides a very natural explicit stratification for π' and ξ' . However, for the purposes of the present paper, it suffices to make explicit the unique dense stratum, which is the set $\mathfrak{g}_{rs}$ of regular semisimple elements.

2.5 Coverings of the regular semisimple elements

Let $\mathfrak{g}_{rs}$ be the set of regular semisimple elements of $\mathfrak{g}$, and let $\xi'' : \tilde{\mathfrak{g}}^P_{rs} \to \mathfrak{g}_{rs}$ be the part of the partial simultaneous resolution lying over it:

$$\begin{array}{ccccc}
\tilde{\mathcal{N}}^P & \subset & \tilde{\mathfrak{g}}^P & \supset & \tilde{\mathfrak{g}}^P_{rs} \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi''} \\
\mathcal{N} & \overset{i}{\subset} & \mathfrak{g} & \supset & \mathfrak{g}_{rs}
\end{array}$$

Let L be a Levi subgroup of P, and let W^P denote the finite group $N_G(L)/L$ (the "partial Weyl group").

LEMMA: <u>The map ξ'' is a covering projection $\tilde{\underline{g}}^P_{rs} \to \underline{g}_{rs}$, on which W^P acts by deck transformations.</u>

In the special case when P is a Borel subgroup B, then L is a maximal torus T and $W^P = W$ is the ordinary Weyl group (relative T), and ξ'' is well known to be a principal W-fibration, so W acts on it by deck transformations. In the general case, a fibre $(\xi'')^{-1}(h)$ consists of those conjugates of P containing the unique maximal torus, $T \subset L$ say, such that $h \in \mathrm{Lie}T$, and these are parametrized by the right cosets of W with respect to $W(L) = N_L(T)/T$. Moreover, the canonical map sending a Borel subgroup to the unique conjugate of P containing it induces a covering map $\tilde{\underline{g}}^B_{rs} \to \tilde{\underline{g}}^P_{rs}$, which is a principal $W(L)$ fibration, so that $\tilde{\underline{g}}^P_{rs}$ identifies with the orbit space $W(L)\backslash \tilde{\underline{g}}^B_{rs}$. Since W acts by deck transformations on $\tilde{\underline{g}}^B_{rs}$, it is now clear that the group $N_W(W(L))/W(L) \cong W^P$ acts by deck transformation on $\tilde{\underline{g}}^P_{rs}$.

<u>Remark</u>: It can be shown that W^P is even the complete group of deck transformations of $\tilde{\underline{g}}^P_{rs}$ over $\underline{g}_{rs}$.

<u>2.6 Generalization of Lusztig's Weyl group action on $\underline{\underline{A}}^{\bullet}$</u>

Denote by i the inclusion of $\mathcal{N}$ in $\underline{g}$. Consider the functor $i^*(\underline{\underline{IC}}^{\bullet}(.))$ from the category of local systems on $\underline{g}_{rs}$ to $D^b(\mathcal{N})$. This associates to a local system $\mathcal{L}$ on $\underline{g}_{rs}$ the restriction to $\mathcal{N}$ of the intersection homology sheaf on $\underline{g}$ with coefficients in $\mathcal{L}$.

<u>Proposition</u>: a) The object $\underline{\underline{A}}^{\bullet P} = R\xi_* \underline{\underline{Q}}(\tilde{\mathcal{N}}^P)$ is obtained by applying the functor

$i^*(\underline{\underline{IC}}^\bullet(.))$ to the local system $\xi''_*\underline{\underline{Q}}(\tilde{\underline{g}}^P_{rs})$. b) The action of W^P on $\tilde{\underline{g}}^P_{rs}$ by deck transformations (2.5) induces a W^P action on object $\underline{\underline{A}}^{\bullet P}$.

Proof: In fact, since $\tilde{\underline{g}}^P$ is smooth (lemma 2.4), and since the map $\xi' : \tilde{\underline{g}}^P \to \underline{g}$ is small, or more precisely is semismall with $\underline{g}_{rs}$ as the only relevant stratum, §6.2 of [GM2] applies to give

$$\underline{\underline{IC}}^\bullet(\xi''_*\underline{\underline{Q}}(\tilde{\underline{g}}^P_{rs})) = R\xi'_*\underline{\underline{Q}}(\tilde{\underline{g}}^P)$$

This implies a) of the proposition, that is

$$i^*(\underline{\underline{IC}}^\bullet(\xi''_*\underline{\underline{Q}}(\tilde{\underline{g}}^P_{rs}))) = R\xi_*\underline{\underline{Q}}'(\tilde{\underline{g}}^P) = \underline{\underline{A}}^{\bullet P}$$

because the left square in the diagram in 2.5 is a fibre square (or in other words $\tilde{\mathcal{N}}^P$ is the full preimage of $\mathcal{N}$ under ξ'). Now b) is an immediate consequence of a).

Remark: For the case when P is a Borel subgroup, this is Lusztig's construction of a W action on $\underline{\underline{A}}^\bullet = R\pi_*\underline{\underline{Q}}(\tilde{\mathcal{N}})$, see [L], [BM] (Note $\pi = \xi$ then.). In this case, one can prove that the endomorphism ring in $D^b(\underline{g})$ of $R\xi'_*\underline{\underline{Q}}(\tilde{\underline{g}})$ (which is just that of $R\xi''_*\underline{\underline{Q}}(\tilde{\underline{g}}_{rs})$) is taken isomorphically onto the endormorphism ring in $D^b(\mathcal{N})$ of $\underline{\underline{A}}^\bullet$ by the function i^* . This behavior of an endomorphism ring in the derived category to be preserved under the image of an inclusion is extremely unusual.

2.7 Invariants of $\underline{\underline{A}}^\bullet$

Let $P = LU$ be the semidirect decomposition of our parabolic subgroup P into its unipotent radical U, and a Levi subgroup L. The Weyl-group of L identifies with a subgroup $W(L)$ of W.

Proposition (notation 2.5):

a) $$R\xi_{\underline{*}}^{!}\underline{\mathbb{Q}}(\tilde{\underline{g}}^{P}) \cong (R\xi_{\underline{*}}^{!}\underline{\mathbb{Q}}(\tilde{\underline{g}}))^{W(L)}$$

b) $$R\xi_{\underline{*}}\underline{\mathbb{Q}}(\tilde{\mathcal{N}}^{P}) \cong (R\xi_{\underline{*}}\underline{\mathbb{Q}}(\tilde{\mathcal{N}}))^{W(L)} .$$

Here the superscripts $W(L)$ denote $W(L)$ invariants. Note that for automorphisms of objects in an abelian category, it makes sense to speak about invariants. The objects considered here are in the abelian categories of perverse sheaves (see §1.6).

Proof: We have seen (cf. 2.5) that the covering $\xi'' : \tilde{\underline{g}}_{rs}^{P} \to \underline{g}_{rs}$ is obtained from the principal W fibration $\pi'' : \tilde{\underline{g}}_{rs} \to \underline{g}_{rs}$ by dividing by the $W(L)$ action: $\tilde{\underline{g}}_{rs}^{P} = W(L)\backslash\tilde{\underline{g}}_{rs}$. Therefore, the local system $\xi''_{\underline{*}}\underline{\mathbb{Q}}(\tilde{\underline{g}}_{rs}^{P})$ on $\underline{g}_{rs}$ is obtained from the local system $\pi''_{\underline{*}}\underline{\mathbb{Q}}(\tilde{\underline{g}}_{rs})$ by taking $W(L)$ invariants:

(1) $$\xi''_{*}\mathbb{Q}(\tilde{\underline{g}}_{rs}^{P}) = (\pi''_{*}\mathbb{Q}(\tilde{\underline{g}}_{rs}))^{W(L)} .$$

Applying the functor $\underline{\underline{IC}}^{\bullet}$, this equation yields part a) of the proposition, using the argument which was used already at the beginning of the proof of 2.6. Next b) follows similarly by applying the functor $i^*(\underline{\underline{IC}}^{\bullet}(.))$ to (1), using 2.6a) for both cases, P a Borel subgroup, and P any parabolic subgroup.

Q.E.D.

2.8 Computation of cohomology of Steinberg varieties

COROLLARY: For all $u \in \mathcal{N}$ and $i \in \mathbb{N}$, we have

$$H^i(\mathcal{P}_u, \mathbb{Q}) \cong H^i(\mathcal{B}_u, \mathbb{Q})^{W(L)} .$$

Proof: This follows from 2.7b), by applying the functor $\underline{\underline{H}}^i(\ldots)_u$, since $\xi^{-1}u \cong \mathcal{P}_u$, while $\pi^{-1}u \cong \mathcal{B}_u$.

Remarks: This result was proved by Hotta and Shimomura for the special case $G = GL_n$, using a spectral sequence argument [HSh]. We shall reobtain this result (even on the sheaf level of 2.7b)) alternatively in §3.

2.9 Completing the proof of theorem 2.3

We note that corollary 2.8 gives immediately Lusztig's lemma (2.3), and hence completes the proof of theorem 2.3. In fact, if applied to the trivial case where $P = G$, then all $\mathcal{P}_u = \mathcal{P}$ reduce to a point, and 2.8 says that the W invariants of the cohomology ring $H^*(\mathcal{B}_u, \mathbb{Q})$ reduce to $\mathbb{Q}$ (in degree 0).

Remark: The reader only interested in 2.3, but not in the P-generalizations above, might as well directly prove that the W invariants of $\underline{\underline{A}}^\bullet = R\pi_*\underline{\underline{\mathbb{Q}}}(\tilde{\mathcal{N}})$ reduce to $\underline{\underline{\mathbb{Q}}}(\mathcal{N})$, using the arguments of 2.6, 2.7.

2.10 Stratification of $\tilde{\mathcal{N}}^P$

Let us turn to the study of our map η, as specified in 2.1. Recall that η maps $Z = \tilde{\mathcal{N}}$ onto $Y = \tilde{\mathcal{N}}^P$. We now specify a stratification of Y by strata $\mathcal{O}_y$ for η. Let us denote $\underline{p} = \underline{\ell} + \underline{n}$ the decomposition of $\underline{p} = \mathrm{Lie}P$ into its nilradical $\underline{n}$ = LieU , and the Levi-subalgebra $\underline{\ell}$ = LieL (cf. 2.7). Let $\mathcal{N}(L) = \bigcup_t \mathcal{O}_t$ denote the decomposition of $\mathcal{N}(L)$, the variety of nilpotent elements in $\underline{\ell}$, into its orbits $\mathcal{O}_t$. This is a stratification for the Springer-resolution $\pi(L)$ of $\mathcal{N}(L)$, with fibres $\pi(L)^{-1}t \cong \mathcal{B}(L)_t$, the variety of Borel-subgroups of L containing t in their Lie-algebra. To each stratum $\mathcal{O}_t$ of $\mathcal{N}(L)$ with base point t , we associate a unique stratum $\mathcal{O}_y$ of $\tilde{\mathcal{N}}^P$ with base point

(1) $\quad y = (t + u, P)$, where $u \in \underline{n}$ = LieU is fixed arbitrarily,

as follows: We use the identification $\tilde{\mathcal{N}}^P \subset \tilde{\underline{g}}^P = G \times^P \underline{p}$ (2.4), and put

$$\mathcal{O}_y = G \times^P (\mathcal{O}_t + \underline{n}) , \tag{2}$$

where we use the notation $A + B = \{a + b \mid a \in A, b \in B\}$ to define $\mathcal{O}_t + \underline{n}$. In particular, for $t = 0$, we write $y = y_0$; in this case

$$\mathcal{O}_{y_0} = G \times^P \underline{n} \cong T^*(\mathcal{P})$$

is the unique closed stratum of $\tilde{\mathcal{N}}^P$, and is isomorphic to the cotangent bundle of $\mathcal{P}$. In general, $\mathcal{O}_y$ is a double fibration with base $\mathcal{P}$, and fibres $\mathcal{O}_t$ resp. $\underline{n}$. To make this more precise, let us introduce the associated fibre bundles

$$\mathcal{V}_t^P = G \times^P \mathcal{O}_t ,$$

$$\mathcal{V}^P = G \times^P \mathcal{N}(L) = \bigcup_t G \times^P \mathcal{O}_t \ ,$$

using the P action on $\underline{\ell} \cong \underline{p}/\underline{n}$ induced by the adjoint action on $\underline{p}$. A point in $\mathcal{V}^P$ is given by a pair (P', n) , P' , a parabolic in $\mathcal{P}$ and n , a nilpotent element of $\mathrm{Lie}P'/\mathrm{Lie}P'^{\perp}$, where $\mathrm{Lie}P'^{\perp}$ is the nilradical of $\mathrm{Lie}P'$.

The Springer resolution $\tilde{\mathcal{N}}(L)$ of $\mathcal{N}(L)$ and the Springer resolution $\tilde{\mathcal{N}}$ of $\mathcal{N}$ fit into a diagram

$$\begin{tikzcd}
\tilde{\mathcal{N}}(L) \arrow[r, hook] \arrow[d, "\pi(L)"'] & \tilde{\mathcal{V}}^P \arrow[d, "\zeta"] & \mathcal{N} \arrow[l] \arrow[d, "\eta"] \arrow[dd, bend left, "\pi"] \\
\mathcal{N}(L) \arrow[r, hook, "i"] & \mathcal{V}^P & \tilde{\mathcal{N}}^P \arrow[l, "q"'] \arrow[d, "\xi"] \\
 & & \mathcal{N}
\end{tikzcd}$$

where i is the inclusion of a fibre of $r : \mathcal{V}^P \to \mathcal{P}$ and $\tilde{\mathcal{V}}^P$ is the variety of pairs (B', n) such that $B' \in \mathcal{B}$ and $n \in \mathrm{Lie}B'/\mathrm{Lie}P'^{\perp}$ where P' is the parabolic in $\mathcal{P}$ containing B'.

Lemma:

a) $\tilde{\mathcal{N}}^P$ is the total space of a double fibration

$$\tilde{\mathcal{N}}^P \xrightarrow{q} \mathcal{V}^P \xrightarrow{r} \mathcal{P},$$

where the fibre of r is $\mathcal{N}(L)$, and the fibre of q over a point $v \in \mathcal{V}^P$ is $T^*_{r(v)}(\mathcal{P}) \cong \underline{n}$.

b) The subvarieties $\mathcal{O}_y = q^{-1}(G \times^P \mathcal{O}_t)$ weakly stratify the map $\eta : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}^P$, with fibres $\eta^{-1}y \cong \mathcal{B}(L)_t$ of dimension $d_y = \dim\eta^{-1}y$, say.

c) Each stratum $\mathcal{O}_y$ is a double fibration

$$\mathcal{O}_y \xrightarrow{q^t} \mathcal{V}^P_t \xrightarrow{r^t} \mathcal{P},$$

where the fibre of q^t is $\underline{n}$, while the fibre of r^t is $\mathcal{O}_t$.

d) The codimension of $\mathcal{O}_y$ in $\tilde{\mathcal{N}}^P$ equals that of $\mathcal{O}_t$ in $\mathcal{N}(L)$.

e) The map η is semismall, with all strata $\mathcal{O}_y$ relevant for η.

f) The decomposition of $H^{2d_y}(\eta^{-1}y, \mathbb{Q})$ into $\pi_1(\mathcal{O}_y)$ isotypical components identifies with the decomposition of $H^{2d_y}(\mathcal{B}(L)_t, \mathbb{Q})$ into $\pi_1(\mathcal{O}_t)$ isotypical components.

Proof: a) The set of nilpotent elements in $\underline{p}$ is $\mathcal{N}(L) + \underline{n}$, which is isomorphic to the product of $\mathcal{N}(L)$ and $\underline{n}$ as an algebraic variety. Using the identification of $\tilde{\underline{g}}^P$ with $G \times^P \underline{p}$ as in lemma 2.4, we identify $\tilde{\mathcal{N}}^P$ as $G \times^P (\mathcal{N}(L) + n)$. Then it becomes obvious, how to define the two step fibration as described in a).

b) By the map $B' \mapsto UB'$, the Borel subgroups of L are in bijection with the Borel subgroups of G contained in P . Using (1) above, we conclude that

$$\eta^{-1}y \cong \{B \in \mathcal{B} | B \subset P,\ t+u \in \mathrm{Lie}B\} = \{UB' | B' \in \mathcal{B}(L),\ t \in \mathrm{Lie}B'\}$$
$$\cong \{B' \in \mathcal{B}(L) | t \in \mathrm{Lie}B'\} = \mathcal{B}(L)_t \ .$$

Next we note that our choice of a base point as in (1) is no restriction. Therefore, the fact that the $\mathcal{O}_y$ weakly stratify η follows from the fact that the $\mathcal{O}_t$ weakly stratify Springer's resolution of $\mathcal{N}(L)$.

c) is similar to a), statement d) is obvious from c), and statement e) follows from b) and d), using the corresponding properties of the Springer resolution of $\mathcal{N}(L)$.

f) The diagram of maps

$$\mathcal{O}_t \xhookrightarrow{i^t} \mathcal{V}_t^P \xrightarrow{q^t} \mathcal{O}_y$$

where i^t is the inclusion of a fibre of r^t, induces a surjection s

$$\begin{tikzcd} \pi_1(\mathcal{O}_y) \arrow[r] \arrow[rr, bend left, "s"] & \pi_1(\mathcal{V}^P_t) & \pi_1(\mathcal{O}_y) \arrow[l] \end{tikzcd}$$

by the long exact sequence in homotopy for the fibration r^t (since $\pi_1(\mathcal{P}) = 0$). But the local system $R^{2d_y}\zeta_*\underline{\underline{Q}}(\tilde{\mathcal{V}}^P)$ on $\mathcal{V}^P_t$ pulls back to $R^{2d_y}\eta_*\underline{\underline{Q}}(\tilde{\mathcal{N}})$ on $\mathcal{O}_y$, whose fibre is $H^{2d_y}(\eta^{-1}y, Q)$, and restricts to $R^{2d_y}\pi(L)_*\underline{\underline{Q}}(\mathcal{N}(L))$, whose fibre is $H^{2d_y}(\mathcal{B}(L)_t, Q)$. Therefore the kernel of s acts trivially on $H^{2d_y}(\mathcal{B}(L)_t, Q)$.

Q.E.D.

2.11 The topology of $\tilde{\mathcal{N}}^P$

PROPOSITION: a) $\tilde{\mathcal{N}}^P$ is homeomorphic to the fibre product $\mathcal{V}^P \times_{\mathcal{P}} T^*(\mathcal{P})$.

b) $\tilde{\mathcal{N}}^P$ is a rational homology manifold.

Proof: Part a) results from trivializing the fibration q over the fibres of r . This can be done since the fibre of r (i.e. $\mathcal{N}(L)$) is contractible. Part b) results from the fact that $\tilde{\mathcal{N}}^P$ is a fibre bundle where both the base and the fibre are rationally smooth. ($\mathcal{N}(L)$ is smooth by theorem 2.3.) Q.E.D.

2.12 Identification of the decomposition of $\underline{\underline{B}}^\bullet$

Now look at the decomposition formula for the map η as considered in 1.10 (2):

(1) $$\underline{\underline{B}}^\bullet = R\eta_*\underline{\underline{Q}}(\tilde{\mathcal{N}}) \cong \bigoplus_{(y,\psi)} Rj^y_* \,\underline{\underline{IC}}^\bullet(L_\psi)[-2d_y] \otimes V_{(y,\psi)} .$$

It is clear from our geometrical description of $Y = \tilde{\mathcal{N}}^P$ in the preceding lemma, that this decomposition is essentially identical with the corresponding formula for the Springer resolution $\pi(L) : \tilde{\mathcal{N}}(L) \to \mathcal{N}(L)$:

$$\underline{\underline{A}}^\bullet(L) = R\pi(L)_* \underline{\underline{Q}}(\tilde{\mathcal{N}}(L)) \cong \bigoplus_{(t,\psi)} Rj^t_* \, \underline{\underline{IC}}^\bullet(\overline{\mathcal{O}}_t, L_\psi)[-2d_t] \otimes V_{(t,\psi)}$$

The relevant pairs (y,ψ) for η are in bijection with the relevant pairs (t,ψ) for $\pi(L)$, the vector-spaces $V_{(y,\psi)}$ resp. $V_{(t,\psi)}$ in the decomposition formulae are identical, etc.

More precisely, consider the perverse sheaf $R\zeta_* \underline{\underline{Q}}\,(\tilde{\mathcal{V}}^P) = \underline{\underline{C}}^\bullet$ on $\mathcal{V}^P$ (see the diagram of §2.10), which is semisimple by theorem 1.7.

Proposition:

a) There are rational identifications

$$\underline{\underline{B}}^\bullet \cong q^* \underline{\underline{C}}^\bullet$$
$$\underline{\underline{A}}^\bullet(L) \cong i^* \underline{\underline{C}}^\bullet$$

b) The functor q^* (resp. i^*) takes the endomorphism ring of $\underline{\underline{C}}^\bullet$ isomorphically onto that of $\underline{\underline{B}}^\bullet$ (resp. $\underline{\underline{A}}^\bullet(L)$).

c) The resulting bijection of isotypical components of $\underline{\underline{B}}^\bullet$ with those of $A^\bullet(L)$ takes the (y,ψ) component to the (t,ψ) component corresponding by lemma 2.10 c) and f).

Proof: Statement a) holds because the two squares of the diagram in 2.10 are fibre squares and $\underline{\underline{Q}}(\tilde{\mathcal{V}}^P)$ pulls back to $\underline{\underline{Q}}(\tilde{\mathcal{N}})$ and restricts to $\underline{\underline{Q}}(\mathcal{N}(L))$.

The map q (resp. i) sends $\overline{\mathcal{O}}_t$ (resp. $\overline{\mathcal{O}}_y$) normally nonsingularly to $\mathcal{V}_t^P$ (see [FM], [GM2] §5.4). Therefore q^*(resp. i^*) takes sheaves $j_* \underline{\underline{IC}}^\bullet(\overline{\mathcal{V}}_t^P, L_\psi)$ to sheaves $j_*^y \underline{\underline{IC}}^\bullet(\overline{\mathcal{O}}_y, L_\psi)$ (resp. $j_*^t \underline{\underline{IC}}^\bullet(\overline{\mathcal{O}}_t, L_\psi)$). So part b) reduces to the statement that q^*(resp. i^*) preserves the irreducibility of L_ψ. This follows from lemma 2.10 f). Part c) is then clear.

$\underline{\underline{2.13}}$ Identification of the Weyl group action on $\underline{\underline{B}}^\bullet$

The map $\eta' : \tilde{\underline{g}} \to \tilde{\underline{g}}^P$ is a principal $W(L)$ fibration over the regular semisimple part of $\underline{g}$. Paralleling Lusztig's construction (§2.6), we obtain a $W(L)$ action on $\underline{\underline{B}}^\bullet$ by restriction of $R\eta'_* \underline{\underline{Q}}(\tilde{\underline{g}})$ to $\tilde{\mathcal{N}}^P$.

Proposition:

a) The Weyl group $W(L)$ acts on $\underline{\underline{B}}^\bullet = R\eta_* \underline{\underline{Q}}(Z)$ by automorphisms in the derived category $D^b(Y)$.

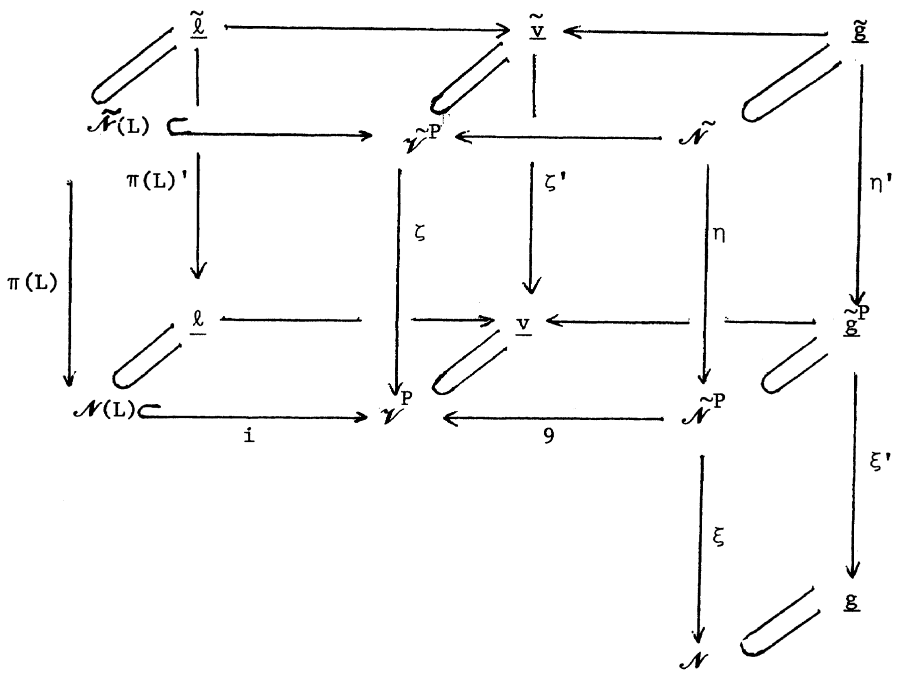
b) The action is given by linear representations $\rho_{(y,\psi)}$ of $W(L)$ on the $V_{(y,\psi)}$ of the formula 2.12 (1).

c) Here $(y,\psi) \mapsto \rho_{(y,\psi)}$ identifies with the Springer-correspondence for the group L (cf. 2.2.).

d) The action a) of $W(L)$ gives an isomorphism $\mathbb{C}[W(L)] \cong \mathrm{End}\ \underline{\underline{B}}^\bullet$.

e) The $W(L)$ action on $\underline{\underline{A}}^\bullet = R\xi_* \underline{\underline{B}}^\bullet$ inherited from $\underline{\underline{B}}^\bullet$ by functoriality of $R\xi_*$ coincides with the restriction of the W-action on $\underline{\underline{A}}^\bullet$.

Proof: We extend the diagram of §2.10 as follows:



where $\underline{v}$ and $\tilde{\underline{v}}$ are defined by dropping the restriction that n be nilpotent in the definitions of $\mathcal{V}^P$ and $\tilde{\mathcal{V}}^P$ respectively in §2.10. Then $\tilde{\underline{v}}$ and $\tilde{\underline{\ell}}$ are W(L) principal fibrations over the generic part of $\underline{v}$ and $\underline{\ell}$, so W(L) acts on $\underline{\underline{A}}^\bullet(L)$, $\underline{\underline{C}}^\bullet$ and $\underline{\underline{B}}^\bullet$ compatibly. Parts a)-d) follow from this. For part e), note that the action of W(L) on the part of $\tilde{\underline{g}}$ over the regular semisimple elements of $\underline{g}$ of this section agrees with that of §2.6, case $P = B$.

[W], #124, m. 3

§3 GROUP THEORETIC APPLICATIONS OF THE DOUBLE DECOMPOSITION FORMULA

3.1 Decomposition of the restriction of Springer's representation

Let us now evaluate what the general results of §1 give for the specific situation introduced in §2. Note that the objects $\underline{\underline{A}}^{\bullet}$ resp. $\underline{\underline{B}}^{\bullet}$, and the vector-spaces $V_{(x,\phi)}$ resp. $V_{(y,\psi)}$ in terms of which the results of §1 were formulated, are now--according to §2-- equipped with a W-resp. W(L)-action as an additional structure. Using this additional structure, the results of §1 may now be reformulated in sharper versions. For example, corollary 1.11 gives, in this reformulation, precisely the information of how Springer's irreducible W-module $V_{(x,\phi)}$ decomposes as a W(L)-module, if restricted to W(L) :

<u>Theorem</u>: (Springer) $V_{(x,\phi)} \cong \bigoplus_{(y,\psi)} V^{(y,\psi)}_{(x,\phi)} \otimes V_{(y,\psi)}$ as W(L)-modules.

<u>Proof</u>: This follows from 1.11 using 2.13 e). Q.E.D.

Alternatively, the theorem may be expressed by:

$$V^{(y,\psi)}_{(x,\phi)} \cong \mathrm{Hom}_{W(L)}(V_{(y,\psi)}, V_{(x,\phi)}) .$$

or also by the formula:

$$(V_{(x,\phi)})^{\rho(y,\psi)} = V^{(y,\psi)}_{(x,\phi)} \otimes V_{(y,\psi)} ,$$

where we denote for any module V of a finite group F, and for any irreducible representation ρ of F, by V^{ρ} the ρ-isotypical component of V.

Remark: This theorem agrees essentially with that of Springer in [S2], Thm. 4.4, which was proved there by completely different methods.

3.2 Spaltenstein's varieties $\mathscr{P}_x^0$, and Springer's partition of Steinberg's varieties $\mathscr{P}_x$

Let us make more explicit, how certain fixed point varieties in generalized flag varieties studied first by Spaltenstein [Sp] resp. Steinberg [St2] resp. Springer [S2] occur in our present situation. We have already noted (2.1) that the Steinberg-variety $\mathscr{P}_x$ of all conjugates $P' = gPg^{-1}$ of P by some $g \in G$ with $x \in \mathrm{Lie}P'$ identifies with the fibre $\xi^{-1}x$ of our partial resolution map ξ. Now for any L-orbit $\mathscr{O}_t$ in $\underline{\ell} \cong \underline{p}/\underline{n}$ (notation of §2.6, §2.7), we may consider the subvariety $\mathscr{P}_x^{(t)}$ of those $P' = gPg^{-1}$ such that $(\mathrm{ad}\, g)^{-1}x$ modulo $\underline{n}$ belongs to $\mathscr{O}_t$. The variety $\mathscr{P}_x^t$ is its closure. The variety $\mathscr{P}_x^{(t)}$ was introduced by Springer in [S2], 4.1; let us call it the Springer t-part of $\mathscr{P}_x$. Let us list a few obvious facts:

a) Each Steinberg variety $\mathscr{P}_x$ is a finite disjoint union of its Springer parts $\mathscr{P}_x^{(t)}$.

b) $\mathscr{P}_x^t = \mathscr{P}_x$ for t a regular nilpotent element of $\underline{\ell}$.

c) $\mathscr{P}_x^0 = \mathscr{P}_x^{(0)} = \{P' \in \mathscr{P} | x \in (\mathrm{Lie}P')^{\perp}\}$.

Since these latter varieties $\mathscr{P}_x^0$ have been studied by Spaltenstein in 1975 [Sp], we call them the Spaltenstein varieties. Note that these are the varieties which were denoted $\mathscr{P}_x$ in [BM], §7, in contrast to our present notation $\mathscr{P}_x^0$.

Now recall that each nilpotent L orbit $\mathscr{O}_t$ in $\underline{\ell}$ corresponds to a stratum $\mathcal{O}_y$ for the map $\eta : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}^P$ (2.10(2)), and that we introduced the notation $\xi_y = \xi j^y : \bar{\mathcal{O}}_y \to \mathcal{N}$ for the restriction of our partial resolution ξ to $\bar{\mathcal{O}}_y$ (see 1.10). With these notations, it is clear that we have:

d) $$\xi_y^{-1}x = \bar{\mathcal{O}}_y \cap \xi^{-1}x \cong \mathcal{P}_x^t .$$

<u>Convention:</u> Since the $\mathcal{O}_t$'s are in bijection with the $\mathcal{O}_y$'s, we shall also write $\mathcal{P}_x^y$ for $\mathcal{P}_x^t$ to simplify notations.

3.3 Computing the cohomology of certain $\mathcal{P}_x^y$

<u>Theorem:</u> Assume that $\bar{\mathcal{O}}_y$ is rationally smooth (1.4) at all points of $\xi_y^{-1}x \cong \mathcal{P}_x^y$ (or in other words: which map onto x). Then

$$H^{i-2d_y}(\mathcal{P}_x^y, \mathbb{Q}) \otimes H^{2d_y}(\mathcal{B}(L)_y, \mathbb{Q}) \cong H^i(\mathcal{B}_x, \mathbb{Q})^{\rho(y,1)}$$

(for all i) gives the $\rho_{(y,1)}$-isotypical component of the Springer representation on $H^i(\mathcal{B}_x, \mathbb{Q})$ restricted to $W(L)$.

<u>Proof:</u> This follows from 1.10 and 3.1. Q.E.D.

3.4 The cohomology of Spaltenstein's varieties in terms of anti-invariants

This theorem applies in particular in the cases where $\mathcal{O}_y$ is the smallest or the biggest stratum, that is where the element t in $\mathcal{N}(L)$ corresponding to y is $= 0$ resp. is regular: In the first case, $\bar{\mathcal{O}}_t \cong T^*\mathcal{P}$ is even smooth (2.10), and in the second case, $\bar{\mathcal{O}}_t = \tilde{\mathcal{N}}^P$ is rationally smooth by 2.11. Moreover, in both cases, $H^{2d_y}(\mathcal{B}(L)_y, \mathbb{Q})$ reduces to $\mathbb{Q}$, and $\rho_{(y,1)}$ is $\varepsilon_{W(L)}$ resp. $1_{W(L)}$. So we obtain:

Corollary: For all nilpotent elements $x \in \mathcal{N}$, and all i:

a) $H^i(\mathcal{P}_x, \mathbb{Q}) \cong H^i(\mathcal{B}_x, \mathbb{Q})^{W(L)}$

b) $H^{i-2d_0}(\mathcal{P}_x^0, \mathbb{Q}) \cong H^i(\mathcal{B}_x, \mathbb{Q})^{\varepsilon W(L))}$

In other words: The cohomology of the Steinberg-varieties is given by W(L)-invariants of the Springer representations (using the sign convention as in [BM]) while the cohomology of the Spaltenstein-varieties is given by the W(L)-anti-invariants (and a dimension-shift).

Comments: a) was proved also in 2.8. In the special case $G = GL_n$, a) was proved by Hotta-Shimomura [HSh] in a completely different way.

3.5 Counting components of $\mathcal{P}_x$ and of $\mathcal{P}_x^0$

If t is a regular nilpotent element of $\underline{\ell}$, then $\mathcal{P}_x^t = \mathcal{P}_x$ is the full Steinberg variety, and $\rho_{(t,1)} = 1_{W(L)}$ is the trivial representation of $W(L)$. If we take $t = 0$, then $\mathcal{P}_x^t$ is the Spaltenstein variety (3.2), and $\rho_{(t,1)} = \varepsilon_{W(L)}$ is the sign representation of $W(L)$. (In both cases, only $\psi = 1$ occurs.) Applying 3.4 to these particular cases, we obtain:

Corollary:

a) A Steinberg-variety $\mathcal{P}_x$ has dimension $\leq d_x$, with equality if and only if $\rho_{(x,1)}$ occurs in $1_{W(L)}^W$ with positive multiplicity. Moreover, the number of C(x)-orbits of d_x-dimensional components equals this multiplicity. This is also the number of irreducible components of $\xi^{-1}(\mathcal{O}_x)$ of dimension $d_x + \dim\mathcal{O}_x$.

b) A Spaltenstein-variety $\mathcal{P}_x^0$ has dimension $\leq d_x - d_0$ (where $d_0 = \dim\mathcal{B} - \dim\mathcal{P}$), with equality if and only if $\rho_{(x,1)}$ occurs in $\varepsilon_{W(L)}^W$ with positive multiplicity. Moreover, this multiplicity gives the number of $C(x)$-orbits of (d_x-d_0)-dimensional components of $\mathcal{P}_x^0$. This is also the number of irreducible components of $\xi_0^{-1}(\mathcal{O}_x)$ of dimension $d_x-d_0+\dim\mathcal{O}_x$.

Comments: a) is due to Springer [S2], Cor. 4.5, while b) seems to be new. In the special case where $G = GL_n$, a) goes back to Steinberg [St2], theorem 5.4, and b) is due to Spaltenstein ([Sp], final corollary), up to the combinatorial observation that the numbers given there coincide with the Kostka numbers [Md], p. 59 which in turn equal the multiplicities in our result b). (We have to thank A. Lascoux for help in verifying this coincidence.) These numbers can also be determined recursively by the formulae given in [BM], §7.

3.6 Are "special" orbits relevant for the moment map of $\mathcal{P}$?

Note that Corollary 3.5 gives a characterization of the strata $\mathcal{O}_x$ relevant for ξ resp. ξ_0. More generally, 3.4 gives:

PROPOSITION: *$\mathcal{O}_x$ is relevant for $\xi_y : \overline{\mathcal{O}}_y \to \mathcal{N}$ if and only if $\rho_{(x,1)}$ occurs in $\rho_{(y,1)}^W$.*

Recall that ξ_0 is the "moment map of $\mathcal{P}$" in the terminology of [BB], and is the map π_P in the notation of [BM], §7. For $G = GL_n$, it turns out that all strata (of the image of π_P) are relevant for π_P. (This goes back to Spaltenstein [Sp]). But in general, this does not hold. (Let us point out here that the statement on fibre dimensions in [BM], line -2 of p. 709 should read correctly "$\leq d_x-d_y$".) Conjecturally, the special nilpotent orbits in the sense of Lusztig [L2] are always relevant for π_P . In order to prove this conjecture using the above result, one has only to verify that $\rho_{(x,1)}$ occurs in $\varepsilon_W^W(L)$ whenever $\rho_{(x,1)}$ is a special representation. This has been done by Gisela Kempken [Ke], proposition 6.7 for G classical, and it could presumably be checked for G exceptional using the tables of Alvis and Lusztig [A], [AL].*

COROLLARY: Let G be classical. If an orbit $\mathcal{O}_x$ in the image of the moment map π_P of $\mathcal{P}$ is special, then $\mathcal{O}_x$ is relevant for π_P .

3.7 Associated parabolics

Two parabolic subgroups are called associated, if they have conjugate Levi subgroups.

Corollary: Let P and P' be associated parabolic subgroups of G , and let $\mathcal{P}$ resp. $\mathcal{P}'$ be the variety of conjugates of P resp. P' . Then we have

a) $H^i(\mathcal{P}_x, \mathbb{Q}) \cong H^i(\mathcal{P}'_x, \mathbb{Q})$, and b) $H^i(\mathcal{P}^0_x, \mathbb{Q}) \cong H^i(\mathcal{P}'^0_x, \mathbb{Q})$ for all nilpotent elements $x \in \mathcal{N}$, and all i .

In fact, this follows from 3.3, since the right hand sides in 3.3 a), b) depend only on the Levi subgroups, and not on the parabolics.

Remark: We note that for $i = 2d_x$, statement a) had been conjectured by Steinberg [St2], and proved by Springer [S2], Cor. 4.6 using different methods.

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*Added in proof: This has been done meanwhile by N. Spaltenstein (A property of special representations of Weyl groups; preprint, Warwick, March 1982), so the corollary holds for G exceptional as well.

3.8 Induced orbits

Given a nilpotent orbit $\mathcal{O}_t$ in $\underline{\ell}$, the orbit induced from $\mathcal{O}_t$ in $\underline{g}$ (in the sense of Lusztig-Spaltenstein [LS]) is denoted $\mathrm{Ind}_{\underline{\ell}}^{\underline{g}}(\mathcal{O}_t)$. This is the orbit $\mathcal{O}_x$ in $\underline{g}$ defined by

$$\overline{\mathcal{O}}_x = \overline{(\mathrm{Ad}\ G)(\mathcal{O}_t + \underline{n})} \qquad \text{(notation 2.10).}$$

Although this definition refers to a parabolic subalgebra with $\underline{\ell}$ as a Levi subalgebra, the result is actually independent of this choice, see [LS], [B2], or also [B1].

Proposition: An orbit $\mathcal{O}_t$ in $\mathcal{N}(L)$ determines a stratum $\mathcal{O}_y$ in $\tilde{\mathcal{N}}^P$ (2.10) and induces an orbit $\mathcal{O}_x$ in $\mathcal{N}$. Choose the base point x of $\mathcal{O}_x$ in $\mathcal{O}_t + \underline{n}$. With these notations:

a) ξ maps $\overline{\mathcal{O}}_y$ onto $\overline{\mathcal{O}}_x$.

b) $\dim\overline{\mathcal{O}}_y = \dim\mathcal{O}_t + 2\dim\underline{g}/\underline{p} = \dim\overline{\mathcal{O}}_x$ (or equivalently: $d_x = d_y$).

c) $\xi_y^{-1}(\mathcal{O}_x)$ is a single G-orbit, contained and dense in $\mathcal{O}_y$.

d) $\xi_y : \overline{\mathcal{O}}_y \to \overline{\mathcal{O}}_x$ is generically a covering of degree $[G_x : P_x] = \#\mathcal{P}_x^y$.

e) The multiplicity of $\rho_{(t,1)}$ in $H^i(\mathcal{B}_x, \mathbb{Q})$ is $[G_x : P_x] = \#\mathcal{P}_x^y$ for $i = 2d_x$, and is zero otherwise.

f) The multiplicity of $\rho_{(t,1)}$ in $\rho_{(x,1)}$ is one.

Proof: a) Since $\mathcal{O}_y = G \times^P (\mathcal{O}_t + \underline{n})$ by 2.10, and since ξ is the canonical (evaluation) map of $G \times^P \underline{p}$ into $\underline{g}$, we have $\xi(\mathcal{O}_y) = (\mathrm{Ad}\ G)(\mathcal{O}_t + \underline{n})$, whose closure is $\mathcal{O}_x$ by definition. Since ξ is proper, $\xi(\overline{\mathcal{O}}_y)$ is closed, hence $= \overline{\mathcal{O}}_x$.

b) The dimension of $\mathcal{O}_y$ was given in 2.10, while the dimension of an induced orbit was computed in [LS] (or more easily in [B2]).

c) Since ξ is G-equivariant, each G-orbit $\mathcal{O} \subset \overline{\mathcal{O}}_y$ mapping to $\mathcal{O}_x$ has dimension $\geq \dim\mathcal{O}_x$. Now b) implies $\dim\mathcal{O} = \dim\mathcal{O}_y$. Since $\overline{\mathcal{O}}_y$ is irreducible, we conclude that $\overline{\mathcal{O}} = \overline{\mathcal{O}}_y$, and that $\xi_y^{-1}(\mathcal{O}_x) = \mathcal{O}$ is a single G-orbit. Finally, $\mathcal{O} \subset \mathcal{O}_y$, since otherwise, $\xi_y(\overline{\mathcal{O}}_y \setminus \mathcal{O}_y)$ meets $\mathcal{O}_x$, so contains $\mathcal{O}_x$ by G-equivariance, and this is impossible for dimension reasons.

d) is clear from c) (cf. [BK], 7.8 (a)).

e) The fibre $\xi_y^{-1}x \cong \mathcal{P}_x^y$ is contained in $\mathcal{O}_y$. In particular, $\overline{\mathcal{O}}_y$ is smooth at all points which map to x . Hence Theorem 3.6 applies to give $\dim H^{i-2d_x}(\mathcal{P}_x^y, \mathbb{Q}) = \mathrm{mtp}(\rho_{(t,1)}, H^i(\mathcal{B}_x, \mathbb{Q}))$. But $\mathcal{P}_x^y$ is a set of $[G_x : P_x]$ points by d), and hence e) follows.

f) This follows from the fact that the component group $C(x) = G_x/G_x^0$ permutes the $[G_x : P_x]$ points in $\mathcal{P}_x^y$ simply transitively, so that $H^0(\mathcal{P}_x^y, \mathbb{Q})^{C(x)}$ is one dimensional. Since the $C(x)$ action on the cohomology of $\mathcal{P}_x^y$ comes from that on $\mathcal{O}_x$, we conclude from e) that $\rho_{(t,1)}$ occurs with multiplicity 1 in $V_{(x,1)} = H^{2d_x}(\mathcal{B}_x, \mathbb{Q})^{C(x)}$. Q.E.D.

<u>Remark:</u> By Frobenius reciprocity, f) says that $\rho_{(x,1)}$ has multiplicity one in the induced representation $\rho_{(t,1)}^W$. More precisely, it can be derived from f) that $\rho_{(x,1)}$ is obtained from $\rho_{(t,1)}$ by "truncated induction", that is to say

$$\rho_{(x,1)} = j_{W(L)}^{W}(\rho_{(t,1)})$$

in the notation of Lusztig's [L2]. This is a result of Lusztig and Spaltenstein, see [LS], theorem 3.5.

3.9 The degree of the moment map of $\mathscr{P}$

Considering the special case $t = 0$ in 3.8, recall that then $\xi_y : \overline{\mathscr{O}}_y \to \overline{\mathscr{O}}_x$ identifies with the "moment map" $\pi_{\mathscr{P}} : T^*(\mathscr{P}) \to \overline{\mathscr{O}}_x \subset \underline{g} = \underline{g}^*$ of the homogeneous space $\mathscr{P}$ (in the terminology of [BB]), which was called a "generalized Springer resolution" in [BM]. This map is in general not birational, but has a finite degree $\deg\pi_{\mathscr{P}} = [G_x : P_x]$. This number seems to play an important role in several different contexts, see e.g. [BK], theorem 7.2, and [BB], theorems 5.5, 5.6, 5.8. Let us therefore point out here the following expression for this number in terms of Springer representations.

<u>Corollary</u>: $\deg\pi_{\mathscr{P}} = \sum_{\phi} \mathrm{mtp}(\rho_{(x,\phi)}, \varepsilon_{W(L)}^{W})\deg\phi$.

<u>Proof</u>: More generally, proposition 3.10e) gives for $\mathscr{O}_t$ arbitrary:

$$\deg\xi_y = \sum_{\phi} \mathrm{mtp}(\rho_{(t,1)}, \rho_{(x,\phi)})\deg\phi$$

since we have $H^{2d_x}(\mathscr{B}_x, \mathbb{Q})$ is $\bigoplus_{\phi} \rho_{(x,\phi)} \otimes \phi$ by definition of $\rho_{(x,\phi)}$. Rewrite this using Frobenius reciprocity:

$$(*) \qquad \deg\xi_y = \sum_{\phi} \mathrm{mtp}(\rho_{(x,\phi)}, \rho_{(t,1)}^{W})\deg\phi \ .$$

The special case $t = 0$, where $\rho_{(t,1)} = \varepsilon_{W(L)}$, gives the corollary. Q.E.D.

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