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Euler products (variation on a theme of Kurokawa’s)

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EULER PRODUCTS (VARIATION ON A THEME OF KUROKAWA'S)

by

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1. Let $k$ be a finite extension of the field $\mathbb{Q}$ of rational numbers, and $K \supseteq k$ is a normal extension of $k$ of degree $d = [K : k]$ with Galois group $G(K/k)$, idèle-class group $C_K$ and Weil group $W(K/k)$. Thus we have an exact sequence

$$1 \longrightarrow C_K \longrightarrow W(K/k) \longrightarrow G(K/k) \longrightarrow 1,$$

and it follows that every irreducible representation of $W(K/k)$ is finite dimensional. Let $\mathbb{Z}$ be the ring of integers, and

$$X = \left\{ \sum_{i=1}^{\ell} m_i \chi_i \mid m_i, \ell \in \mathbb{Z}, \ell \geq 1, \chi_i \text{ is an irreducible character of } W(K/k) \text{ for any } i \right\}$$

is the ring of virtual characters of $W(K/k)$. For any polynomial

$$H(t) = 1 + \sum_{j=1}^{n} a_j t^j \in \mathbb{C}[t],$$

and $g \in W(K/k)$ we set $H_g(t) = 1 + \sum_{j=1}^{n} a_j(g) t^j \in \mathbb{C}[t]$, where $C$ is the complex number field. Let now $\sigma_p$ and $I_p$ be the Frobenius class and the inertia subgroup of $W(K/k)$ at the prime divisor $p$ of $k$ [1], and $\rho$ a finite dimensional representation of $W(K/k)$ with representation space $V$ and character $\chi = \text{tr } \rho$. Consider the subspace

$$V^p = \{ v \mid \rho(g) v = v \text{ for } g \in I_p, v \in V \}$$

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of $I_p$-invariant elements of $V$ and choose a representative $\tilde{\sigma}_p \in \sigma_p$ of the Frobenius class. Then the trace of the operator

$$\rho(\tilde{\sigma}_p) : V^p \rightarrow V^p$$

does not depend on the choice of $\tilde{\sigma}_p$ in $\sigma_p$; we set

$$\chi(\sigma_p) = \text{tr} \rho(\tilde{\sigma}_p) |_{V^p}$$

and extend this definition to $X$ by linearity. Thus we can define

$$H_p(t) = 1 + \sum_{j=1}^{n} a_j(\sigma_p) t^j,$$

and for $\text{Re } s > 1$ consider an Euler product

$$L(s, H) = \prod_p H_p (|p|^{-s})^{-1}, \quad (1)$$

where $p$ runs over prime divisors of $k$ and $|p| = N_{k/Q} p$. In particular, for $H(t) = \det (I - t \rho)$ we get $[ZJ]$ $L(s, H) = L_W(s, \rho)$, where $L_W(s, \rho)$ is the Weil $L$-function associated to a representation $\rho$ of $W(K/k)$.

**PROPOSITION 1.** - The function $s \mapsto L(s, H)$ defined by (1) can be meromorphically continued to the half-plane $C^+ = \{ s \mid \text{Re } s > 0 \}$.

**DEFINITION 1.** - Representation $\rho$ of $W(K/k)$ is said to be of Galois type, if $C_K \subseteq \text{Ker } \rho$. We denote by $X_o \subseteq X$ the subring of $X$ generated by the characters of representations of Galois type.

**DEFINITION 2.** - A polynomial $H \in X[t]$ is called unitary, if for any $g \in W(K/k)$ the condition $H(\alpha) = 0$ implies $|\alpha| = 1$, and non-unitary otherwise.

**PROPOSITION 2.** - If $H$ is unitary, the function $L(s, H)$ can be meromorphically continued to the whole complex plane $C$; if $H \in X_o[t]$ and is non-unitary, then $L(s, H)$ has $C^0$ as its natural boundary.

To state the next proposition we recall the Generalised Riemann Hypothesis (GRH): every $L$-function Hecke ("mit Grössencharakteren") has all its roots with $\text{Re } s > 0$ on the line $\text{Re } s = 1/2$. 

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DEFINITION. - For any positive $\varepsilon, c, x$ let $\xi(x, \varepsilon, c)$ denote the number of prime divisors $p$ in $k$ satisfying two conditions:

a) $N_{k/Q} p < x$, and

b) there exists $\mu_p$ such that $H_p(\mu_p) = 0$ and $|\log |\mu_p| - \log (1+c)| < \varepsilon$.

We call the polynomial $H$ strongly non-unitary, if one can find $c > 0$ such that for any $\varepsilon > 0$ there exists

$$\lim_{x \to \infty} \frac{\xi(x, \varepsilon, c)}{\pi(x)} = \alpha(\varepsilon, c) > 0;$$

where

$$\pi(x) = \sum_{N_k < x} 1.$$ 

PROPOSITION 3. - If the GRH holds and $H$ is strongly non-unitary, then $C^0$ is the natural boundary of $L(s, H)$.

2. - As an application of these results, let us mention the following problem discussed by several authors [3-10]. Consider $r$ finite extensions $k_1, \ldots, k_r$ of $k$ and the Galois hull $K$ of these fields over $k$, and fix a Hecke character $\chi_i$ in $k_i$. One can associate to $\chi_i$ an $L$-function

$$L(s, \chi_i) = \sum_{\mathfrak{a}} \chi_i(\mathfrak{a}) N_{k_i/k}^{-s} = \sum_n c_n(\chi_i) N_{k_i/k}^{-s},$$

where $\mathfrak{a}$ (accordingly $n$) runs over all the integral ideals of $k_i$ (accordingly $k$) and $c_n(\chi_i) = \sum \chi_i(\mathfrak{a})$. We define the scalar product of these $L$-functions as

$$a \text{ Dirichlet series}$$

$$L(s; \chi_1, \ldots, \chi_r) = \sum_n c_n(\chi_1) \ldots c(\chi_r) N_{k/Q}^{-s}$$

convergent for $\Re s > 1$. It turns out [6, 8, 10] that up to a finite number of Euler factors

$$L(s; \chi_1, \ldots, \chi_r) = L_W(s, \rho) L(s, H)^{-1}$$

for some representation $\rho$ of $\mathcal{W}(K/k)$ and a polynomial $H \in \mathbb{X}[t]$. It can be proved that $H$ is either unitary, or strongly non-unitary. Moreover, $H$ is unitary, if and only if either no more than one of the fields $k_i$ does not coincide with $k$.
or two of these fields are quadratic extensions of \( k \) and all the others coincide with \( k \); in this case the function (2) can be easily evaluated [9]. The propositions 1 - 3 show that the function (2) can be continued to \( C^+ \) and in most cases has a natural boundary \( C^0 \). We refer to the work of Kurokawa's [6-8] for further applications of the propositions 1 and 2.

3. - To outline the method of proof of propositions 1 - 3 let us consider the most simple case \( k = \mathbb{Q} = K \). The following proposition is, in fact, a classical result [11].

PROPOSITION 4. - Let \( h(t) = 1 + \sum_{j=1}^{n} a_j \prod_{i=1}^{n} (1 - \alpha_i t) \) and \( a_j \in \mathbb{Z} \). Then the function

\[
L(s, h) = \prod_p h(p^{-s})^{-1}
\]

defined by (3) for \( \text{Re } s > 1 \) can be meromorphically continued to \( C^+ \). If \( |\alpha_i| = 1 \) for any \( i \), then \( L(s, h) = \prod_{m=1}^{M} \zeta(m) \beta_m \) for some \( \beta_m \in \mathbb{Z} \) and, therefore, \( L(s, h) \) is meromorphic in \( C \); if \( |\alpha_i| \neq 1 \) for some \( i \), then \( C^0 \) is the natural boundary of \( L(s, h) \).

Proof. - Let us consider the ring \( \mathbb{C}[[t]] \) of formal power series and define by induction a sequence

\[
\{b_k \mid k = 1, 2, \ldots \} \subseteq \mathbb{Z}
\]

in such a way that

\[
h(t) = \prod_{k=1}^{\infty} (1 - t^k)^{b_k} \text{ in } \mathbb{C}[[t]].
\]

(4)

This sequence is uniquely determined; in fact,

\[
b_k = \frac{1}{k} \sum_{i=1}^{n} \mu(i) u(x),
\]

where \( u(x) = \sum_{i=1}^{n} \alpha_i x^i \), \( \mu \) is the Möbius function. In particular, it follows from (5) that

\[
|b_k| \leq n \left( \frac{\tau(k)}{k} \right)^{\gamma^k},
\]

where \( \tau(k) = \sum_{i|k} 1 \), \( \gamma = \max |\alpha_i| \). Therefore, the product (4) converges in \( C^0 \) in the disk \( |t| < 1/\gamma \). For any \( M, N > 1 \) we set
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\[ \psi_M(s) = \prod_{p \leq M} (1-p^{-s})^{-b_k} \]

For \( \text{Re } s \) large enough

\[ L(s,h) = \psi_M(s) \prod_{p \leq N} h(p^s)^{-1} \psi_M(\frac{1}{2}-is) \]

We use now (7) to continue \( L(s,h) \) to \( \mathbb{C}^+ \). The functions \( \psi_N \) and \( R_{N,M} \) are obviously meromorphic in \( \mathbb{C} \) and so is the function

\[ \psi_M(s) = \prod_{n \leq M} \zeta(n)^{b_n} . \]

We prove that if \( N > \gamma M \), then the product expansion for \( T_{N,M} \) converges absolutely for \( \text{Re } s > 1/M \). In fact, (6) implies

\[ \log T_{N,M}(s) \leq \sum_{p \leq N} \sum_{k > M} \frac{\tau(k)}{k} \gamma k \log(1-p^{-sk}) \leq n \log \zeta(\gamma) + \sum_{p \leq N} \sum_{k > M} \tau(k) \log(1-p^{-sk}) \leq n \log \zeta(\gamma) + \sum_{p \leq N} \sum_{k > M} \tau(k) \log(1-p^{-sk}) . \]

and the last series converges absolutely for \( \text{Re } s > 1/M \), \( N > \gamma M \). Taking \( M \to \infty \) we get the desired result.

If \( |a_i| = 1 \) for any \( i \), then \( \gamma = 1 \), and it follows from (6) that \( b_k = 0 \) as soon as \( n \tau(k) k^{-1} < 1 \); therefore, expansion (4) contains only a finite number of terms, so that \( L(s,h) \) is a product of a finite number of \( \zeta \)-functions, as it has been claimed. Assume that \( \gamma > 1 \). We prove that in this case any point in \( \mathbb{C}^0 \) is a limit point of poles of \( L(s,h) \) in \( \mathbb{C}^+ \). Suppose that \( |a_1| = \gamma \), and set \( a_1 = \gamma e^{i\phi} \). Consider the sequence

\[ \left\{ s_k(p) = \frac{\log \gamma + i(\phi + 2\pi k)}{\log p} \right\} \]

of roots of the functions \( s \mapsto h(p^{-s}) \) and count the number \( S(\nu, \delta) \) of \( s_k(p) \) in the region

\[ D_{\nu}(\delta) = \left\{ s \left| \frac{1}{\nu+1} < \text{Re } s < \frac{1}{\nu}, \ \text{t}_0 < \text{Im } s \leq t_0 + \delta \right\} \]

where \( \nu \) is a positive integer, \( \delta > 0 \) and \( t_0 > 0 \). If \( \frac{2\pi}{\log p} < \delta \) and

\[ \frac{1}{\nu+1} < \frac{\log \gamma}{\log p} < \frac{1}{\nu}, \]

then there exists \( k \) such that \( s_k(p) \in D_{\nu}(\delta) \). For
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\[ \delta > \frac{2\pi}{v \log \gamma} \]

we get \[ S(v, \delta) \geq \sum_{1 \leq \gamma < \delta} 1 = \pi(\gamma^{v+1}) - \pi(\gamma^v) - 1 \]

so that

\[ S(v, \delta) > A \gamma^v \]

for some \( A > 0 \) independent of \( v \). On the other hand, if \( N \gamma^{v+1} \)

and \( p < \gamma^{v+1} \), the number \( s_k(p) \) is a pole of \( \frac{\zeta(n)}{\log \gamma} \) for any \( k \).

Since \( s_k(p) \neq s_k(p') \) for \( p \neq p' \), we conclude that \( \frac{\zeta(n)}{\log \gamma} \) has at least \( A \gamma^v \) distinct poles in \( D_\gamma(\delta) \) as soon as \( N \gamma^{v+1} \)

and \( \delta \geq \frac{1}{\gamma} (2\pi / \log \gamma) \). Take \( M = \gamma \),

then \( R_N, M(s) \neq 0 \) and \( T_N, M(s) \neq 0 \) for \( s \in D_\gamma(\delta) \). Finally, the function

\[ \psi_M(s) = \prod_{n=1}^{M} \zeta(ns) \]

cannot have more than \( \sum_{n=1}^{M} N(n(t_0 + \delta)) = O(M^3) \) distinct zeros

in \( D_\gamma(\delta) \), where \( N(T) \) denotes the number of zeros of \( \zeta(s) \) in the region

\[ 0 < \text{Im } s \leq T \]

We see, therefore, that for large enough \( v \) and \( \delta \geq \frac{1}{\gamma} (2\pi / \log \gamma) \) the function \( L(s, h) \) has poles in \( D_\gamma(\delta) \). Thus any neighbourhood of a point \( t_0 \in \mathbb{C}_0 \)

contains a pole of \( L(s, h) \). This completes the proof of proposition 4.

We should mention another classical result [12] responsible for the ideas discussed here.

**Proposition 5.** - The function

\[ P(s) = \sum_{P} p^{-s} \]

defined for \( \text{Re } s > 1 \) can be continued to \( \mathbb{C}^+ \) and has \( \mathbb{C}_0 \) as its natural boundary.

**Proof.** - The standard expansion for \( \log \zeta(s) \) and Möbius inversion formula give

\[ P'(s) = \sum_{m=1}^{\infty} \mu(m) \frac{\zeta'(m \cdot s)}{\zeta(m \cdot s)} \]

so that \( P' \) is meromorphic in \( \mathbb{C}^+ \). Let \( v(s) \) denote the multiplicity of a zero \( s \) of \( \zeta(s) \); since \( N(T + 1) - N(T) = o(T) \), it follows that \( v(s) < A_1 \log |\text{Im } s| \)

for some \( A_1 \) independent on \( s \) (assuming \( |\text{Im } s| \geq 2 \)). Moreover, for any \( \delta > 0 \) and \( t_0 > 0 \) we have

\[ N(m(t_0 + \delta)) - N(m t_0) > 0 \]

as soon as \( m > A_2(t_0, \delta) \).

Keeping these facts in mind, consider a region

\[ D(\delta) = \{ s | 0 < \text{Re } s < \delta, \ t_0 < \text{Im } t < t_0 + \delta \} \]

and choose a rational prime \( q \) satisfying inequalities

\[ q > 1/\delta, \quad q > 2/t_0, \quad q > A_1 \log((t_0 + \delta)q), \quad q > A_2(t_0, \delta) \]

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Then one can find a root \( s_1 \) of \( \zeta(s) \) such that
\[
\frac{1}{2} \leq \text{Re} \ s_1 < 1, \quad q t_0 < \text{Im} \ s_1 \leq q(t_0 + \delta), \quad \forall(s_1) < q.
\] (9)

Obviously, \( s_1/q \in D_\nu(\delta) \). To prove that \( s_1/q \) is, in fact, a pole of \( P'(s) \) we notice that \( \zeta(m s_1/q) \neq 0 \), \( \forall m \geq 2q \), and, therefore, it is enough to show (see (8)) that
\[
\sum_{m=1}^{2q-1} \frac{\mu(m)}{m} v(m s_1/q) \neq 0.
\]

But (10) follows from (9) because
\[
\sum_{m=1}^{2q-1} \frac{\mu(m)}{m} v(m s_1/q) = \frac{v(s_1)}{q} + \frac{a}{b},
\]
where \( a/b = \sum_{m=1}^{2q} \frac{\mu(m)}{m} v(m s_1/q) \), so that \( q \nmid b \) whenever \( (a,b) = 1 \).

Thus the point \( t_0 \in D_\nu \) is a limit point of poles of \( P'(s) \), and the proposition follows.

For a generalisation of Propositions 4 and 5 we refer to a paper by G. Dahlquist [13].

4. - The proof of the results discussed in n°1 can be obtained along the same lines \([6-8, 10]\) with the help of the following lemma (whose proof we omit).

**Lemma.** Let \( H(t) \in \mathbb{X}[t] \), \( H(0) = 1 \) and \( H_i(t) = \prod_{i=1}^{n} (1 - \alpha_i(g) t) \) for \( g \in W(K/k) \); set \( \gamma = \sup \{ |\alpha_i(g)| | 1 \leq i \leq n, \ g \in W(K/k) \} \). Then

1) there exists a sequence of integers \( \{a_{m,j} \mid m, j = 1, 2, \ldots \} \) such that
\[
H(t) = \prod \det(I - t^m \Phi_j)^{a_{m,j}} \text{ in } \mathbb{X}[t],
\]
where \( \Phi_1, \Phi_2, \ldots \) are the irreducible representations of \( W(K/k) \);

2) dimension of \( \Phi_i \) does not exceed \( (K : k) = d \);

3) \( \sum_i a_{m,i} \text{tr}(\Phi_i(g)) \leq \frac{\tau(m)}{m} (d-1) m \) for any \( m \) and \( g \in W(K/k) \);

4) \( \sum_i a_{m,i}^2 \leq \gamma^2 \frac{\tau(m)}{m} (d-1)^2 \) for any \( m \).
5) the product

\[ H_p(t) = \prod_{m,i} (1 - t^m \Phi_i (\sigma))^{a_{m,i}} \]

converges absolutely in the disk \(|t| < \gamma^{-2}\).

REFERENCES


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