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Astérisque, tome 94 (1982), p. 143-151
<http://www.numdam.org/item?id=AST_1982__94__143_0>

EULER PRODUCTS (VARIATION ON A THEME OF KUROKAWA'S)

by

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1. - Let k be a finite extension of the field Q of rational numbers, and $K \supseteq k$ is a normal extension of k of degree $d = (K : k)$ with Galois group $G(K/k)$, idèle-class group C_K and Weil group $W(K/k)$. Thus we have an exact sequence

$$1 \longrightarrow C_K \longrightarrow W(K/k) \longrightarrow G(K/k) \longrightarrow 1 ,$$

and it follows that every irreducible representation of $W(K/k)$ is finite dimensional. Let Z be the ring of integers, and

$$X = \left\{ \sum_{i=1}^{\ell} m_i \chi_i \;\middle|\; m_i, \ell \in Z, \ \ell \geq 1, \ \chi_i \text{ is an irreducible character of } W(K/k) \text{ for any } i \right\}$$

is the ring of virtual characters of $W(K/k)$. For any polynomial

$$H(t) = 1 + \sum_{j=1}^{n} a_j t^j \in X[t]$$

and $g \in W(K/k)$ we set $H_g(t) = 1 + \sum_{j=1}^{n} a_j(g)\, t^j \in C[t]$, wher C is the complex number field. Let now σ_p and I_p be the Frobenius class and the inertia subgroup of $W(K/k)$ at the prime divisor p of k [1], and ρ a finite dimensional representation of $W(K/k)$ with representation space V and character $\chi = \operatorname{tr} \rho$. Consider the subspace

$$V^{I_p} = \{ v \mid \rho(g)\, v = v \ \text{ for } \ g \in I_p \, , \ v \in V \}$$

of I_p invariant elements of V and choose a representative $\tilde{\sigma}_p \in \sigma_p$ of the Frobenius class. Then the trace of the operator

$$\rho(\tilde{\sigma}_p) : V^{I_p} \longrightarrow V^{I_p}$$

does not depend on the choice of $\tilde{\sigma}_p$ in σ_p ; we set

$$\chi(\sigma_p) = \operatorname{tr} \rho(\tilde{\sigma}_p) |_{V^{I_p}}$$

and extend this definition to X by linearity. Thus we can define

$$H_p(t) = 1 + \sum_{j=1}^{n} a_j(\sigma_p)\, t^j ,$$

and for $\operatorname{Re} s > 1$ consider en Euler product

$$L(s,H) = \prod_p H_p(|p|^{-s})^{-1} , \tag{1}$$

where p runs over prime divisors of k and $|p| = N_{k/Q}\, p$. In particular, for $H(t) = \det(I - t\rho)$ we get [2] $L(s,H) = L_W(s,\rho)$, where $L_W(s,\rho)$ is the Weil L-function associated to a representation ρ of $W(K/k)$.

PROPOSITION 1. - <u>The function</u> $s \longmapsto L(s,H)$ <u>defined by</u> (1) <u>can be meromorphically continued to the half-plane</u> $C^+ = \{ s \mid \operatorname{Re} s > 0 \}$.

DEFINITION 1. - Representation ρ of $W(K/k)$ is said to be of Galois type, if $C_K \subseteq \operatorname{Ker} \rho$. We denote by $X_o \subset X$ the subring of X generated by the characters of representations of Galois type.

DEFINITION 2. - A polynomial $H \in X[t]$ is called unitary, if for any $g \in W(K/k)$ the condition $H_g(\alpha) = 0$ implies $|\alpha| = 1$, and non-unitary otherwise.

PROPOSITION 2. - <u>If</u> H <u>is unitary</u>, <u>the function</u> $L(s,H)$ <u>can be meromorphically continued to the whole complex plane</u> C ; if $H \in X_o[t]$ <u>and is non-unitary</u>, <u>then</u> $L(s,H)$ <u>has</u> C^o <u>as its natural boundary</u>.

To state the next proposition we recall the Generalised Riemann Hypothesis (GRH) : every L-function Hecke ("mit Grössencharakteren") has all its roots with $\operatorname{Re} s > 0$ on the line $\operatorname{Re} s = 1/2$.

DEFINITION. - For any positive ε, c, x let $\mathcal{L}(x, \varepsilon, c)$ denote the number of prime divisors p in k satisfying two conditions :

α) $N_{k/Q}\, p < x$, and

β) there exists μ_p such that $H_p(\mu_p) = 0$ and $\left|\log|\mu_p| - \log(1+c)\right| < \varepsilon$.

We call the polynomial H strongly non-unitary, if one can find $c > 0$ such that for any $\varepsilon > 0$ there exists

$$\lim_{x \to \infty} \frac{\mathcal{L}(x, \varepsilon, c)}{\pi(x)} = \alpha(\varepsilon, c) > 0 ;$$

where

$$\pi(x) = \sum_{N_{k/Q}\, p < x} 1 .$$

PROPOSITION 3. - <u>If the</u> GRH <u>holds and</u> H <u>is strongly non-unitary, then</u> C^o <u>is the natural boundary of</u> $L(s, H)$.

2. - As an application of these results, let us mention the following problem discussed by several authors [3-10]. Consider r finite extensions $k_1, \dots, k_r$ of k and the Galois hull K of these fields over k, and fix a Hecke character χ_i in k_i. One can associate to χ_i an L-function

$$L(s, \chi_i) = \sum_{\mathfrak{a}} \chi_i(\mathfrak{a})\, N_{k_i/k}\, \mathfrak{a}^{-s} = \sum_{\mathfrak{n}} c_{\mathfrak{n}}(\chi_i)\, N_{k/Q}\, \mathfrak{n}^{-s} ,$$

where $\mathfrak{a}$ (accordingly $\mathfrak{n}$) runs over all the integral ideals of k_i (accordingly k) and $c_{\mathfrak{n}}(\chi_i) = \sum_{N_{k_i/k}\mathfrak{a} = \mathfrak{n}} \chi_i(\mathfrak{a})$. We define the scalar product of these L-functions as a Dirichlet series

$$L(s ; \chi_1, \dots, \chi_r) = \sum_{\mathfrak{n}} c_{\mathfrak{n}}(\chi_1) \dots c(\chi_r)\, N_{k/Q}\, \mathfrak{n}^{-s} \qquad (2)$$

convergent for $\mathrm{Re}\, s > 1$. It turns out [6, 8, 10] that up to a finite number of Euler factors

$$L(s ; \chi_1, \dots, \chi_r) = L_W(s, \rho)\; L(s, H)^{-1}$$

for some representation ρ of $W(K\, k)$ and a polynomial $H \in X[t]$. It can be proved that H is either unitary, or strongly non-unitary. Moreover, H is unitary, if and only if either no more than one of the fields k_i does not coincide with k , or two of these fields are quadratic extensions of k and all the others coincide with k ; in this case the function (2) can be easily evaluated [9]. The propositions 1 - 3 show that the function (2) can be continued to C^+ and in most cases has a natural boundary C^o. We refer to the work of Kurokawa's [6 - 8] for further applications of the propositions 1 and 2.

3. - To outline the method of proof of propositions 1 - 3 let us consider the most simple case $k = Q = K$. The following proposition is, in fact, a classical result [11].

PROPOSITION 4. - Let $h(t) = 1 + \sum_{j=1}^{n} a_j t^j = \prod_{i=1}^{n} (1 - \alpha_i t)$ and $a_j \in \mathbb{Z}$. Then the function

$$L(s, h) = \prod_p h(p^{-s})^{-1} \tag{3}$$

defined by (3) for $\operatorname{Re} s > 1$ can be meromorphically continued to C^+. If $|\alpha_i| = 1$ for any i, then $L(s, h) = \prod_{m=1}^{M} \zeta(ms)^{\beta_m}$ for some $\beta_m \in \mathbb{Z}$ and, therefore, $L(s,h)$ is meromorphic in C ; if $|\alpha_i| \neq 1$ for some i, then C^o is the natural boundary of $L(s,h)$.

Proof. - Let us consider the ring $C[[t]]$ of formal power series and define by induction a sequence

$$\{b_k \mid k = 1, 2, \dots\} \subseteq \mathbb{Z}$$

in such a way that

$$h(t) = \prod_{k=1}^{\infty} (1 - t^k)^{b_k} \quad \text{in } C[[t]] . \tag{4}$$

This sequence is uniquely determined ; in fact,

$$b_k = \frac{1}{k} \sum_{\ell \mid k} \mu(\ell)\, u\left(\frac{k}{\ell}\right) , \tag{5}$$

where $u(x) = \sum_{i=1}^{n} \alpha_i^x$, μ is the Möbius function. In particular, it follows from (5) that

$$|b_k| \leq n \left(\frac{\tau(k)}{k}\right) \gamma^k , \tag{6}$$

where $\tau(k) = \sum_{\ell \mid k} 1$, $\gamma = \max_i |\alpha_j|$. Therefore, the product (4) converges in in the disk $|t| < 1/\gamma$. For any $M, N > 1$ we set

$$\mathfrak{U}_N(s) = \prod_{p<N} h(p^{-s})^{-1}, \qquad \Psi_M(s) = \prod_p \prod_{k\leq M} (1-p^{-sk})^{-b_k}$$

$$T_{N,M}(s) = \prod_{p\geq N} \prod_{k>M} (1-p^{-sk})^{-b_k}, \quad R_{N,M} = \prod_{p<N} \prod_{k\leq M} (1-p^{-sk})^{b_k} .$$

So that for Re s large enough

$$L(s,h) = \mathfrak{U}_N(s)\, \Psi_M(s)\, T_{N,M}(s)\, R_{N,M}(s) . \tag{7}$$

We use now (7) to continue $L(s,h)$ to C^+. The functions $\mathfrak{U}_N$ and $R_{N,M}$ are obviously meromorphic in C and so is the function

$$\Psi_M(s) = \prod_{n\leq M} \zeta(ns)^{b_n} .$$

We prove that if $N > \gamma^M$, then the product expansion for $T_{N,M}$ converges absolutely for Re $s > 1/M$. In fact, (6) implies

$$|\log T_{N,M}(s)| \leq \sum_{p\geq N} \sum_{k>M} \left(n \frac{\tau(k)}{k} \gamma^k\right) |\log(1-p^{-sk})| \leq$$

$$\leq n \sum_{p\geq N} \sum_{k>M} \sum_{m=1}^{\infty} \frac{\tau(k)}{km} \gamma^k p^{-km(\mathrm{Re}\, s)} \leq n \sum_{p\geq N} \sum_{k>M} \gamma^k (\tau(k))^2 k^{-1} p^{-k\,\mathrm{Re}\, s} ,$$

and the last series converges absolutely for Re $s > 1/M$, $N > \gamma^M$. Taking $M \to \infty$ we get the desired result.

If $|\alpha_i| = 1$ for any i, then $\gamma = 1$, and it follows from (6) that $b_k = 0$ as soon as $n\,\tau(k)\,k^{-1} < 1$; therefore, expansion (4) contains only a finite number of terms, so that $L(s,h)$ is a product of a finite number of ζ-functions, as it has been claimed. Assume that $\gamma > 1$. We prove that in this case any point in C^o is a limit point of poles of $L(s,h)$ in C^+. Suppose that $|\alpha_1| = \gamma$, and set $\alpha_1 = \gamma e^{i\Phi}$. Consider the sequence

$$\left\{ s_k(p) = \frac{\log \gamma + i(\Phi + 2\pi k)}{\log p} \;\middle|\; k \in Z \right\}$$

of roots of the functions $s \longmapsto h(p^{-s})$ and count the number $S(\nu, \delta)$ of $s_k(p)$ in the region

$$D_\nu(\delta) = \left\{ s \;\middle|\; \frac{1}{\nu+1} < \mathrm{Re}\, s < \frac{1}{\nu},\ t_o < \mathrm{Im}\, s \leq t_o + \delta \right\} ,$$

where ν is a positive integer, $\delta > 0$ and $t_o > 0$. If $\frac{2\pi}{\log p} < \delta$ and $\frac{1}{\nu+1} < \frac{\log \gamma}{\log p} < \frac{1}{\nu}$, then there exists k such that $s_k(p) \in D_\nu(\delta)$. For

$\delta > \frac{2\pi}{\nu \log \gamma}$ we get $S(\nu, \delta) \geq \sum_{\gamma^\nu < p < \gamma^{\nu+1}} 1 = \pi(\gamma^{\nu+1}) - \pi(\gamma^\nu) - 1$, so that $S(\nu, \delta) > A\gamma^\nu$ for some $A > 0$ independent of ν. On the other hand, if $N > \gamma^{\nu+1}$ and $p < \gamma^{\nu+1}$, the number $s_k(p)$ is a pole of $\mathfrak{U}_N(s)$ for any k. Since $s_k(p) \neq s_{k'}(p')$ for $p \neq p'$, we conclude that $\mathfrak{U}_N(s)$ has at least $A\gamma^\nu$ distinct poles in $D_\nu(\delta)$ as soon as $N > \gamma^{\nu+1}$ and $\delta > \frac{1}{\nu}(2\pi/\log\gamma)$. Take $M = \nu$, then $R_{N,M}(s) \neq 0$ and $T_{N,M}(s) \neq 0$ for s $D_\nu(\delta)$. Finally, the function $\psi_M(s) = \prod_{n=1}^{M} \zeta(ns)^{b_n}$ cannot have more than $\sum_{n=1}^{M} N(n(t_o + \delta)) = O(M^3)$ distinct zeros in $D_\nu(\delta)$, where $N(T)$ denotes the number of zeros of $\zeta(s)$ in the region $0 < \mathrm{Im}\, s \leq T$ We see, therefore, that for large enough ν and $\delta > \frac{1}{\nu}(\frac{2\pi}{\log\gamma})$ the function $L(s,h)$ has poles in $D_\nu(\delta)$. Thus any neighbourhood of a point $t_o \in C_o$ contains a pole of $L(s,h)$. This completes the proof of proposition 4.

We should mention another classical result [12] responsible for the ideas discussed here.

PROPOSITION 5. - The function

$$P(s) = \sum_p p^{-s}$$

defined for $\mathrm{Re}\, s > 1$ can be continued to C^+ and has C^o as its natural boundary.

Proof. - The standard expansion for $\log \zeta(s)$ and Möbius inversion formula give

$$P'(s) = \sum_{m=1}^{\infty} \mu(m) \frac{\zeta'}{\zeta}(ms), \tag{8}$$

so that P' is meromorphic in C^+. Let $v(s)$ denote the multiplicity of a zero s of $\zeta(s)$; since $N(T+1) - N(T) = 0(\log T)$, it follows that $v(s) < A_1 \log|\mathrm{Im}\, s|$ for some A_1 independent on s (assuming $|\mathrm{Im}\, s| \geq 2$). Moreover, for any $\delta > 0$ and $t_o > 0$ we have

$$N(m(t_o + \delta)) - N(m t_o) > 0 \text{ as soon as } m > A_2(t_o, \delta).$$

Keeping these facts in mind, consider a region

$$D(\delta) = \{ s \mid 0 < \mathrm{Re}\, s < \delta, \ t_o < \mathrm{Im}\, t < t_o + \delta \}$$

and choose a rational prime q satisfying inequalities

$$q > 1/\delta, \quad q > 2/t_o, \quad q > A_1 \log((t_o + \delta)q), \quad q > A_2(t_o, \delta).$$

Then one can find a root s_1 of $\zeta(s)$ such that

$$\frac{1}{2} \leq \operatorname{Re} s_1 < 1 \,, \quad qt_o < \operatorname{Im} s_1 \leq q(t_o + \delta) \,, \quad v(s_1) < q \quad . \qquad (9)$$

Obviously, $s_1/q \in D_\nu(\delta)$. To prove that s_1/q is, in fact, a pole of $P'(s)$ we notice that $\zeta(m\, s_1/q) \neq 0$, ∞ for $m \geq 2q$, and, therefore, it is enough to show (see (8)) that

$$\sum_{m=1}^{2q-1} \frac{\mu(m)}{m} v(\frac{m s_1}{q}) \neq 0 \,.$$

But (10) follows from (9) because

$$\sum_{m=1}^{2q-1} \frac{\mu(m)}{m} v(\frac{m s_1}{q}) = \frac{v(s_1)}{q} + \frac{a}{b} \,,$$

where $a/b = \sum_{\substack{m \neq q \\ m < 2q}} \frac{\mu(m)}{m} v(\frac{m\, s_1}{q})$, so that $q \nmid b$ whenever $(a, b) = 1$.

Thus the point $t_o \in D_\nu$ is a limit point of poles of $P'(s)$, and the proposition follows.

For a generalisation of Propositions 4 and 5 we refer to a paper by G. Dahlquist [13].

4. - The proof of the results discussed in n° 1 can be obtained along the same lines [6-8, 10] with the help of the following lemma (whose proof we omit).

LEMMA. - <u>Let</u> $H(t) \in X[t]$, $H(0) = 1$ <u>and</u> $H_g(t) = \prod_{i=1}^{n} (1 - \alpha_i(g)t)$ <u>for</u> g $W(K/k)$; <u>set</u> $\gamma = \sup \{ |\alpha_i(g)| \mid 1 \leq i \leq n \,,\ g \in W(K/k) \}$. <u>Then</u>

1) <u>there exists a sequence of integers</u> $\{a_{m,j} \mid m, j = 1, 2, \dots\}$ <u>such that</u>

$$H(t) = \prod_{m,j} \det(I - t^m \Phi_j)^{a_{m,j}} \quad \text{in} \quad [X[[t]] \,,$$

<u>where</u> $\Phi_1, \Phi_2, \dots$ <u>are the irreducible representations of</u> $W(K/k)$;

2) <u>dimension of</u> Φ_i <u>does not exceed</u> $(K : k) = d$;

3) $\left| \sum_i a_{m,i} \operatorname{tr}(\Phi_i(g)) \right| \leq \frac{\tau(m)}{m} (d-1) \gamma^m$ <u>for any</u> m <u>and</u> g $W(K/k)$;

4) $\sum_i a_{m,i}^2 \leq \gamma^{2m} (\frac{\tau(m)}{m} (d-1))^2$ <u>for any</u> m ;

5) the product

$$H_p(t) = \prod_{m,i} (I - t^m \Phi_i(\sigma_p))^{a_{m,i}}$$

converges absolutely in the disk $|t| < \gamma^{-2}$.

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