Astérisque

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Astérisque, tome 87-88 (1981), p. 221-247
<http://www.numdam.org/item?id=AST_1981__87-88__221_0>

WEIERSTRASS POINTS ON CURVES

Dan Laksov

Introduction.

In the following article we give a definition of Weierstrass points of complete linear systems on non-singular curves which has three main features. Firstly, it is a natural definition in the sense that it interprets Weierstrass points in terms of rank conditions on maps between vector bundles on the curve and brings out the often overlooked connection between Weierstrass points and properties of associated curves. Secondly, the definition takes into account the multiplicities of the Weierstrass points. To associate multiplicities to the Weierstrass points we construct a global wronskian determinant associated to the linear system. This construction is the central part in the global study of Weierstrass points and leads to a formula for the total weight of the Weierstrass points and to a generalization of the Brill-Segre formula for $(r+1)$-tuple points of the linear system. The existence of a formula for the total weight of the Weierstrass points is rather surprising as it is well known that there are curves of arbitrary high genus with only one Weierstrass point (see e.g. Example 3, § 6). Thirdly, our treatment is independent of the characteristic and shows how the classical formulas carry over to the case of non-classical gap sequences. The most surprising part of this article is that there exists a formula, mentioned above, for the total weight of the Weierstrass points whereas there is no natural generalization of the classical formula for the local weight (Remark § 6).

When the characteristic of the ground field is zero our definition gives the traditional Weierstrass points with their multiplicities. However, even in this case our point of view that puts the Weierstrass points and the associates curves on an equal footing, contributes to the understanding of the geometry of the curve. We shall not, however, exploit this connection below. In arbitrary characteristic Weierstrass points have been defined previously by F.K. Schmidt [10] for the canonical linear system and by K.R. Mount and O.E. Villamayor [7] for arbitrary linear systems and also for higher dimensional varieties. Our definition specializes to give the same point set and the same gap sequences as those given by Schmidt. He was however mainly interested in the Weierstrass points as a point set and in spite of using a wronskian determinant his treatment is completely different from ours. On the other hand, although the Weierstrass points as we define them differ from those of Mount and Villamayor even as point sets, our treatment is in spirit much closer to their approach. They try to interpret the Weierstrass points as singularities of mappings of bundles and it was the attempt to understand the difference of their point sets with those of Schmidt that was the starting point of the work presented below.

Section by section the contents of this article is as follows:

§ 1. We recall the main properties of the bundle of principal parts and define the wronskian bundles.

§ 2. Weierstrass points are defined and interpreted in terms of the rank of the wronskian bundles.

§ 3. The wronskian of a linear system is constructed and the global enumerative formulas and the Brill-Segre formula for $(r+1)$-tuple points of the linear system is discussed.

§ 4. We give relations between the local invariants of the linear system and express the maps of principal parts locally.

§ 5. When the characteristic is zero or greater than the degree of the linear system we show that we obtain the classical formulas for Weierstrass points.

§ 6. We present three examples that illustrate the pathologies discussed in the article.

I would like to thank S.L. Kleiman for a very careful reading of the manuscript and for his valuable comments.

§ 1. The bundle of principal parts.

Let C be a non-singular curve of genus g and D a positive divisor of degree d and (projective) dimension r.

Denote by I the ideal defining the diagonal in $C \times C$ and by $C(m)$ the subscheme of $C \times C$ defined by I^{m+1}. We have

$$(\Omega_C^1)^{\otimes m} = I^m/I^{m+1} .$$

(See e.g. [2] (17.12.4). In [2] (16.3.1) this relation for $m = 1$ is used to define the Kähler differentials.) Denote by p and q the projections of $C \times C$ onto the first and second factor. The exact sequence

$$0 \to I^{m+1} \to \mathcal{O}_{C\times C} \to \mathcal{O}_{C(m)} \to 0$$

tensored by $q^*\mathcal{O}(D)$ gives, after passing to cohomology, a long exact sequence

$$(1) \quad 0 \to p_*(I^{m+1} \otimes q^*\mathcal{O}(D)) \to p_*q^*\mathcal{O}(D) \to p_*(q^*\mathcal{O}(D)|C(m)) \to$$

$$\to R^1p_*(I^{m+1} \otimes q^*\mathcal{O}(D)) \to R^1p_*q^*\mathcal{O}(D) \to 0 \ .$$

Here we have zero to the right because $p|C(m)$ is affine.

The bundle $P^m(D)$ of m'th order principal parts of D is defined by

$$P^m(D) = p_*(q^*\mathcal{O}(D)|C(m)) \ .$$

Via p the principal parts have a natural structure as $\mathcal{O}_C$-modules and $P^o(D) = \mathcal{O}(D)$.

From the exact sequence

$$0 \to I^m/I^{m+1} \to \mathcal{O}_{C(m+1)} \to \mathcal{O}_{C(m)} \to 0$$

we obtain an exact sequence

$$(2) \quad 0 \to (\Omega_C^1)^{\otimes m} \otimes \mathcal{O}(D) \to P^m(D) \to P^{m-1}(D) \to 0 \ .$$

We see that $P^m(D)$ is a locally free $\mathcal{O}_C$-module of rank $(m + 1)$.

By flat base change we have $R^ip_*q^*\mathcal{O}(D) = H^i(C, D) \otimes \mathcal{O}_C(D)$. Let

$$v^m : H^o(C, D)_C \to P^m(D)$$

be the map defined by the sequence (1) and let $B^m(D)$ and $V^m(D)$ be the image and cokernel of v^m. Since $P^m(D)$ is locally free we have that $B^m(D)$ is also a locally free $\mathcal{O}_C$-module.

From the map $P^m(D) \to P^{m-1}(D)$ of sequence (2) we obtain a natural commutative diagram,

$$\text{(3)}\qquad \begin{array}{ccccccccc} 0 & \to & B^m(D) & \to & P^m(D) & \to & V^m(D) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^{m-1}(D) & \to & P^{m-1}(D) & \to & V^{m-1}(D) & \to & 0 \end{array}$$

with surjective vertical maps. Moreover, we obtain from the sequence (1) an exact sequence

$$\text{(4)}\qquad 0 \to V^m(D) \to R^1p_*(I^{m+1} \otimes q^*\mathcal{O}(D)) \to H^1(C, D)_C \to 0 .$$

By the principle of exchange we have for all points $x \in C$ an isomorphism $R^1p_*(I^{m+1} \otimes q^*\mathcal{O}(D))(x) = H^1(C, D - (m + 1)x)$. Consequently we obtain from the sequence (4) a natural commutative diagram of vector spaces

$$\text{(5)}\qquad \begin{array}{ccccccccc} 0 & \to & V^m(x) & \to & H^1(C, D - (m+1)x) & \to & H^1(C, D) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V^{m-1}(x) & \to & H^1(C, D - mx) & \to & H^1(C, D) & \to & 0 . \end{array}$$

Here the horizontal sequences are exact because $H^1(C, D)_C$ is free.

Proposition 1. There are integers $0 = b_o < b_1 < \ldots < b_r \leqq d < b_{r+1} = \infty$ such that rank $B^j(D) = (m+1)$ for $b_m \leqq j < b_{m+1}$ and $m = 0, \ldots, r$.

Proof. It follows from the Riemann-Roch theorem that $h^1(C, D - (m + 1)x) = g - d + m$ when $m \geqq d$. Consequently, when $m \geqq d$, we see from the sequence (4) that $V^m(D)$ is locally free of rank $g - d + m - h^1(C, D) = m - r$. Hence rank $B^m(D) = (m + 1) - (m - r) = r + 1$ when $m \geqq d$. The bundle $B^o(D)$ is of rank 1. Indeed it is a subbundle of $P^o(D) = \mathcal{O}(D)$ and is nontrivial because D is positive. It follows that the rank of $B^m(D)$ lies between 1 and $(r + 1)$. Moreover we see from diagram (3) that (rank $B^m(D)$ - rank $B^{m-1}(D)) \leqq 1$. Hence, in the chain $B^o(D) \subseteq B^1(D) \subseteq \dots \subseteq B^d(D)$ of $\mathcal{O}_C$-modules, there are exactly r jumps in the ranks, each jump increasing the rank by 1. The integers $b_1, \dots, b_r$ are the indices where the jumps appear.

Definition. For each m = 0, 1, ..., r we denote the bundle $V^{b_m}(D)$ by $W^m(D)$ and call it the m'th Weierstrass-module of D.

§ 2. Weierstrass points.

Proposition 2. Fix an integer $m \geqq 0$ and a point $x \in C$. The following three assertions are equivalent;

(i) The canonical surjection $H^1(C, D - (m+1)x) \to H^1(C, D - mx)$ of diagram (5) is an isomorphism.

(ii) The canonical surjection $V^m(x) \to V^{m-1}(x)$ of diagram (5) is an isomorphism.

(iii) The kernel $(\Omega_C^1)^{\otimes m}(x)$ of the map $P^m(x) \to P^{m-1}(x)$ of diagram (3) is contained in the image of $v^m(x)$.

Proof. The equivalence of (i) and (ii) is immediate by diagram (5) and the equivalence of (ii) and (iii) follows after an easy chase in the diagram obtained from diagram (3) at the point x.

Definition. An integer $(m + 1) \geqq 1$ satisfying the three equivalent conditions of Proposition 3 is called a gap of D at x.

Remark. When $m \geqq d$ it follows from the Riemann-Roch theorem that $h^1(C, D - (m + 1)x) = g - d + m$. Hence if (m + 1) is a gap, then $1 \leqq (m + 1) \leqq (d + 1)$. On the other hand $h^1(C,D) = g-d+r$, so there are exactly $d + 1 - (g - (g - d + r)) = r + 1$ gaps of D at x that we denote by

$$1 \leqq g_1(x) < g_2(x) < \ldots < g_{r+1}(x) \leqq (d+1) .$$

Definition. If $g_{i+1}(x) = b_i + 1$ for $i = 0, 1, \ldots, r$ we call x an ordinary point of D. A point which is not ordinary is called a Weierstrass point of D.

Proposition 3. With the above notation the following assertions hold,

(i) We have $g_{i+1}(x) \geqq (b_i + 1)$ for $i = 0, 1, \ldots, r$ and for all points x of C.

(ii) We have $g_{i+1}(x) = (b_i + 1)$ for $i = 0, 1, \ldots, r$ for all but a finite number of points x of C. In other words, there are only a finite number of Weierstrass points of D.

(iii) A point x of C is a Weierstrass point of D if and only if

$$\dim W^m(x) > b_m - m \quad \text{for some} \quad m = 0, \ldots, r .$$

Proof. (i) Assume that $b_{s-1} \leqq m < b_s$. Then rank $B^m(D) = s$ and we have an inequality $\dim V^m(x) \geqq m+1-s$ with equality for all but a finite number of points x of C. Fix a point $x \in C$ and let $g(m) = \# \{g_i(x) \mid g_i(x) \leqq m+1\}$, that is $g(m)$ is the number j such that $g_1(x), \ldots, g_j(x) \leqq (m+1)$ but $g_{j+1}(x) > (m+1)$. Then by the definition of Weierstrass points, we have that $\dim V^m(x) = m+1-g(m)$. This equality together with the above inequality show that $s \geqq g(m)$ and in particular that $s \geqq g(b_s - 1)$. Hence, $g_{s+1}(x) > b_s$.

(ii) In the proof of part (i) we observed that $\dim V^m(x) = m+1-s$ for nearly all points x of C. It follows, by the definition of Weierstrass points, that the only value of m in the interval

$b_{s-1} \leqq m < b_s$ for which $m + 1$ is a gap is $m = b_{s-1}$. The gap values are therefore $b_o + 1, b_1 + 1, \ldots, b_r + 1$ for nearly all points of C.

(iii) In the proof of part (i) we observed that $\dim W^m(x) = b_m + 1 - g(b_m)$. If $\dim W^m(x) > b_m - m$, then we obtain that $m \geqq g(b_m)$ and consequently that $g_{m+1}(x) > b_m + 1$. That is, x is a Weierstrass point.

Conversely, if x is a Weierstrass point of D, then $g_{m+1}(x) > b_m+1$ for some m and consequently $g(b_m) < m + 1$. Hence we obtain the inequality

$$\dim W^m(x) = b_m + 1 - g(b_m) > b_m - m .$$

<u>Remark</u>. Mount and Villamayor ([7] Definition 2.7 p. 353) called a point x of C a Weierstrass point of the divisor D if the single condition $\dim W^r(x) > b_r - r$ is fulfilled. In section 5 (Example 1) below we show that even in the traditional case, that is when D is a canonical divisor, there may be points x of C that are Weierstrass points of D in the above sense, but with $\dim W^r(x) = b_r - r$. After having computed a large number of examples we are however amazed of how often the condition $\dim W^r(x) > b_r - r$ is necessary for a point to be Weierstrass even when the gap sequence is non-classical, that is when $b_i = i$ do not hold for all i. In the case that $b_i = i$ for $i = 1, \ldots, r + 1$ the condition $\dim W^r(x) > b_r - r$ is clearly necessary for a point to be Weierstrass. We shall show in § 5 (Theorem 11 (i)) that when char $k = 0$ or char $k > d$ then the gap sequence is always classical.

The definition given above of a Weierstrass point of a divisor D on a curve C is equivalent to the definition given by

F.K. Schmidt [10] in the case when D is a member of the canonical linear system. Indeed in [10] (p. 77) a point x of C is defined to be a Weierstrass point of C (or of the canonical system) if there are at most a finite number of points that have the same gap sequence as x and by Proposition 3 (ii) this happens if and only if $g_{i+1}(x) \neq b_i + 1$ for some i.

It is remarkable that Schmidt only observed ([10] p. 78) that the first non-zero integer in the sequence $g_1(x) - b_o - 1, g_2(x) - b_1 - 1, \ldots$ $\ldots, g_{r+1}(x) - b_r - 1$ is positive. The fact that they are all non-negative, which follows naturally from our approach (Proposition 3 (i)) was first proved by H.B. Matzat ([6] p. 17).

§ 3. The wronskian.

Theorem 4. Let $c_t < c_{t+1} < \ldots < c_s$ be integers satisfying $c_i \geqq i$ for $i = t, t+1, \ldots, s$ and let Q^i for $i = c_t, c_t + 1, \ldots, c_s$ be vector bundles on C with rank $Q^i = i + 1$. Moreover, let V be a vector space of dimension $n + 1$ and $v(i) : V_C \to Q^i$ be maps such that rank $v(i) \leqq m + 1$ when $c_m \leqq i < c_{m+1}$. Finally we let $q(i) : Q^i \to Q^{i-1}$ be surjective maps such that $q(i)v_i = v_{i-1}$ for $i = c_t + 1, \ldots, c_s$ and we denote the kernel of q(i) by Ω^i.

Then there exists a unique map

$$w(c_t, \ldots, c_s) : \overset{s+1}{\wedge} V_C \to (\overset{t+1}{\wedge} Q^{c_t}) \otimes \Omega^{c_{t+1}} \otimes \ldots \otimes \Omega^{c_s}$$

satisfying the following two properties;

(i) For all points x of C we have that

$$w(c_t, \ldots, c_s)(x) = 0 \iff \text{rank } v(c_m)(x) < m + 1$$

for some $m = t, t + 1, \ldots, s$.

(ii) Choose a basis $e(0), e(1), \ldots, e(c_s)$ of Q^{c_s} at a point x of C in such a way that $q(i+1)q(i+2)\ldots q(c_s)e(i)$ generates Ω^i at x for $i = c_{t+2}, \ldots, c_s$. Moreover, choose a basis $f(0), \ldots, f(n)$ of V and denote by $d(i_o, \ldots, i_v; j_o, \ldots, j_v)$ the vector $e(i_o, \ldots, i_v) = e(i_o) \wedge \ldots \wedge e(i_v)$ of $\overset{v+1}{\wedge} Q^{c_s}$ multiplied by the determinant of the $(v+1) \times (v+1)$-matrix formed from the rows $i_o + 1, \ldots, i_v + 1$ and columns $j_o + 1, \ldots, j_v + 1$ of the $(c_s + 1) \times (n+1)$-matrix obtained by expressing $v(c_s)$ in the above choice of bases'.

Then locally at x we have

$$w(c_t, \ldots, c_s)(f(j_o) \wedge \ldots \wedge f(j_s))$$
$$= \sum_{i_o<\ldots<i_t\leq c_t} d(i_o, \ldots, i_t, c_{t+1}, \ldots, c_s; j_o, \ldots, j_s).$$

<u>Proof</u>. The unicity of the map $w(c_t, \ldots, c_s)$ follows from assertion (ii) so it suffices to show the existence of the map. When $s = t$ we choose $w(c_s) = \overset{s+1}{\wedge} v(c_s)$. Then all the assertions of the theorem are fulfilled. We shall first, proceeding by induction with respect to $s - t$, simultaneously show the existence of $w(c_t, \ldots, c_s)$ and prove assertion (ii).

Let $\Omega_t = \Omega^{c_{t+1}} \otimes \ldots \otimes \Omega^{c_s}$. Assume that we have defined a map $w(c_{t+1}, \ldots, c_s) : \overset{s+1}{\wedge} V_C \to \overset{t+2}{\wedge} Q^{c_{t+1}} \otimes \Omega_{t+1}$ satisfying property (ii) of the theorem. Then the image of $w(c_{t+1}, \ldots, c_s)$ is contained in the kernel $\overset{t+1}{\wedge} Q^{c_{t+1}-1} \otimes \Omega^{c_{t+1}} \otimes \Omega_{t+1}$ of the map

$$\overset{t+2}{\wedge} q(c_{t+1}) \otimes \mathrm{id} : (\overset{t+2}{\wedge} Q^{c_{t+1}}) \otimes \Omega_{t+1} \to (\overset{t+2}{\wedge} Q^{c_{t+1}-1}) \otimes \Omega_{t+1}.$$

Indeed, since rank $v(c_{t+1} - 1) \leqq t + 1$, the elements

$$\left(\overset{t+2}{\Lambda}\ q(c_{t+1}) \otimes \mathrm{id}\right) w(c_{t+1}, \ldots, c_s)(f(j_o) \wedge \ldots \wedge f(j_s))$$

$$= \sum_{i_o<\ldots<i_{t+1}<c_{t+1}} d(i_o, \ldots, i_{t+1}, c_{t+2}, \ldots, c_s ; j_o, \ldots, j_s)$$

are zero for all choices of $j_o < \ldots < j_s$. It follows that we can define $w(c_t, \ldots, c_s)$ canonically as the composite of the map $\overset{s+1}{\Lambda}\ V_C \to \overset{t+1}{\Lambda}\ Q^{c_{t+1}-1} \otimes \Omega^{c_{t+1}} \otimes \Omega_{t+1}$, induced by $w(c_{t+1}, \ldots, c_s)$, with the map

$$\left(\overset{t+1}{\Lambda}\ q(c_{t+1} - 1)\ q(c_{t+1} - 2)\ \ldots\ q(c_t + 1)\right) \otimes \mathrm{id} :$$

$$: \left(\overset{t+1}{\Lambda}\ Q^{c_{t+1}-1}\right) \otimes \Omega_t \to \left(\overset{t+1}{\Lambda}\ Q^{c_t}\right) \otimes \Omega_t \ .$$

Then the following equalities hold:

$$w(c_t, \ldots, c_s)(f(j_o) \wedge \ldots \wedge f(j_s))$$

$$= \left(\overset{t+1}{\Lambda}\ q(c_{t+1} - 1) \otimes \ldots \otimes q(c_t + 1) \otimes \mathrm{id}\right)$$

$$\sum_{i_o<\ldots<i_t<c_{t+1}} d(i_o, \ldots, i_t, c_{t+1}, c_{t+2}, \ldots, c_s ; j_o, \ldots, j_s)$$

$$= \sum_{i_o<\ldots<i_t\leqq c_t} d(i_o, \ldots, i_t, c_{t+1}, \ldots, c_s ; j_o, \ldots, j_s) \ ,$$

the first equality being a consequence of the equalities $d(i_o, \ldots, i_{t+1}, c_{t+2}, \ldots, c_s ; j_o, \ldots, j_s) = 0$ for all indices $j_o < \ldots < j_s$ that we observed above.

It remains to prove the first assertion of the theorem. Assume that rank $v(c_m)(x) < m + 1$ for some $m = t, \ldots, s$. Then $d(i_o, \ldots, i_m, c_{m+1}, \ldots, c_s ; j_o, \ldots, j_s) = 0$ whenever $i_o < \ldots < i_m \leqq c_m$. Hence $w(c_m, \ldots, c_s)$ is zero by part (ii) of the theorem. However, if the mapping $w(c_m, \ldots, c_s)$ is zero, then certainly the mapping $w(c_t, \ldots, c_s)$ is zero, being the composite of the former with other mappings.

Conversely, assume that $w(c_t, \ldots, c_s)$ is zero at $x \in C$. We shall prove that rank $v(c_m)(x) < m + 1$ for some $m = t, \ldots, s$ by induction with respect to $s - t$. Assume that the assertion (i) holds for $s - t + 1$. Then if $w(c_{t+1}, \ldots, c_s)(x) = 0$ we have that rank $v(c_m)(x) = 0$ for some $m = t + 1, \ldots, s$ and we have finished. Hence we may assume that $w(c_{t+1}, \ldots, c_s)(x)$ is non-zero and consequently that rank $v(c_m)(x) = m + 1$ for $m = t + 1, \ldots, s$. The problem is to show that rank $v(c_t)(x) < t+1$.

Note that we have that rank $v(i)(x) = m + 1$ for $c_m \leqq i < c_{m+1}$ and all $m = t + 1, \ldots, s$. Write $K^i = \ker(q(i+1)q(i+2) \ldots q(c_s)(x))$ and $M = \operatorname{im} v(c_s)(x)$. Then $\dim M = s + 1$ and $M \cap K^{c_m} = M \cap K^{c_m+1} = \ldots = M \cap K^{c_{m+1}-1}$ for $m = t + 1, \ldots, s$. Moreover we have that $\dim (M \cap K^{c_m}) = s - m$ for $m = t + 1, \ldots, s$ and that $\dim (M \cap K^{c_t}) \geqq s - t$. We can consequently choose a basis $e(0), \ldots, e(c_s)$ of Q^{c_s} at x and a basis $f(0), \ldots, f(n)$ of V as in part (ii) of the theorem and satisfying the additional requirements that $e(c_m) \in (M \cap K^{c_m-1}) \setminus (M \cap K^{c_m})$ for $m = t + 1, \ldots, s$ and that $v(c_s)(f(n - i)) = e(c_{s-i})$ for $i = 0, \ldots, s-t-1$. With this choice of basis we have that

$$d(i_o, \ldots, i_s ; j_o, \ldots, j_t, n - s - t+1, \ldots, n) = 0$$

whenever $i_o < \ldots < i_s$ and $j_o < \ldots < j_t < n - s - t + 1$ unless $i_{t+1} = c_{t+1}, \ldots, i_s = c_s$ and that

$$d(i_o, \ldots, i_t, c_{t+1}, \ldots, c_s ; j_o, \ldots, j_t, n-s-t+1, \ldots n)$$

is the determinant formed from the rows $i_o, \ldots, i_t$ and columns $j_o, \ldots, j_t$ of the matrix $v(c_t)(x)$. Moreover, we have that the last $s - t$ columns of $v(c_{t+1} - 1)(x)$ are zero. Consequently the above determinants give all the $(t+1) \times (t+1)$-minors of $v(c_{t+1} - 1)(x)$. These are however all zero, because by assumption $w(c_t, \ldots, c_s)(x) = 0$ such that the determinants

$$d(i_o, \ldots, i_t, c_{t+1}, \ldots, c_s ; j_o, \ldots, j_s)$$

are zero for all indices $j_o < \ldots < j_s$ by part (ii) of the theorem. We conclude that $v(c_{t+1} - 1)$, and consequently $v(c_t)$, is of rank strictly less than $t + 1$ at x.

Corollary 5. With the notation of the previous sections there exists a canonical map

$$w : \stackrel{r+1}{\wedge} H^o(C, D)_C \to (\Omega_C^1)^{\otimes(b_o+\ldots+b_r)} \otimes \mathcal{O}(D)^{r+1}$$

such that $w(x) = 0$ if and only if $\dim W^m(x) > b_m - m$ for some $m = 0, \ldots, r$, that is, if and only if x is a Weierstrass point of D.

Fix a point x of C and a uniformizing parameter t at x. Choose a basis of $P^{c_r}(D)$ whose elements maps to the generators $(dt)^i$ of $(\Omega_C^1)^{\otimes i}$ for $i = 0, \ldots, c_r$ and fix any basis for $H^o(C, D)$. Then the determinant of the $(r+1) \times (r+1)$-matrix taken from rows $b_o + 1, b_1 + 1, \ldots, b_r + 1$ of the mapping $v^{b_r}: H^o(C,D)_C \to P^{c_r}(D)$ expressed in the above bases, vanishes to the same order as w, at x.

Proof. The corollary is a direct translation of Theorem 4 in the case when $t = 0$, $c_i = b_i$ for $i = 0, \ldots, r$ and when $v(i) = v^i : H^o(C, D)_C \to P^i(D)$ for $i = 0, \ldots, b_r$. Proposition 1 states that the mappings $v(i)$ and the integers c_i then satisfy the conditions of the theorem.

The assertions of the corollary are then seen to be exactly the same as those of the theorem once we observe that $\dim V^{c_m}(x) < m+1$ if and only if $\dim W^m(x) > b_m + 1 - (m+1)$.

Definition. The canonical map w of the line bundle $(\Omega^1_C)^{\otimes(b_o+\ldots+b_r)} \otimes \mathcal{O}(D)^{r+1}$ described in Corollary 5 is called the wronskian of the divisor D. The order to which w vanishes at a point x is called the weight of x with respect to D and the sum of the weights of all points of C is called the total weight of the Weierstrass points of D.

Theorem 6. The total weight of the Weierstrass points of D is

$$(2g-2)\sum_{i=o}^{r} b_i + (r+1)d \ .$$

Proof. By definition the total weight of the Weierstrass points of D is the weighted sum of the zeroes of the section w. It follows from Proposition 3 (ii) and the description of the zeroes of w in Corollary 5 that w is not identically zero. Consequently the total weight is the degree $(2g-2)\sum_{i=o}^{r} b_i + (r+1)d$ of the line bundle $(\Omega^1_C)^{\otimes(b_o+\ldots+b_r)} \otimes \mathcal{O}(D)^{r+1}$.

Remark. The formula of Theorem 6 is an immediate consequence of the existence of the wronskian and may therefore appear to be merely a matter of definition. Its content is however well illustrated by the class of hyperelliptic curves of Example 3, (§ 6) below. This class contains curves of arbitrarily high genus with classical gap sequence and with only one Weierstrass point. The formula then give the multiplicity of this point.

The wronskian determinant, considered as a section of a line bundle, was constructed in characteristic zero by G. Galbura [1] (proof of Teorema p. 351) by patching locally defined determinants and he observed (§ 4 p. 355) that the patching argument could be performed with suitable modifications over a field of positive characteristics. In the characteristic zero case we shall prove later (Theorem 11 (iv)) that the wronskian is simply the determinant of the mapping v^r so that in this case the existence and the properties of the wronskian do not require any particular construction. However, it was Galbura's observation about wronskians in positive characteristic that was the point of departure of the construction of Theorem 4.

Galbura used the wronskian to interpret a classical result of Brill and C. Segre (see [11] § 11 n. 44 p. 89) about $(r+1)$-tuple points of the linear system $H^o(C, D)$, in terms of sections of line bundles. We shall show how these results can be treated in arbitrary characteristic.

A point x of C is called an $(m+1)$-tuple point of the linear system $H^o(C, D)$ if there is a member of the linear system that vanishes to the order at least $(m + 1)$ at x. It follows from the definition of principal parts that if we let R denote the local ring of C at x and M its maximal ideal, then $P_C^m(D)(x) \xrightarrow{\sim} R/M^{m+1}$ and that if a section $s \in H^o(C, D)$ is represented at x by a function $f \in R$, then $v^m(x)(s)$ is the class of f in R/M^{m+1}. In particular s has an $(m+1)$-tuple zero at x if and only if the class of f is zero. We see that the space of sections of multiplicity at least $m + 1$ is $r + 1 - j$ dimensional if and only if $\dim(\operatorname{im} v^m(x)) \leqq j$. In particular, there is an $(r+1-m)$-dimensional space of sections of multiplicity at least $b_m + 1$ if and only if $\dim W^m(x) \geqq b_m + 1 - m$. We shall in analogy with the classical case call a point x a <u>strictly</u> $(r+1)$-<u>tuple point</u> of the linear system if the space of divisors that vanish to order at least $b_m + 1$ is of dimension at least $(r + 1 - m)$ for some $m = 0, \ldots, r$. When $b_i = i$ for $i = 0,\ldots, r$ we have that a point x is strictly $(r+1)$-tuple if and only if there is a section of the linear system that vanishes to order at least $r + 1$ at x. Indeed, in this case if there is an $(r + 1 - j)$-dimensional space of sections that vanish, at a point x, to order at least $j + 1$ for some j, then there is at least an $(r - j)$-dimensional space of sections that vanish to order at least $j + 2$ at x. We shall prove in § 5 (Theorem 11 (i)) that the equalities $b_i = i$ for $i = 0, \ldots, r$ hold when char $k = 0$ or char $k > d$. Hence in these cases our definition of strictly $(r+1)$-tuple points coincides with the traditional one of $(r+1)$-tuple points.

By the above discussion it is clear that a strictly $(r+1)$-tuple point is nothing but a Weierstrass point and we can give these points the weight given by the order of vanishing of the wronskian section. Theorem 6 can then be reformulated in the following way:

<u>Theorem 7</u> (Brill-Segre). With the above definitions, the number of strictly $(r+1)$-tuple points of the linear system $H^o(C, D)$ (counted with multiplicity) is given by the expression

$$(2g - 2) \sum_{i=o}^{r} b_i + (r + 1)d \ .$$

Severi ([12]) expressed the number of $(r+1)$-tuple points in terms of a functional equation in a canonical divisor K and a hyperplane section H. In arbitrary characteristic we obtain

$$\sum_{i=o}^{r} b_i K + (r + 1)H \ .$$

This equation is immediate from the form of the line bundle that has w as a section.

§ 4. Local computations.

Fix a point x of C and a local parameter of C at x. Let R be the local ring of C at x and let I be the kernel of the multiplication map $R \otimes R \to R$. Then we have that $P^m(D)_x \simeq R \otimes R/I^{m+1}$ and that under the identification $(\Omega_C^1)^{\otimes m} = I^m/I^{m+1}$ the generator $(dt)^m$ maps to $(t \otimes 1 - 1 \otimes t)^m$. The R-module structure on $P^m(D)_x$ corresponding to the projection p makes $P^m(D)_x$ into a left R-module and as such it is free with basis $1, dt, \ldots, (dt)^m$. Denote by $d_R^m : R \to R \otimes R/I^{m+1}$ the map induced by the other projection q, that is $d_R^m f$ is the class of the element $1 \otimes f$. For each element $f \in R$ we denote by $d^i f$ the coefficient of $(dt)^i$ in the expression of $d_R^m f$ in the above basis. We consider $P^m(D)_x$ as an R-module via the first factor and write simply f instead of $f \otimes 1$. We then have $d_R^m t = t + dt$ and since $d_R^m t^h = (d_R t)^h$ we obtain the formula

$$(6) \quad d_R^m t^h = \sum_{i=o}^{h} \binom{h}{i} t^{h-i} (dt)^i .$$

In particular we have that $d^i t^h = \binom{h}{i} t^{h-i}$.

Lemma 8. Let $f \in R$ and suppose that $f \equiv at^h \pmod{t^{h+1}}$ with a in the ground field. Then $d^i f \equiv a\binom{h}{i} t^{h-i} \pmod{t^{h-i+1}}$.

Proof. Write $f = at^h + bt^{h+1}$ with $b \in R$. Since d_R^m is linear it follows from the formula (6) that it is sufficient to prove that $d^i(bt^{h+1}) = t^{h+1-i}c$ for some element $c \in R$. However, since $d_R^m(bt^{h+1}) = d_R^m b \cdot d_R t^{h+1}$ we have that $d^i(bt^{h+1}) = \sum_{j=o}^{i} (d^{i-j}b)(d^j t^{h+1})$ and by formula (6) $d^j t^{h+1} \equiv 0 \pmod{t^{h+1-i}}$ for $j = 0, 1, \ldots, i$.

The equality of the lemma is therefore a consequence of formula (6).

Lemma 9. Let $x_o, x_1, \ldots, x_r$ be variables. Let M be the $(r+1) \times (r+1)$-matrix with entry $\binom{x_j}{i}$ in the i'th row and j'th column. Then we have that

$$\prod_{i=o}^{r} (i!)(\det M) = \prod_{o \leqq j < i \leqq r} (x_i - x_j) .$$

Proof. (See e.g. [8], § 327 p. 324.) The determinant of the $(r+1) \times (r+1)$-matrix of the lemma is clearly an alternating function in the variables $x_o, \ldots, x_r$. Hence the determinant is divisible by the polynomial $\prod_{o \leqq j < i \leqq r} (x_i - x_j)$. The two polynomials are homogenous of the same degree. Hence the expression of the lemma follows by equating the coefficients of a particular monomial, e.g. $x_o^r \, x_1^{r-1} \ldots x_r$, in the two polynomials.

Choose a basis $v_o, \ldots, v_r$ of $H^o(C, D)$. At a point x of C this basis determines functions $f_o, \ldots, f_r$ in the local ring R of C at x. We can clearly choose the basis in such a way that the order of vanishing of the functions f_i at x form a sequence $\text{order}_x f_o < \ldots < \text{order}_x f_r$. The integers $h_i = \text{order}_x f_i$ will be called the Hermite invariants of D at x. (That they are local invariants is easily verified and moreover follows from Theorem 10 below.)

With respect to the latter choice of basis of $H^o(C, D)$ and the basis $1, (dt), \ldots, (dt)^m$ of $P^m(D)_x$ the map

$$v_x^m : H^o(C, D) \otimes R \to P_x^m(D)$$

is expressed by the $(m+1) \times (r+1)$-matrix with coordinate $d^i f_j$ in row i and column j. Writing only the lowest order terms in t it follows from Lemma 7 that this matrix takes the form,

$$(7)\qquad \begin{vmatrix} a_o\binom{h_o}{0}t^{h_o}+\ldots, & a_1\binom{h_1}{0}t^{h_1}+\ldots, & \ldots, & a_r\binom{h_r}{0}t^{h_r}+\ldots \\ a_o\binom{h_o}{1}t^{h_o-1}+\ldots, & a_1\binom{h_1}{1}t^{h_1-1}+\ldots, & \ldots, & a_r\binom{h_r}{1}t^{h_r-1}+\ldots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_o\binom{h_o}{m}t^{h_o-m}+\ldots, & a_1\binom{h_1}{m}t^{h_1-m}+\ldots, & \ldots, & a_r\binom{h_r}{m}t^{h_r-m}+\ldots \end{vmatrix}$$

where the a_i's are non-zero elements of k and $f_i \equiv a_i t^{h_i}$ (mod t^{h_i+1}) for $i = 0, \ldots, r$.

Remark. The above local computations are well known. Slightly less detailed accounts can be found in articles by A. Kato ([4] Lemma (2.3) and proof of Theorem (2.4)) and R. Piene ([9] (§ 2 and § 6)). Piene claims that (7), with a suitable change of basis, can be brought into a normal form whose entries in the $(i+1)$'st row and column is $\binom{h_i}{i}$ for $i = 0, 1, \ldots, m$ and and all the other entries are zero and she uses this (proof of Theorem (3.2) p. 481) to show that the lowest order of the determinants of the $(m+1) \times (m+1)$-submatrices of (7) is $\sum_{i=o}^{m}(h_i-i)$ when the characteristic of the ground field is zero or does not divide the $\binom{h_i}{i}$ for $i = 0, \ldots, m$. It is however not always possible to obtain such normal forms when $h_i \neq i$ for some i. In such cases Theorem 10 (i) below, which is an immediate consequence of Lemma 9, shows that the lowest order of the determinants is indeed $\sum_{i=o}^{m} (h_i - i)$ under appropriate restrictions on the characteristic of the ground field.

Theorem 10. With the above notation the following four assertions hold:

(i) $$\left(\prod_{i=o}^{r} i!\right)\det v_x^r \equiv \prod_{i=o}^{r} a_i \prod_{o\leq j<i\leq r} (h_i - h_j)t^h \pmod{t^{h+1}}$$

where $h = \sum_{i=o}^{r} (h_i - i)$.

(ii) $\dim(\operatorname{im} v^m(x)) = i$ for $h_{i-1} \leq m < h_i$ and $i = 0, \ldots, r+1$.

(iii) $\ker(P^m(x) \to P^{m-1}(x)) \subseteq \operatorname{im} v^m(x)$ if and only if $m = h_i$ for some $i = 0, \ldots, r$.

(iv) $g_{i+1}(x) = h_i + 1$ for $i = 0, \ldots, r$.

Proof. From expression (7) we see that modulo t^{h+1} it is only the lowest order terms of the coordinates that contribute to the determinant of v_x^r. Assertion (i) consequently follows from Lemma 9.

At the point $t = 0$ the expression (7) shows that if $h_{i-1} \leq m < h_i$ then the columns $i + 1, i + 2, \ldots, r + 1$ of the matrix representing $v^m(x)$ are zero. Since $\binom{h_j}{h_j} = 1$ in all characteristics the $(h_j + 1, j + 1)$-entry is nonzero and the expression (7) shows that the first h_j entries of column $(j + 1)$ is zero, it follows that the first i columns are linearly independent. In other words, assertion (ii) of the theorem holds.

More precisely, the above argument shows that $\operatorname{im} v^m(x)$ is generated by elements $(dt)^{h_o} + \sum_{j=h_o+1}^{m} b_{o,j}(dt)^j, \ldots$. $(dt)^{h_{i-1}} + \sum_{j=h_{i-1}+1}^{m} b_{i-1,j}(dt)^j$ with $b_{i,j} \in k$. We see the element $(dt)^m$ which generates the kernel of the map $P^m(x) \to P^{m-1}(x)$ is in $\operatorname{im} v^m(x)$ if and only if $m = h_{i-1}$. Hence assertion (iii) holds.

Finally, assertion (iii) of Proposition 2 states that the integer $(m+1)$ is a gap if and only if $\ker(P^m(x) \to P^{m-1}(x)) \subseteq V^m(x)$. Consequently assertion (iv) follows from assertion (iii).

§ 5. Characteristic $(k) = p$ with $p = 0$ or $p > d$.

Theorem 11. Let $p = \operatorname{char}(k)$ and assume that $p = 0$ or $p > d$. Then the following assertions hold;

(i) We have $b_i = i$ for $i = 0, \ldots, r$.

(ii) For a general point x of C we have $g_i(x) = i$ for $i = 1, \ldots, r+1$.

(iii) A point x of C is a Weierstrass point of D if and only if

$$\dim W^r(x) > 0 .$$

(iv) The wronskian determinant is the composite of the canonical map $\overset{r+1}{\wedge} v^r : \overset{r+1}{\wedge} H^o(C,D)_C \to \overset{r+1}{\wedge} P^r(D)$ with the canonical isomorphism $\overset{r+1}{\wedge} P^r(D) \cong \bigotimes_{i=o}^{r} (\Omega_C^1)^{\otimes i} \otimes \mathcal{O}(D)$ obtained from the sequences (2) of § 1.

(v) (C. Segre [11] § 11 n. 43 p. 86, see also C. Galbura [1] Teorema 2 p. 352). The weight of a point x of C is $\sum_{i=o}^{r} (g_{i+1}(x) - i - 1)$.

(vi) The total weight of D is $((g-1)r + d)(r+1)$.

Proof. (i). We observed in the definition of gaps in § 2 that for all points x of C we have $g_i(x) \leqq (d+1)$ for $i = 1, \ldots, r+1$. Consequently it follows from Theorem 10 (iv) that $h_i \leqq d$ for $i = 0, \ldots, r$. Moreover, it follows from the Riemann-Roch theorem that $r \leqq d$. Hence we see from the formula of Theorem 10 (i) that det v_x^r is not identically zero. It follows that v^r is generically surjective and consequently that rank $B^r(D) = r$.

(ii). This assertion follows from (i) and Proposition 3 (ii).

(iii). It follows from (i) and Proposition 3 (iii) that a point x of C is Weierstrass if and only if dim $W^m(x) > 0$ for some $m = 0, \ldots, r$. The latter assertion is however equivalent to the assertion that dim $W^r(x) > 0$ because of the surjections $W^m(D) \to W^{m-1}(D)$ of diagram (3) of § 1.

(iv). This assertion is an easy consequence of property (ii) of Theorem 4.

(v). The order of vanishing of the wronskian w at x is by (iv) equal to the order of vanishing of det v^r at x. It follows from Theorem 10 (i) and (iv) that this order is equal to $\sum_{i=o}^{r} (h_i - i) = \sum_{i=o}^{r} (g_{i+1}(x) - i - 1)$.

(vi). This assertion follows from Proposition 6 and assertion (i) above.

Remark. Curves having property (i) of Theorem 11 are said to have a classical gap sequence. Several classes of such curves are known (see e.g. [6] and [10]). For curves with a non-classical gap sequence the properties (i) - (vi) all fail. However, as we have seen in the previous sections, the properties (i) - (iv) and (vi) all generalize naturally to this case. The most unfortunate feature of the non-classical case is that property (v), which is very important for the computation of multiplicities of Weierstrass points, do not generalize to state that the local multiplicity is $\sum_{i=o}^{r} (g_{i+1}(x) - b_i - 1)$ (see example 2 of § 6 below). This makes it the more surprising that the global formula (property (vi)) globalizes (Theorem 6).

§ 6. Examples.

1. The following example was discussed by K. Komiya ([5] § 4 p. 390) who showed that the curves $x^3 + y^3 + z^3 + \lambda = 0$ with $\lambda^2 \neq \lambda$ are the only curves of genus 4 in characteristic 2 with non-classical gap sequences. We shall prove that they give examples of curves having a Weierstrass point x (or more precisely having a point x such that $\dim W^2(x) > b_2 - 2$) with the property that $\dim W^r(x) = \dim W^3(x) = b_3 - 3$. That is x is not a Weierstrass point in the sense of Mount and Villamayor.

Assume that char $k = 2$. Let $C \subseteq \mathbb{P}^3$ be the complete intersection of the two hypersurfaces $x_0x_3 = x_1x_2$ and $x_1^3 + x_2^3 + x_3^3 + \lambda x_0^3$ where $\lambda^2 \neq \lambda$. Then C is non-singular of degree 6 and genus 4 and consequently is canonically embedded ([3] Chapter 4, § 5 Example 5.2.2). We choose D to be a canonical divisor.

Map the curve C into $\mathbb{P}^2$ by the projection of $\mathbb{P}^3$ to $\mathbb{P}^2$ with center at the point (0; 0; 0; 1). We obtain a plane model of C with equation $x_1^3x_2^3 + x_0^3x_1^3 + x_0^3x_2^3 + \lambda x_0^6$. The plane model is non-singular in the affine piece $x_0 \neq 0$. At the point (1; a; c) of the plane model $x^3y^3 + x^3 + y^3 + \lambda$ we can choose $x - a = t$ as local parameter. Write $y - c \equiv \sum_{i=1}^{4} c_it^i \pmod{t^5}$ and substitute the expressions for x and y into the equation of the plane model. Comparing coefficients of the powers of t we obtain when $b \neq 0$ the following equations

$$c_1 = \frac{(c^3+1)a^2}{(a^3+1)c^2}, \quad c_2 = \frac{a(c^3+1)\lambda}{c^5(a^3+1)^2}, \quad c_3 = \frac{\lambda^2(c^3+1)}{c^8(1+a^3)^3} \quad \text{and}$$

$$c_4 = \frac{a^2(c^3+1)}{c^8(a^3+1)^4}(\lambda + a^3c^3 + a^6c^6) .$$

At a point (1; a; c; ac) with $a^3c^3 + a^3 + c^3 + \lambda = 0$ and $c \neq 0$

the embedding $C \subseteq \mathbb{P}^3$ is given by the functions

$$e_o = 1,\ e_1 = a + t,\ e_2 = c + \sum_{i=1}^{\infty} c_i t^i \text{ and } e_3 = ct + \sum_{i=1}^{\infty} c_i t^{i+1}$$

or after a change of basis

$$f_o = 1,\ f_1 = t,\ f_2 = c_2 t^2 + c_3 t^3 + \dots \quad \text{and} \quad f_3 = c_1 t^2 + c_2 t^3 + \dots .$$

From the above formulas we obtain the two equations $c_2^2 = c_1 c_3$ and $c_3 c_2 - c_1 c_4 = \dfrac{(c^3 + 1)^2 a}{c^{13}(a^3 + 1)^5} (\lambda^3 - a^3 c^3 (\lambda + a^3 c^3 + a^6 c^6))$.

Using the relation $(c^3 + 1)a^3 = c^3 + \lambda$ it is easily checked that the latter equation is non-zero for most choices of c. Consequently the Hermite invariants at a general point of C are 0, 1, 2 and 4. Again it follows from Theorem 10 (iv) and Proposition 3 (ii) that $b_o = 0$, $b_1 = 1$, $b_2 = 2$ and $b_3 = 4$. The point (1; 0; d) with $d^3 + \lambda = 0$ is on C. At this point $c_1 = c_2 = c_4 = 0$ and $c_3 = \dfrac{d^3 + 1}{d^2} \neq 0$ and the Hermite invariants are 0, 1, 3 and 4. Consequently (1; 0; d) is a Weierstrass point of the canonical divisor d.

On the other hand the matrix expressing v^4 at the point (1; 0; c) is

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_3 \end{vmatrix} .$$

We see that rank $v^4(1; 0; c) = 4$ so that $\dim W^3(1; 0; c) = 5 - 4 = b_3 - 3$.

2. The purpose of the following example is to show that the weight of a Weierstrass point x is not necessarily equal to $\sum_{i=o}^{r} g_{i+1}(x) - b_i - 1$.

Assume that char $k = 3$ and let C be the elliptic plane curve given by the equation $y = x^3 + xy^2$. Then
$(x) = (0; 0; 1) + (0; 1; 0) - (i; 1; 0) - (-i; 1; 0)$ and
$(y) = 3(0; 0; 1) - (0; 1; 0) - (i; 1; 0) - (-i; 1; 0)$.
We let D be the hyperplane section $(0; 1; 0) + (i; 1; 0) + (-i; 1; 0)$. Then $\deg D = 3$, $\dim H^o(C, D) = 3$ and $H^o(C, D)$ is spanned by 1, x and y. At a point $(a; c)$ of the curve we choose a local parameter $t = x - a$ and we let $y = c + c_1 t + c_2 t^2 \ldots$. Substitution of x and y into the equations of the curve allows us to determine $c_1, c_2, \ldots$. We obtain $c_1 = c^2(1 - 2ac)^{-1}$, $c_2 = c_1(ac_1 + 2c)(1 - 2ac)^{-1}$ and $c_3 = (1 + 2ac_1c_2 + c_1^2 + 2cc_2)(1 - 2ac)$. Hence $b_o = 0$, $b_1 = 1$ and $b_2 = 2$, and at the point $(0; 0)$ we have $h_o = 0$, $h_1 = 1$ and $h_2 = 3$.

At $(0, 0)$ we have that $\sum_{i=o}^{r} g_{i+1}(0, 0) - b_i - 1 = 1$. However the wronskian matrix is of the form

$$\begin{vmatrix} 1 & a + t & t^3 + t^7 + 2t^{11} + \ldots \\ 0 & 1 & t^6 + t^{10} + \ldots \\ 0 & 0 & 2t^9 + \ldots \end{vmatrix} .$$

Consequently the multiplicity of the wronskian is 9. It is easily checked that the point $(0; 0)$ is the only Weierstrass point in the affine piece $z = 1$ and considering the equation $z^2 = x^3 + x$ of the curve in the affine piece $y = 1$ one sees that there are no Weierstrass points in the latter affine space. Hence the curve has only one Weierstrass point and this point has multiplicity 9.

3. It is well known (see e.g. F.K. Schmidt [10]) that the hyperelliptic curves furnish examples of curves of arbitrary high genus and with classical general gap sequences, but with only one Weierstrass point. We shall give a treatment of a class of such curves suited to our purposes.

Assume that char $k = 2$. Let m be an odd integer and let C be the hyperelliptic curve of genus $\frac{m-1}{2}$ having $y^2 + y = x^m$ as a plane model. The map $(x, y) \to x$ of the plane model onto $\mathbb{A}^1$ induces a map $C \to \mathbb{P}^1$ which is unramified of degree 2 over $\mathbb{A}^1$ and has a single ramification point P_∞ over the point at infinity. At all points (a, b) of $\mathbb{A}^1$ we can take $(x + a)$ as a local parameter. We find that $(x) = (0,0) + (0,1) - 2P_\infty$ and $(y) = m(0, 0) - mP_\infty$. Hence if t is a local parameter at P_∞ then at this point $x = ut^{-2}$ and $y = vt^{-m}$ with u and v units. The function $y + b$ is a local parameter at all points (a, b) of $\mathbb{A}^1$ except at $(0, 0)$ and $(1, 0)$ where it has multiplicity m. We obtain that $(dy) = (m-1)(0, 1) + (m-1)(0, 0) - (m+1)P_\infty$ and that $K = (x^{-m+1} dy) = m - 3$. Moreover, we see that $1, x, x^2, \ldots, x^{\frac{m-3}{2}}$ is a basis for $H^0(C, K)$. Since the function $x + a = t$ is a local parameter at all points (a, b) of $\mathbb{A}^1$ and the functions $1, (a + t), (a + t)^2, \ldots, (a + t)^{\frac{m-3}{2}}$ are linearly independent for all $a \in k$ we have that $b_i = h_i = i$ for $i = 0, 1, \ldots, r$. Hence P_∞ is the only Weierstrass point and by Proposition 6 it has weight $(g + 1)g(g - 1)$ $= \frac{1}{8}(m^3 - 3m^2 - m + 3)$.

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