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Young tableaux and P.I. Algebras


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INTRODUCTION

This is a report about the relations between the theory of algebras satisfying polynomial identities (P.I. algebras) and the representation theory of the symmetric groups ($S_n$-rep.). The relations between $S_n$-reps and the Procesi-Razmyslov theory of trace identities is not discussed here. A review of these results can be found in [3].

The sequence $\{c_n^A\}$ of a P.I. algebra $A$ was introduced in [15] in order to prove that $A \otimes B$ is P.I. It was later understood that these dimensions are the degrees of certain $S_n$-characters, called the cocharacters of the P.I. algebra [18]. These sequences enable one to apply $S_n$-reps to study many questions about P.I. algebras. For example, Amitsur's $s_\ell^k[x]$ theorem was known with an upper bound for $\ell$ but not for $k$. Applying $S_n$-characters we can re-prove it, together with such bounds for both $k$ and $\ell$. This as well as other applications are discussed here. One application of P.I. theory to $S_n$-characters is also described in §5.

We summarize here most of the results which are known to us and which are relevant to that relation between the two theories. Most of the results which are due to Amitsur are unpublished yet, although some of them can be found in [3]. Detailed proofs are avoided here, but we do give some proofs when they are both short and illuminating. For more results on T-ideals see [23], [24].
To simplify the presentation we assume here the characteristic of the base field to be zero.

§1. Some P.I. Algebras and Identities

Let $S_n$ be the symmetric group on $1, \ldots, n$. The following are two important non-commutative (associative) polynomials:

$$s_n[x_1, \ldots, x_n] = \sum_{\sigma \in S_n} (\text{sgn } \sigma)x_\sigma(1) \cdots x_\sigma(n)$$

is the $n$-th standard polynomial (of degree $n$), and

$$d_n[x_1, \ldots, x_n; y_1, \ldots, y_{n-1}] = \sum_{\sigma \in S_n} (\text{sgn } \sigma)x_\sigma(1)y_1x_\sigma(2)y_2 \cdots y_{n-1}x_\sigma(n)$$

is the $n$-th Capelli polynomial (of degree $2n-1$).

The definition of a P.I. algebra is well known. The most important P.I. algebra is $F_k$, the algebra of $k \times k$ matrices over $F$. Since the Capelli polynomial $D_n[x_1, \ldots, x_n; y_1, \ldots, y_{n-1}] = d_n[x;y]$ is alternating in $x_1, \ldots, x_n$ and is multilinear, a "determinant" type argument shows that $d_n[x;y]$ is an identity for any algebra $A$ of dimension $\dim A < n$. In particular, $F_k$ satisfies $d_{2n}[x;y]$. It is shown in [2] that $F_k$ does not satisfy $d_{k+1}[x;y]$. This completely answers which Capelli identities are satisfied by $F_k$: since $d_{n+1}[x;y]$ is a combination of $n$-th Capelli polynomials, if $A$ satisfies $d_n[x;y]$ then it also satisfies $d_m[x;y]$ for all $m \geq n$.

Next we note that $d_n[x_1, \ldots, x_n; 1, \ldots, 1] = s_n[x_1, \ldots, x_n]$, so if $A$ satisfies $d_n[x;y]$ then it satisfies also $s_n[x_1, \ldots, x_n]$ (the converse is discussed below). The Amitsur-Levitski theorem says that $F_k$ satisfies...
s_{2k}[x_1,\ldots,x_{2k}] \text{ (as a minimal identity). Roset [22] used the Grassmann algebra to give a very short proof of that theorem. Kemer [8], used the Grassmann algebra in a different way to prove that if an algebra satisfies } s_k[x] \text{ then it satisfies some } d_n[x;y], \ n = n(k) . \text{ Thus (if } \text{Char } F = 0) , \text{ an algebra satisfies a standard identity if and only if it satisfies a Capelli identity. It is interesting to mention that the Grassmann algebra was one of the earliest examples, given by Cohn, of an algebra that does not satisfy any standard identity.}

We close this section with a classical theorem of Amitsur [1], to be revisited in §6.

**Theorem 1.** If A satisfies an identity of degree d, then A satisfies

$$(s_{2k}[x_1,\ldots,x_{2k}])^k \text{ where } k \leq [d/2] .$$

§2. P.I. Algebras and S-representations

The identities I(A) of a P.I. algebra A are elements of F<x>, the free algebra in infinitely many variables {x}. Also, I(A) = Q is a two-sided ideal in F<x>, closed under substitutions (a T-ideal).

A basic P.I. result says that every identity can be multilinearized and, since Char F = 0, the multilinear identities determine the others. We therefore restrict our attention to multilinear identities. Those of degree n are completely determined by such identities in n fixed variables. We are led to the following construction:

Fix $x_1, x_2, \ldots \in \{x\}$ . For each n, let $V_n$ be the vector space of all
the multilinear polynomials in $x_1, \ldots, x_n$:

$$V_n = \left\{ \sum_{\sigma \in S_n} a_\sigma x_\sigma(l) \cdots x_\sigma(n) \mid a_\sigma \in F \right\}.$$  

Clearly,

$$\sum_{\sigma \in S_n} a_\sigma \sigma \leftrightarrow \sum_{\sigma \in S_n} a_\sigma x_\sigma(l) \cdots x_\sigma(n)$$

is an isomorphism between the group algebra $FS_n$ and $V_n$, as vector spaces over $F$.

If $Q = I(A)$ are the identities of $A$, then $Q_n = Q \cap V_n$ are the multilinear identities of degree $n$ in $x_1, \ldots, x_n$ and $\{Q_n\}_{n=1}^\infty$ determines $Q$ (char $F = 0$).

Since $FS_n$ is an algebra, the above isomorphism (*) induces an algebra structure on $V_n$. It is convenient to use (*) to identify $V_n$ with $FS_n$:

$$\sigma \equiv x_\sigma^\text{def} = x_\sigma(l) \cdots x_\sigma(n).$$

The algebra structure of $V_n$ is determined by rule

$$x_\sigma x_\eta \equiv \sigma \eta \equiv x_{\sigma \eta} \text{ for } \sigma, \eta \in S_n,$$

Elements of $FS_n$ are now realized as polynomials in $V_n$, and we proceed to describe the polynomials realizing the idempotents corresponding to some Young Tableau $T_\lambda$, [3], [16], [19]. Here $\lambda \in \text{Par}(n)$ is a partition of $n$, $D_\lambda$ the corresponding Young diagram and $T_\lambda$ a chosen Young Tableau.

**Example 1.** $\lambda = (n)$, $D_\lambda = \begin{array}{c} \cdots \end{array}$

$$T_\lambda = \begin{array}{c} 1 \ 2 \ \cdots \ n \end{array}.$$ The corresponding polynomial is

$$e_T(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_\sigma(l) \cdots x_\sigma(n).$$

This is a multilinearization of $x^n$, hence an identity for all rings satisfying $x^n = 0$. 

338
Example 2. \( \lambda = (1^n) \), \( T_\lambda = \begin{array}{c} 1 \\ n \end{array} \)

\[ e_T = \frac{1}{n!} \sum_{\sigma} (\text{sgn} \sigma) x_\sigma(1) \cdots x_\sigma(n) = \frac{1}{n!} s_n[x_1, \ldots, x_n]. \]

Example 3. \( \lambda \in \text{Par}(n) \) arbitrary with conjugate \( \lambda' = (b_1, b_2, \ldots) \).

Choose

\[
T_\lambda = \begin{array}{cccc}
1 & b_1 + 1 & & \\
2 & b_1 + 2 & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
b_1 & & & \\
\end{array}
\]

After evaluating \( e_T(x_1, \ldots, x_n) \), identify

\[ x_1 = x_{b_1 + 1} = x_{b_1 + b_2 + 1} = \cdots \]

\[ x_2 = x_{b_1 + 2} = x_{b_1 + b_2 + 2} = \cdots \]

\[ \vdots \]

\( e_T(x) \) becomes a scalar multiple of

\[ s_{b_1}[x_1, \ldots, x_{b_1}] \cdot s_{b_2}[x_1, \ldots, x_{b_2}] \cdots. \]

If \( D_\lambda \) is an \( \ell \times k \) rectangle, we obtain in this way the polynomial \((s_\lambda[x_1, \ldots, x_\ell])^k\).
Example 4. Start with an arbitrary tableau

\[
T_\lambda = \begin{array}{ccc}
  a_{11} & a_{12} & \cdots \\
  a_{21} & a_{22} & \cdots \\
  \vdots & \vdots & \ddots \\
\end{array}
\]

Distinguish the variables corresponding to the first column by \( z_1, \ldots, z_h \), then \( e_\lambda(x_1, \ldots, x_n) \) is a combination of Capelli polynomials

\[ q_0 q_h[z_1, \ldots, z_h; q_1, \ldots, q_{h-1}] q_h \]

where \( q_0, \ldots, q_h \) are polynomials in the other variables.

We now turn back to \( \{Q_n\}_1^\infty \). If \( f(x_1, \ldots, x_n) \in \mathcal{V}_n \) then

\[ sf(x_1, \ldots, x_n) = f(x_{0(1)}, \ldots, x_{0(n)}) \]

which implies that \( Q_n \) is a left ideal in \( \mathcal{V}_n \equiv \mathcal{F}_n \). It is almost never two-sided. Thus \( Q_n \) determines an \( S_n \)-representation so an \( S_n \)-character \( \chi(Q_n) \), which can be determined by complements:

\[ \mathcal{F}_n \] is semi-simple, \((\text{char } F = 0)\), so \( \mathcal{F}_n = Q_n \oplus J_n \) for some (not necessarily unique) complementary left ideal \( J_n \) which determines a unique \( S_n \)-character \( \chi(J_n) \), and \( \chi(Q_n) = \chi(\mathcal{F}_n) - \chi(J_n) \).

**Definition.** Let \( Q = I(A) \subseteq F<x> \) be the identities of the algebra \( A \), \( Q_n = \mathcal{V}_n \cap Q \) and \( \mathcal{V}_n \equiv \mathcal{F}_n = Q_n \oplus J_n \) as above, then \( \chi(J_n) \) is the \( n \)-th co-character of \( A \) (or of \( Q \)), denoted by \( \chi(J_n) = \chi_n(A) \). We call \( \{\chi_n(A)\}_{n=1}^\infty \) the cocharacter sequence (c.c.s.) of \( A \). Also \( c_n(A) = \deg \chi_n(A) = \dim \mathcal{V}_n/Q_n \) is the \( n \)-th codimension of \( A \), and \( \{c_n(A)\}_{1}^\infty \) is the sequence of codimensions (c.d.s.) of \( A \).
These sequences are tools for obtaining information about the identities of a P.I. algebra. Although \( Q_n^\omega \) determines \( Q \), its computation has so far been next to impossible. Since characters are much easier to handle than their representations, \( \{\chi_n(A)\}_1^\infty \) does look like the right invariant to begin with. Several examples will be discussed later.

§3. Codimensions

The sequence of codimensions \( \{c_n(A)\} \) is a significant invariant of \( A \), which is also useful in determining \( \{\rho_n(A)\} \). Codimensions were introduced to show that if \( A \) and \( B \) are P.I. then so is \( A \oplus B \), [15]. The main tool there is the exponential estimate \( c_n(A) \leq d^n \) [15, Th.4.7], the proof of which was considerably simplified by Latyshev (see [16]). We describe now his proof, which is further simplified by the Robinson-Schensted correspondence.

**Definition.** \( \sigma \in S_n \) is "d-bad" (\( d \leq n \)) if there exist \( 1 \leq i_1 < ... < i_d \leq n \) such that \( \sigma(i_1) > ... > \sigma(i_d) \). Otherwise \( \sigma \) is "d-good".

**Lemma.** If \( A \) satisfies an identity of degree \( d \), then \( V_n \) is spanned, modulo \( Q_n \), by the d-good permutations.

Latyshev then bounds the number of d-good permutations by a direct combinatorial argument, to conclude that

\[
c_n(A) \leq \text{number of d good permutations} \leq (d-1)^{2n}.
\]

A detailed proof of the above appears in [16, §1].

Using the Robinson-Schensted correspondence, we can actually count that number \( g_d(n) \) of d-good permutations.
Let $\lambda = (a_1, \ldots, a_r) \in \text{Par}(n)$ be a partition of $n$: $a_1 + \ldots + a_r = n$, $a_1 \geq \ldots \geq a_r > 0$. Clearly, $h(\lambda) = r$ is the height of the Young diagram $D_\lambda$. Let $\chi_\lambda$ be the corresponding $S_n$-irreducible character, $d_\lambda$ its degree (given by the hook formula) and $I_\lambda \subset \mathcal{F}_S n$ the corresponding minimal two-sided ideal: $\dim I_\lambda = d_\lambda^2$. Let $U$ be an $\ell$-dimensional vector space. Construct $U^{\otimes n} \overset{\text{def}}{=} \left(U \otimes \ldots \otimes U\right)_n$, map $\varphi: S_n \to \text{End}(U^{\otimes n})$ by $\varphi(\sigma) = \delta$, $\delta(u_1 \otimes \ldots \otimes u_n) = u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(n)}$ and extend $\varphi$ from $\mathcal{F}_S n$ onto the algebra $B(\ell, n)$ spanned by the $n!$ elements $\hat{\sigma}: \varphi: \mathcal{F}_S n + B(\ell, n) \subseteq \text{End}(U^{\otimes n})$.

A basic result in this construction of Schur is

**Theorem 2.** (H. Weyl): $B(\ell, n) = \sum_{\lambda \in \text{Par}(n) \atop h(\lambda) \leq \ell} d_\lambda^2 I_\lambda^{(2)}(n)$, and since $B(\ell, n) \subseteq \text{End}(U^{\otimes n})$, $S_\ell^{(2)}(n) \leq \ell^{2n}$.

The Robinson-Schensted correspondence, [9], maps each $\sigma \in S_n$ to a pair $(P, Q)$ of standard Young tableaux having the same shape. It is one-to-one, onto, and has the following property (among others): the height $h(P) = h(Q) = d$ is the length of a maximal chain $1 \leq i_1 < \ldots < i_d \leq n$ such that $\sigma(i_1) > \ldots > \sigma(i_d)$. In other words, $\sigma \leftrightarrow (P, Q)$ is "$d$-good" if and only if $h(P) \leq d-1$. It follows that $S_{d-1}^{(2)}(n)$ is the number of $d$-good permutations in $S_n$. By the above and by Latyshev's lemma we thus have

**Theorem 3.** If $A$ satisfies an identity of degree $d$ then $c_n(A) \leq \text{No. of } d \text{ good permutations} = S_{d-1}^{(2)}(n) \leq (d-1)^{2n}$, which proves the exponential bound for codimensions.
Kemer [7] characterized the algebras \( A \) such that \( c_n(A) = O(n^k) \) as those satisfying some very specific identities. This indicates that codimensions almost always have exponential rate of growth.

In the very few examples that have been done so far (to be discussed later), \( c_n(A) \) is exponentially smaller than \( (d-1)^{2n} \). However, unless the estimate "\( c_n(A) \leq \text{No. of d-good permutations} \)" is improved, one cannot significantly lower the bound \( c_n(A) \leq (d-1)^{2n} \): it is shown, [21], that as \( n \to \infty \),

\[
S_{\ell}^{(2)}(n) \sim_c \frac{1}{n} e^{\ell^2 n}
\]

where \( c \) is some (interesting) constant and \( e = \frac{1}{2}(\ell^2 - 1) \).

We note also that the constant appearing in the asymptotic formula for \( S_{\ell}^{(\beta)}(n) \) relates the theory of Young tableaux to a very interesting conjecture of I.G. Macdonald on the invariants of finite reflection groups (see [12], [21]).

§4. Cocharacters

Any \( S_n \) character \( \chi_n \) can be written as \( \chi_n = \sum_{\lambda \in \text{Par}(n)} m_{\lambda} \chi_{\lambda} \) where \( m_{\lambda} \) is the multiplicity of the irreducible character \( \chi_{\lambda} \) (since \( \text{char} F = 0 \)). This in particular applies to the cocharacter \( \chi_n(A) \), and we are looking for information about its \( m_{\lambda} \)'s.

Example. [10], [14]: The infinite dimensional Grassmann algebra \( E \) satisfies \( [[x_1,x_2],x_3] = 0 \) \( ([a,b] = ab-ba) \). The polynomial \( f(x) \in V_d \) is of type \( J_d \) if
\[ f(x_1, \ldots, x_d) = x_1 \cdots x_d + \sum_{\sigma(1) \neq 1} a_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(d)}. \]

By [10], if \( \Lambda \) satisfies a \( \mathcal{J}_d \)-identity, then \( c_n(\Lambda) \leq (d-1)^n \), so \( c_n(E) \leq 2^{n-1} \). The reverse inequality follows from studying the cocharacters \( \chi_n(E) \). The Partition \( \lambda = (k, 1^{n-k}) \in \text{Par}(n) \) defines a \( \Gamma \)-shaped Young diagram. For such \( \lambda \), it is easy to choose a polynomial in \( \mathcal{I}_\lambda \) which is not an \( E \)-identity. Hence \( \chi_n(E) = \sum_{\lambda \in \text{Par}(n)} m_\lambda \chi_\lambda \) and for each \( \lambda = (k, 1^{n-k}) \), \( 1 \leq k \leq m, \; m_\lambda > 1 \). Thus

\[ c_n(E) = \deg \chi_n(E) \geq \sum_{k=1}^{n} d \cdot (k, 1^{n-k}) = \sum_{k=1}^{n} (n-1)_k = 2^{n-1}. \]

It clearly follows that \( c_n(E) = 2^{n-1} \) and that \( \chi_n(E) = \sum_{k=1}^{n} \chi_{(k, 1^{n-k})} \).

The only other cocharacters which have been determined are those of \( Q = T(s_3[x_1, x_2, x_3]) \), the \( T \) ideal generated by \( s_3[ x ] \) , [6], [17]. Because of the importance of \( F_k \), the main goal in this direction should be to estimate \( c_n(F_k) \) and the multiplicities in \( \chi_n(F_k) \). So far only partial information had been obtained when \( k \geq 3 \). The results for \( \chi_n(F_2) \) are quite satisfactory and appear in [19]. We here summarize the main results:

**Theorem 4.** \( \Lambda \) satisfies the Capelli polynomial \( d_{k+1}[x;y] \) if and only if for all \( n \),

\[ \chi_n(\Lambda) = \sum_{\lambda \in \text{Par}(n), h(\lambda) \neq k} m_\lambda \chi_\lambda . \]

**Corollary.** \( \chi_n(F_k) = \sum_{\lambda \in \text{Par}(n), h(\lambda) \neq k^2} m_\lambda \chi_\lambda \). In particular, \( \chi_n(F_2) = \sum_{\lambda \in \text{Par}(n), h(\lambda) \neq 4} m_\lambda \chi_\lambda \).
Let now $X \in \text{Par}(n)$ with $h(X) \leq 4$ and write

$$
\lambda = (w_1^{\lambda_1}w_2^{\lambda_2}w_3^{\lambda_3}w_4^{\lambda_4}, w_2^{\lambda_5}w_3^{\lambda_6}w_4^{\lambda_7}, w_3^{\lambda_8}w_4^{\lambda_9}, w_4^{\lambda_{10}}).
$$

Let $m_\lambda$ be its multiplicity in $\chi_n(F_2)$. It is shown that $m_\lambda$ is very close to $w_1 \cdot w_2 \cdot w_3$. It then follows [19, Cor. 5.5] that $c_n(F_2)$ is asymptotically $(n \to \infty)$ sandwiched as follows:

$$
\frac{\sqrt{2}}{\pi} \left[ \frac{4}{\sqrt{\pi}} \frac{1}{n^{\frac{3}{2}}} 4^n \right] \leq c_n(F_2) \leq \frac{4}{\sqrt{\pi}} \frac{1}{n^{\frac{3}{2}}} 4^n
$$

There are indications, [19, Rem. 5.6] that the general results for $\chi_n(F_k)$ and $c_n(F_k)$ are similar.

$\S 5$. Applications of Codimensions and Cocharacters

These sequences can be defined for any algebra $A$, and $A$ is P.I. iff $c_n(A) \leq n!$ for some $n$. The theorem that $A \otimes B$ is P.I. if $A$ and $B$ are, clearly follows from the exponential bounds ($c_n(A) \leq a^n,...$) and from the inequality $c_n(A \otimes B) \leq c_n(A) \cdot c_n(B)$, because $n!$ exceeds any $(a \cdot b)^n$.

This codimension's inequality has an interesting cocharacter interpretation: Define $\sum _{\lambda \in \text{Par}(n)} m_\lambda \chi_\lambda \leq \sum _{\lambda \in \text{Par}(n)} m_\lambda^{'} \chi_\lambda$ if $m_\lambda \leq m_\lambda^{'}$ for all $\lambda$. Given two $S_n$-characters $\chi_n, \psi_n$, let $\chi_n \otimes \psi_n$ denote their Kronecker (inner) product. In [20] we proved

**Theorem 5.** $\chi_n(A \otimes B) \leq \chi_n(A) \otimes \chi_n(B)$ ($A \otimes B$ is the usual tensor product of the algebras $A$ and $B$.) Apply "degree" to both sides to deduce the previous codimensions inequality. Theorem 5 strongly relates P.I. algebras to Kronecker products of $S_n$-character. In what follows we bring applications in both directions.
Let $\chi_n = \sum_{\lambda \in \text{Par}(n)} m_{\lambda} \chi_{\lambda}$ and define $h(\chi_n) = \max\{h(\lambda) \mid m_{\lambda} \neq 0 \text{ in } \chi_n\}$.

By applying (Weyl’s) Theorem 2 one can prove (see [20])

**Theorem 6.** $h(\chi_n \otimes \psi_n) \leq h(\chi_n) \cdot h(\psi_n)$ for any two $S_n$-characters $\chi_n, \psi_n$.

We first apply $S_n$-characters to P.I. theory:

**Theorem 7.** If $A$ satisfies $d_k[x;y]$ and $B$ satisfies $d_1[x;y]$ then $A \circ B$ satisfies $d_{k+1}[x;y]$.

**Proof.** By Theorem 4, $h(\chi_n(A)) \leq k$, $h(\chi_n(B)) \leq \ell$, hence $h(\chi_n(A \circ B)) \leq h(\chi_n(A)) \cdot h(\chi_n(B)) \leq k \cdot \ell$, so by 4, $A \circ B$ satisfies $d_{k+1}[x;y]$.

By Kemer’s theorem (§1), existence of standard and Capelli identities is equivalent, so we obtain another proof of a result of E. Berman [4].

**Corollary.** (Berman): If $A$ and $B$ satisfies standard identities, then so does $A \circ B$.

Examine now the inequality $h(\chi \otimes \psi) \leq h(\chi)h(\psi)$ of Theorem 6. Tables in [13] show that in many cases, $h(\chi \otimes \psi) \leq h(\chi)h(\psi)$. The proper question should be

**Question H1:** Given two heights $h_1, h_2$ is there $N = N(h_1, h_2)$ such that for any $n \geq N$ there are two $S_n$-characters $\chi_n, \psi_n$ satisfying:

$h(\chi_n) = h_1$, $h(\psi_n) = h_2$ and $h(\chi_n \otimes \psi_n) = h_1h_2 = h(\chi_n) \cdot h(\psi_n)$ ?!

Note that for outer products $\chi \otimes' \psi$ we do have $h(\chi \otimes' \psi) = h(\chi) + h(\psi)$ as a consequence of the Littlewood-Richardson rule. Missing yet a rule for inner products, this we think, makes $H$ rather intriguing. We do conjecture "yes" to $H$! At the moment we can prove it when $h_1 = k^2$, $h_2 = k^2$ are.
squares, by applying P.I. theory. We sketch the proof: First, if \( n \geq 2k^2 - 1 \) then \( h(x_n(F_k)) = k^2 \). This follows from the fact that \( F_k \) satisfies \( d_{k+1} \), but not \( d_{k^2} \), by an argument of Amitsur's which is also applied to prove (half of) Theorem 4. Given \( h_1 = k^2 \), \( h_2 = \ell^2 \), let \( N = 2k^2\ell^2 - 1 \), so
\[
h(x_n(F_k)) = k^2\ell^2 \quad \text{if} \quad n \geq N : \text{there exists} \quad \lambda \in \text{Par}(n), \quad h(\lambda) = k^2\ell^2 \quad \text{and} \quad \chi_\lambda \quad \text{has a non-zero multiplicity in} \quad \chi_n(F_{k\ell}). \quad \text{Since} \quad \chi_n(F_k) = \chi_n(F_k \otimes F_{\ell}) \subseteq \chi_n(F_k) \otimes \chi_n(F_{\ell}), \quad \text{there must be} \quad \chi_{\lambda_1} \quad \text{in} \quad \chi_n(F_k), \quad \chi_{\lambda_2} \quad \text{in} \quad \chi_n(F_{\ell}), \quad \text{both with non-zero multiplicity, such that} \quad \chi_\lambda \quad \text{appears in} \quad \chi_{\lambda_1} \otimes \chi_{\lambda_2}. \quad \text{But} \quad h(\lambda_1) \leq k^2, \quad h(\lambda_2) \leq \ell^2 \quad \text{and} \quad h(\lambda) = k^2\ell^2, \quad \text{so necessarily} \quad h(\lambda_1) = k^2 \quad \text{and} \quad h(\lambda_2) = \ell^2, \quad \text{as was to be shown.}

§6. Explicit Identities

We begin with the following "Structure" argument: For each \( \ell \) assume
\[
f_{\ell}(x_1, \ldots, x_n(\ell)) \quad \text{is an identity for} \quad F_{\ell}. \quad \text{Let} \quad A \quad \text{be an arbitrary P.I. algebra and mode out its nil radical} \quad N. \quad \text{Since} \quad A/N \quad \text{is semi-simple, there exists} \quad \ell \quad \text{such that it satisfies all the identities of} \quad F_{\ell}, \quad \text{hence in particular}
\[
f_{\ell}(x_1, \ldots, x_n(\ell)) \quad \text{by using "generic elements" (or other methods) one can lift}
\[
f_{\ell}(x) \quad \text{back to} \quad A: \quad \text{there exists a power} \quad k \quad \text{such that} \quad A \quad \text{satisfies}
\[
(f_{\ell}(x_1, \ldots, x_n(\ell)))^k \quad \text{for example, assume} \quad A \quad \text{satisfies an identity of degree}
\[
d. \quad \text{Choose} \quad f_{\ell}(x) = s_{2\ell}[x_1, \ldots, x_{2\ell}] \quad \text{to conclude that} \quad A \quad \text{satisfies}
\[
s_{2\ell}[x]^k, \quad \ell \leq \frac{d}{2}; \quad \text{Choose} \quad f_{\ell}(x) = d_{\ell+1}[x;y] \quad \text{to conclude that} \quad A \quad \text{satisfies}
\[
d_{\ell+1}[x;y]^k, \quad \ell \leq d + 1.
\]

This method, due to Amitsur [1], usually yeilds a bound for \( \ell \) but not for \( k \). We now apply Young tableaux, to re-prove these results with explicit bounds for both indices \( k \) and \( \ell \).
It is well known that the minimal two-sided ideal $I_{\lambda} \subset \mathcal{P}_{n}$ ($\lambda \in \text{Par}(n)$) is a direct sum of $(d_{\lambda})$ minimal left ideals $J_{\lambda}$ and $(\dim J_{\lambda})^{2} = d_{\lambda}^{2} = \dim I_{\lambda}$.

As before, $Q = I(A)$, and our basic tool is

**Lemma 8.** If $d_{\lambda} < c_{n}(A)$ then $Q \nsubseteq I_{\lambda}$.

**Proof.** $I_{\lambda} = \emptyset J_{\lambda}$, $J_{\lambda}$ minimal left ideals. If some $J_{\lambda} \nsubseteq Q_{n}$ then $Q_{n} \cap J_{\lambda} = 0$, so $c_{n}(A) = \dim V_{n}/Q_{n} \geq \dim J_{\lambda} = d_{\lambda}$, a contradiction.

Q.E.D.

As in Theorem 3, let $c_{n}(A) \leq (d-1)^{2n}$. We are therefore looking for $n = n(d)$ and $\lambda \in \text{Par}(n)$ such that $d_{\lambda} > (d-1)^{2n}$; all the elements of $I_{\lambda}$ are then $A$-identities. If that $\lambda$ is "rectangular" $\lambda = (k^{d})$, we deduce from §2, Expl. 3 that $A$ satisfies $s_{k}^{n}[x]$.

Such $\lambda$ is found in [16] by analytic methods. Amitsur [3], gave a very short and simple method for finding such $\lambda$, which we now describe.

By the hook formula, $d_{\lambda} = n!/\pi \sum h_{ij}$ where $\{h_{ij}\}$ are the $n$ hook numbers. Replace $d_{\lambda} > (d-1)^{2n}$ by the equivalent inequality $(n!/\pi \sum h_{ij})^{1/n} > (d-1)^{2}$. It is well known that $(n!)^{1/n} > n/e$, and since the geometric mean is smaller than the arithmetic mean, $(\pi \sum h_{ij})^{1/n} \leq \frac{1}{n} \sum h_{ij}$. It is therefore enough to find $\lambda \in \text{Par}(n)$ such that $n^{2} / \sum h_{ij} \geq e(d-1)^{2}$.

Let $\lambda$ be "rectangular": $n = k \cdot \ell$, $\lambda = (k^{\ell})$. The hook numbers in $D_{\lambda}$ are

<table>
<thead>
<tr>
<th>$k+\ell-1$</th>
<th>$k+\ell-2$</th>
<th>$\vdots$</th>
<th>$k+1$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$k+1$</td>
<td>$\vdots$</td>
<td>$\ell+1$</td>
<td>$\ell$</td>
<td>$\ell$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\vdots$</td>
<td>$2$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

348
so $\sum h_{ij} = k \frac{k+\ell}{2} = n \frac{k+\ell}{2}$. If $\frac{k \ell}{k+\ell} > \frac{e}{2} (d-1)^2$ we then conclude that all the polynomials in $I_\lambda$, $\lambda = (k^\ell)$, are identities, hence also $s_\ell^k [x]$ (Expl. 3).

There are many such $k$ and $\ell$: Denote $a = \frac{e}{2} (d-1)^2$. Since $k, \ell > \frac{ek+\ell}{k+\ell}$, one must have $k, \ell > a$, so choose $\ell > a$. If $k > \frac{a \ell}{\ell-a}$ then clearly $\frac{k \ell}{k+\ell} > a$.

We summarize: Assume $A$ satisfies an identity of degree $d$, and let $a = \frac{e}{2} (d-1)^2$. If $\ell > a$ and $k > \frac{a \ell}{\ell-a}$ then $A$ satisfies the identity $(s_\ell^k [x_1, \ldots, x_\ell])^k$. Notice the gap: "Structure" yields the same result, with $\ell \leq d$ (but no bound for $k$).

Using the codimensions inequality $c_n (A \otimes B) \leq c_n (A) \cdot c_n (B)$ one can construct similar such identities for $A \otimes B$ by replacing $\frac{e}{2} (d-1)^2$ by $\frac{e}{2} (d_1-1)^2 (d_2-1)^2$.

Next, consider the analog of Amitsur's $s_\ell^k [x]$ theorem, but with Capelli instead of standard polynomials. The following construction is essentially due to Amitsur (unpublished) and is based on the branching theorem.

Construct the Capelli polynomial $d_\ell^k [x; y]$ in two steps: first write $s_\ell^k [x_1, \ldots, x_\ell] x_{\ell+1} \ldots x_{2\ell-1}$ and denote $x_{\ell+1} = y_1, \ldots, x_{2\ell-1} = y_{\ell-1}$. Now, there exists $\sigma \in S_{2\ell-1}$ such that $(s_\ell^k [x_1, \ldots, x_\ell] x_{\ell+1} \ldots x_{2\ell-1}) \sigma = d_\ell^k [x; y]$ , [3], [16]. The construction of a product of Capelli polynomials is done similarly.

Let $\lambda \in \text{Par}(n)$, $\mu \in \text{Par}(n+k)$ such that $D_\mu$ extends $D_\lambda$, i.e.: $D_\mu$ is obtained from $D_\lambda$ by adding $k$ boxes. A trivial consequence of the branching theorem implies that $d_\mu \succeq d_\lambda$.

Start with a P.I. algebra $A$, $Q = I(A)$, $c_n (A) \leq (d-1)^{2n}$. Next, find $n = k \ell$ such that $(\lambda = (k^\ell))$, $d_\lambda \succeq (d-1)^{4n}$. Let $\mu \in \text{Par}(2n)$ such that $D_\mu$ extends $D_\lambda$, then $d_\mu \succeq d_\lambda \succeq (d-1)^{4n} \succeq c_2 n (A)$, hence $I_\mu \subseteq Q_{2n}$. Since
A. REGEV

this is true for any such \( \mu \), the branching theorem implies that

\[
\text{FS}_{2n} \cdot I \cdot \text{FS}_{2n} \subseteq Q_{2n}.
\]

Choose

\[
\begin{array}{cccc}
1 & k+1 & \ldots & \\
\vdots & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \\
1 & 2k & \ldots & k^2
\end{array}
\]

(\( k = n \))

so, in \( \text{FS}_{2n} \),

\[
e_T(x_1, \ldots, x_{2n}) = \sum_{P \in \mathcal{P}} p \cdot s_k[x_1, \ldots, x_{k}] \cdot s_{k}[x_{k+1}, \ldots, x_{2k}] \ldots s_k[\ldots, x_{k^2}]x_{n+1} \ldots x_{2n}.
\]

Denote \( x_{n+1} = y_1, \ldots, x_{2n} = y_n \). There exists \( \sigma \in S_{2n} \) such that

\[
(s_k[x_1, \ldots, x_{k}] \ldots s_{k}[\ldots, x_{k^2}] \cdot y_1 \ldots y_n)^\sigma =
\]

\[
= d_k[x_1, \ldots, x_{k}; y_1, \ldots, y_{k-1}]y_1 \cdot d_k[x_1, \ldots, x_{k}; y_{k+1}, \ldots, y_{2k-1}] \cdot y_{2k} \ldots
\]

Now equate \( x_1 = x_{k+1} = x_{2k+1} = \ldots, x_2 = x_{k+2} = x_{2k+2} = \ldots, \)

\[
y_1 = y_{k+1} = y_{2k+1} = \ldots, y_2 = y_{k+2} = y_{2k+2} = \ldots, \)

\[
\text{and } y_k = y_{2k} = \ldots = y_{kk} = 1
\]

to conclude that \( \Lambda \) satisfies \( (d_k(x;y))^k \) (it is easy to deduce stronger results from these same arguments. See [3]). Recall that \( \lambda = (k^2) \in \text{Par}(n) \)

should only satisfy \( d_\lambda > (d-1)^{4n} \). As before, let \( \beta = \frac{e}{2} \cdot (d-1)^4 \) (instead of \( \alpha = \frac{e}{2} (d-1)^2 \)), then choose \( k > \beta \) and \( k > \frac{\beta^2}{k-\beta} \) to obtain explicit \( k \) and \( k \).

350
REFERENCES


17. ———— The T-ideal generated by the standard identity $s_3[x_1,x_2,x_3]$. Israel J. Math. 26 (1977), 105-125.


