P. LOUSBERG

Backward parabolic equations


<http://www.numdam.org/item?id=AST_1981__89-90__213_0>
I. INTRODUCTION

This paper is devoted to the study of the singularities of the solutions of backward parabolic pseudo-differential equations.

Let $\mathbb{R}^n$ denote the n-dimensional euclidean space and write $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$. Let $\Omega'$ be an open subset of $\mathbb{R}^{n-1}$ and $S$ a positive constant. Suppose that the extendible distribution $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_N)$ of $D^*(\Omega'|0, SL)$ satisfies

$$
(1.1) \quad \mathcal{T} \cdot [(D_{x_n} + Q(x, D_{x'}))] \in C_0(\Omega' \times [0, SL])
$$

where $Q(x, D_{x'})$ is a first order properly supported $(N \times N)$ pseudo-differential operator in $\Omega'$ depending smoothly on $x_n \in [0, SL]$ and with principal symbol $Q_1(x, \xi')$ homogeneous of degree 1 in $\xi'$.

It follows that

$$
(1.2) \quad \mathcal{T} \cdot \Phi = \int_{\Omega'} \mathcal{T}_{x_n} \Phi \, dx_n
$$

with $\mathcal{T}_{x_n} \in C_0([0, SL] ; D^*(\Omega'))$.

We assume that the operator $D_{x_n} + Q$ is backward parabolic at $(x_n', \xi_n') \in T^*(\Omega') \setminus 0$, that is

$$
(1.2) \quad \text{all the eigenvalues of the matrix } Q_1(x_n', 0, -\xi_n') \text{ have positive real parts.}
$$
By extension, we say that the equation (I.1) is backward parabolic at \((x'_0, \xi'_0)\). The condition (I.2) still holds if \((x, \xi')\) belongs to a conic neighborhood \(\omega' \times [0, s] \times \gamma\) of \((x'_0, 0, -\xi'_0)\).

We examine the behaviour of the singularities of \(\tau\) near \((x'_0, \xi'_0)\). As is well known, [4], \(\tau\) is microlocally \(C_\infty\) if \(x_n > 0\); more precisely,
\[
WF \tau \cap [(\omega' \times ]0, s[) \times (-\gamma \times \mathbb{R})] = \emptyset.
\]

Moreover, all the traces of \(\tau\) are regular at \((x'_0, \xi'_0)\). This is the main result of the present paper which we prove in section III. We obtain it by constructing in section II a microlocal parametrix at \((x'_0, \xi'_0)\) for the Cauchy problem
\[
\begin{aligned}
D_{x_n} \tau + Q(x, D_x) \tau &= 0, \\
\tau |_{x_n=0} &= g(x').
\end{aligned}
\]

J. Polking has obtained in [2] other regularity theorems for parabolic operators, using \(L^2\) methods, (see also [3]).

II. CONSTRUCTION OF A MICROLOCAL PARAMETRIX

We first introduce an auxiliary space.

Let us set
\[
q(x, \xi', W) = dtm \left( Q_1(x, \xi') + iWQ_2(x, \xi') \right), \quad W \in \mathcal{C}.
\]

It follows from (I.2) that all the roots \(W\) of \(q\) have positive imaginary parts when \((x, \xi') \in \omega' \times [0, s] \times \gamma\). We denote by \(\phi_{x, \xi'}\), a closed curve containing these roots in its interior.

**Definition II.1.** The space \(\Sigma_m\) is the linear hull of the functions
where \( \mathbf{A}_k \) is a classical \((N \times N)\) symbol of order \( k \) in \( \omega' \times [0,s[ \) with support in \( \xi' \) contained in a closed subcone of \( \gamma \).

The essential property of this space is presented in the following theorem.

Theorem II.1.: If \( F \) is an element of \( \Sigma_m \), then the function

\[
A(x,\xi') = \int_{\phi, x, \xi'} \Phi(x,\xi',\omega) d\omega
\]

belongs to the space

\[
S^{m+1}_m = S^{p,\sigma}_{m+1}(\omega' \times [0,s[ \times \mathbb{R}^n) \cap S_n(\omega' \times [0,s[ \times \mathbb{R}^n)
\]

with \( p = (1,\ldots,1) \), \( \sigma = (0,\ldots,0,1) \), [1].

Proof: If \( K = K' \times \{ \varepsilon_0 , \varepsilon_1 \} \) is a compact subset of \( \omega' \times [0,s[ \) , we have, uniformly for \( x \in K \),

\[
|A(x,\xi')| \leq \begin{cases} 
C|\xi'|^{m+1} & \text{if } \varepsilon_0 = 0 , \\
\frac{CN}{|\xi'|^N} & \text{if } \varepsilon_0 > 0 . 
\end{cases}
\]

Let \( \gamma' \) denote a closed subcone of \( \gamma \) containing \([F(x,.,\omega)]\).

It clearly suffices to prove that

\[
\sup_{x \in K} \left| \int_{\phi, x, \xi'} \Phi(x,\xi',\omega) d\omega \right| \leq \begin{cases} 
C|\xi'|^{j-1N+1} & \text{if } \varepsilon_0 = 0 , \\
\frac{C_N}{|\xi'|^N} & \text{if } \varepsilon_0 > 0 . 
\end{cases}
\]

in \( \gamma' \).

Note that there exists a closed curve \( \phi \) enclosing the compact set

\[
\{ \omega : \exists (x,\xi') \in K \times \gamma' , |\xi'| = 1 : q(x,\xi',\omega) = 0 \}
\]

and contained in

\[
215
\]
Hence, for \((x, \xi') \in K \times y'\), we obtain

\[
\phi_{x, \xi'} \left[ \frac{ix_n W^j}{\left[ q(x, \xi', W) \right]^1} dW \right] = e^{ix \cdot W} \left[ \frac{ix_n W^j}{\left[ q(x, \xi', W) \right]} \right] dW = |\xi'| \left[ e^{ix \cdot W} \right]_{\xi'}^{1} \left[ q(x, \xi', W) \right]^{1} dW.
\]

The absolute value of this expression is bounded by

\[
C e^{-c} |\xi'|^{j-1N+1}.
\]

We then easily obtain (II.1).

It follows that the expression

\[
\sum_{p=0}^{\alpha_n} \left( e^{ix_n W} d^{\alpha_n - p}_{x_n} A(x, \xi') \right) = \sum_{p=0}^{\alpha_n} C_{p} \int e^{ix_n W} d^{\alpha_n - p}_{x_n} A(x, \xi') dW
\]

gives the required estimate since

\[
W^{p} d^{\alpha_n - p}_{x_n} d^{\beta'}_{\xi'} A(x, \xi') F \in \mathcal{E}_{m+p-|\beta'|} C \mathcal{E}_{m+p-|\beta'|}.
\]

Now, we shall construct a microlocal parametrix at \((x'_0, \xi'_0)\) for the Cauchy problem (I.3).

**Theorem II.2.** There exists a smooth family of \((N \times N)\) pseudo-differential operators \(P(x, D_x)\) and such that

\[
P(x, D_x) \psi = \iint e^{i(x'-y') \cdot \xi'} A(x, \xi') \psi (y') dy' d\xi'
\]

with

\[
x_n \in [0, s[ \), \(A \in \mathcal{S}_0\),
\]

and such that
(i) \((D x_n + Q)P\) is an integral operator with kernel in \(C_0(\omega' \times [0,\varepsilon] \times \omega')\),

(ii) \(P(x',0,D x_n)\) is elliptic at \((x'_0,-\xi'_0)\).

**Proof:** Let us define the amplitude by

\[
A(x,\xi') \sim \sum_{p,q=0}^{\infty} A_{pq}(x,\xi')
\]

where \(A_{pq} \in \mathcal{S}^{-(p+q)}\).

More precisely, we set

\[
A_{pq}(x,\xi') = \int_{\phi_{x,\xi'}} e^{\pi i x W} F_{pq}(x,\xi',W) dW
\]

with \(F_{pq} \in \mathcal{S}^{-1-(p+q)}\).

In particular, we take

\[
(II.2) \quad F_{pq} = (Q_1(x,\xi') + i W I_n)^{-1} F_q(x',\xi')
\]

with \(F_q \in \mathcal{S}^{-q}(\omega' \times \mathbb{R}^n)\).

Applying \(D x_n + Q\) to \(P\) yields, \([3]\),

\[
(D x_n + Q)P \psi = \iint e^{i(x'-y') \cdot \xi'} [D x_n A(x,\xi') + B(x,\xi')] \psi(y') dy' d\xi'
\]

where \(B(x,\xi')\) is a symbol of \(\mathcal{S}_1\) defined by the following asymptotic expansion

\[
B(x,\xi') \sim \sum_{\alpha} \frac{(-\alpha)^{-1}}{\alpha!} D^\alpha_{\xi'} Q(x,\xi') D^\alpha_{x_n} A(x,\xi')
\]

Writing for large \(\xi'\),

\[
Q = Q_1 + Q_0
\]

with \(Q_0 \in \mathcal{S}_0\), we obtain

\[
D x_n A + B \sim \sum_{k=0}^{\infty} T_{1-k} A
\]

where
\[
T_1(x, \xi', D_{x_n}) = D_{x_n} + Q_1,
\]
\[
T_0(x, \xi', D_{x}) = Q_0 + \sum_{|\alpha|=1} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha Q \ D_{x'}^\alpha,
\]
\[
T_{1-k}(x', \xi', D_{x'}) = \sum_{|\alpha|=k} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha Q \ D_{x'}^\alpha, \quad \text{if } k \geq 2,
\]
are differential operators which map \( \mathcal{G}_m \) into \( \mathcal{G}_{m+1-k} \).

Noting that
\[
A \sim \sum_{r=0}^{\infty} \left( \sum_{p+q=r} A_{pq} \right)
\]
we get
\[
D_{x_n} A + B \sim \sum_{r=0}^{\infty} \left( \sum_{k=0}^{r} \sum_{p+q=r-k} T_{1-k} A_{pq} \right).
\]

In order to realize condition (i), we annihilate each term of the asymptotic expansion of \( D_{x_n} A + B \). We obtain
\[
\sum_{q=0}^{r-1} \sum_{k=0}^{r-q} T_{1-k} A_{r-q-k,q} = 0, \quad \text{for } r \geq 1,
\]
if we remark that
\[
T_1 A_{\ast q} = \left( \int \phi_{x_n, \xi'} \ e^{-ix \ W} (iW_I + Q_1)(iW_N + Q_1)^{-1} dW \right) F_q = 0.
\]

The conditions (II.3) are satisfied if the functions \( F_{pq} \) are given by
\[
F_{pq} = -(iW_N + Q_1)^{-1} \sum_{k=1}^{p} T_{1-k} F_{p-k,q}, \quad p \geq 1, \ q \in \mathbb{N}.
\]

These relations determine \( F_{pq} \) from \( F_{0q} \).

Furthermore, we have
\[
P(x', 0, D_{x'}) \psi = \int\int e^{i(x'-y') \cdot \xi'} A(x', 0, \xi') \psi(y') dy' d\xi'.
\]
Here $A(x',0,\xi')$ is a classical symbol of order 0 having the following asymptotic expansion

$$A(x',0,\xi') \sim A_\infty(x',0,\xi') + \sum_{q=1}^\infty \left( \sum_{p=1}^\infty A_{p,q}(x',0,\xi') + A_{q,q}(x',0,\xi') \right)$$

The condition (ii) is satisfied if we take

$$A_{\infty}(x',0,\xi') = \alpha(x') \chi(\xi') I_N,$$

$$A_{pq}(x',0,\xi') = -\sum_{p=1}^\infty A_{p,q}(x',0,\xi'), \text{ for } q \geq 1,$$

where $\alpha \in D(\omega')$ is equal to 1 in a neighborhood of $x'^0$ and $\chi \in C_\infty(\mathbb{R}^n)$ is homogeneous of degree 0 for large $\xi'$, equal to 1 in a conic neighborhood of $-\xi'^0$ for $|\xi'| \geq \frac{1}{2}|\xi'^0|$ and with support contained in a closed subcone $\gamma'$ of $\gamma$. Noting that

$$A_{pq}(x',0,\xi') = \left( \int_{\phi_{x',0,\xi'}} (Q_1(x',0,\xi') + i\omega N)^{-1} d\omega \right) F_q(x',\xi') = 2\pi F_q(x',\xi')$$

we obtain

$$F_q = -\frac{1}{2\pi} \sum_{p=1}^\infty A_{p,q}(x',0,\xi'), \text{ for } q \geq 1.$$

The relations (II.2), (II.4), (II.5) determine the functions $F_{pq}$. It is easy to prove by induction that $F_{pq} \in \Sigma_{-1-(p+q)}$. Let us remark that the support in $(x',\xi')$ of $F_{pq}$ is contained in $[\alpha] \times \gamma'$; hence

$$[A(\cdot,x_n',\cdot)] \subset [\alpha] \times \gamma'.$$

Furthermore, if $x_n > 0$, $P(x,D_{x_n})$ is an integral operator with kernel

$$\in C_\infty(\omega' \times ]0,\infty[ \times \omega').$$
III. MAIN THEOREM

Lemma III.1.: If the distribution $\tau$ satisfies the equation (I.1), we have

(i) $\frac{D}{dx_n} \tau \cdot \phi_n - \tau \cdot Q(x, D_{x_n}) \phi_n + \int \frac{d}{dx'} \phi' dx' = 0$, if $x_n \in [0, SL]$, $\phi' \in D(\Omega')$

and where $\tau \in C_0^\infty (\Omega' \times [0, SL])$.

(ii) $\int_0^{+\infty} \tau \cdot (D_{x_n} + Q(x, D_{x_n})) \phi_n dx_n + \int \left[ \int_0^{+\infty} \tau \cdot \phi dx = -\tau_0 \cdot \phi(x', 0) \right]$, for every $\phi \in D(\Omega' \times [0, SL])$.

Proof: Integrating by parts, we obtain

(III.1) $\left[ \int_0^{+\infty} \tau \cdot (D_{x_n} + Q(x, D_{x_n})) \phi_n dx_n = \int_0^{+\infty} \left[ -D_{x_n} \tau \cdot \phi_n + \tau \cdot Q(x, D_{x_n}) \phi_n \right] dx_n + \tau_0 \cdot \phi(x', 0) \right]$

In particular, if we take

$\phi = \psi \phi'$, $\phi' \in D(\Omega')$, $\psi \in D([0, SL])$,

we obtain

$\int \psi dx_n \left[ \frac{d}{dx'} \phi' \right] = \int_0^{+\infty} \psi \left[ -D_{x_n} \tau \cdot \phi_n + \tau \cdot Q(x, D_{x_n}) \phi_n \right] dx_n$,

where $\tau \in C_0^\infty (\Omega' \times [0, SL])$.

Hence we deduce (i) and using (III.1), we get (ii).

Theorem III.1.: If the equation (I.1) is backward parabolic at $(x_0', \xi_0')$, all the traces of the distribution $\tau$ are regular at $(x_0', \xi_0')$.

Proof: Let us introduce in the relation (ii) of Lemma III.1 the function

$\alpha(x_n) P(x, D_{x_n}) \psi$

where $P$ is the microlocal parametrix constructed in Theorem II.2 and $\alpha$ is a function in $D([-s, s[)$ equal to 1 in a neighborhood of the origin.
We obtain
\[ T_0 \cdot (D_{x} \alpha) P(x, D_{x}, t_0) \psi + \int g \cdot \psi \, dx' = -T_0 \cdot P(x', 0, D_{x'}) \psi, \]
where \( g \in C_{\infty}(w') \).

Hence
\[ T_0 \cdot P(x', 0, D_{x'}) \in C_{\infty}. \]

Since \( P(x', 0, D_{x'}) \) is elliptic at \((x'_0, \epsilon_0)\), it follows that
\[ (x'_0, \epsilon_0) \notin WF T_0. \]

To complete the proof, it remains to note that
\[ WF T_0 = \bigcup_{k=0}^{\infty} WF D_{x_n}^{k} T_0 \bigg|_{x_n=0} \]
by relation (i) of Lemma III.1.

REFERENCES


Pierre LOUSBERG,
Institute of Mathematics,
University of Liège,
Avenue des Tilleuls 15,
B-4000 Liège,
Belgium.